

16.4 Green's Theorem

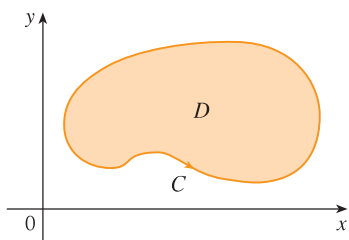


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C . (See Figure 1. We assume that D consists of all points inside C as well as all points on C .) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C . Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C . (See Figure 2.)

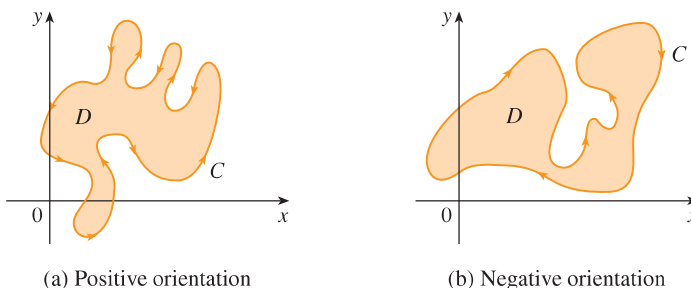


FIGURE 2

Recall that the left side of this equation is another way of writing $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$.

Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

NOTE The notation

$$\oint_C P dx + Q dy \quad \text{or} \quad \oint_C P dx + Q dy$$

is sometimes used to indicate that the line integral is calculated using the positive orientation of the closed curve C . Another notation for the positively oriented boundary curve of D is ∂D , so the equation in Green's Theorem can be written as

$$\boxed{1} \quad \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{\partial D} P dx + Q dy$$

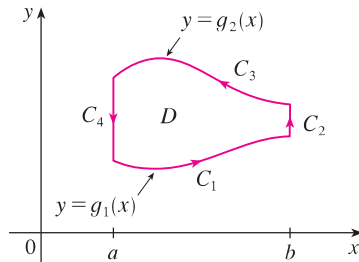
Green's Theorem should be regarded as the counterpart of the Fundamental Theorem of Calculus for double integrals. Compare Equation 1 with the statement of the Fundamental Theorem of Calculus, Part 2, in the following equation:

$$\int_a^b F'(x) dx = F(b) - F(a)$$

In both cases there is an integral involving derivatives (F' , $\partial Q/\partial x$, and $\partial P/\partial y$) on the left side of the equation. And in both cases the right side involves the values of the original functions (F , Q , and P) only on the *boundary* of the domain. (In the one-dimensional case, the domain is an interval $[a, b]$ whose boundary consists of just two points, a and b .)

George Green

Green's Theorem is named after the self-taught English scientist George Green (1793–1841). He worked full-time in his father's bakery from the age of nine and taught himself mathematics from library books. In 1828 he published privately *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism*, but only 100 copies were printed and most of those went to his friends. This pamphlet contained a theorem that is equivalent to what we know as Green's Theorem, but it didn't become widely known at that time. Finally, at age 40, Green entered Cambridge University as an undergraduate but died four years after graduation. In 1846 William Thomson (Lord Kelvin) located a copy of Green's essay, realized its significance, and had it reprinted. Green was the first person to try to formulate a mathematical theory of electricity and magnetism. His work was the basis for the subsequent electromagnetic theories of Thomson, Stokes, Rayleigh, and Maxwell.

**FIGURE 3**

Green's Theorem is not easy to prove in general, but we can give a proof for the special case where the region is both type I and type II (see Section 15.2). Let's call such regions **simple regions**.

PROOF OF GREEN'S THEOREM FOR THE CASE IN WHICH D IS A SIMPLE REGION Notice that Green's Theorem will be proved if we can show that

$$\boxed{2} \quad \int_C P \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

and

$$\boxed{3} \quad \int_C Q \, dy = \iint_D \frac{\partial Q}{\partial x} \, dA$$

We prove Equation 2 by expressing D as a type I region:

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

where g_1 and g_2 are continuous functions. This enables us to compute the double integral on the right side of Equation 2 as follows:

$$\boxed{4} \quad \iint_D \frac{\partial P}{\partial y} \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}(x, y) \, dy \, dx = \int_a^b [P(x, g_2(x)) - P(x, g_1(x))] \, dx$$

where the last step follows from the Fundamental Theorem of Calculus.

Now we compute the left side of Equation 2 by breaking up C as the union of the four curves C_1 , C_2 , C_3 , and C_4 shown in Figure 3. On C_1 we take x as the parameter and write the parametric equations as $x = x$, $y = g_1(x)$, $a \leq x \leq b$. Thus

$$\int_{C_1} P(x, y) \, dx = \int_a^b P(x, g_1(x)) \, dx$$

Observe that C_3 goes from right to left but $-C_3$ goes from left to right, so we can write the parametric equations of $-C_3$ as $x = x$, $y = g_2(x)$, $a \leq x \leq b$. Therefore

$$\int_{C_3} P(x, y) \, dx = - \int_{-C_3} P(x, y) \, dx = - \int_a^b P(x, g_2(x)) \, dx$$

On C_2 or C_4 (either of which might reduce to just a single point), x is constant, so $dx = 0$ and

$$\int_{C_2} P(x, y) \, dx = 0 = \int_{C_4} P(x, y) \, dx$$

Hence

$$\begin{aligned} \int_C P(x, y) \, dx &= \int_{C_1} P(x, y) \, dx + \int_{C_2} P(x, y) \, dx + \int_{C_3} P(x, y) \, dx + \int_{C_4} P(x, y) \, dx \\ &= \int_a^b P(x, g_1(x)) \, dx - \int_a^b P(x, g_2(x)) \, dx \end{aligned}$$

Comparing this expression with the one in Equation 4, we see that

$$\int_C P(x, y) \, dx = - \iint_D \frac{\partial P}{\partial y} \, dA$$

Equation 3 can be proved in much the same way by expressing D as a type II region (see Exercise 30). Then, by adding Equations 2 and 3, we obtain Green's Theorem. ■

EXAMPLE 1 Evaluate $\int_C x^4 \, dx + xy \, dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

SOLUTION Although the given line integral could be evaluated as usual by the methods of Section 16.2, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead. Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4). If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

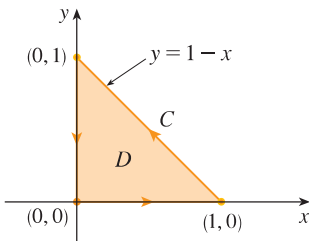


FIGURE 4

$$\begin{aligned} \int_C x^4 \, dx + xy \, dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_0^1 \int_0^{1-x} (y - 0) \, dy \, dx \\ &= \int_0^1 \left[\frac{1}{2} y^2 \right]_{y=0}^{y=1-x} \, dx = \frac{1}{2} \int_0^1 (1-x)^2 \, dx \\ &= -\frac{1}{6} (1-x)^3 \Big|_0^1 = \frac{1}{6} \end{aligned}$$

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy$, where C is the circle $x^2 + y^2 = 9$.

SOLUTION The region D bounded by C is the disk $x^2 + y^2 \leq 9$, so let's change to polar coordinates after applying Green's Theorem:

$$\begin{aligned} \oint_C (3y - e^{\sin x}) \, dx + (7x + \sqrt{y^4 + 1}) \, dy &= \iint_D \left[\frac{\partial}{\partial x} (7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y} (3y - e^{\sin x}) \right] \, dA \\ &= \int_0^{2\pi} \int_0^3 (7 - 3) r \, dr \, d\theta = 4 \int_0^{2\pi} d\theta \int_0^3 r \, dr = 36\pi \end{aligned}$$

Instead of using polar coordinates, we could simply use the fact that D is a disk of radius 3 and write

$$\iint_D 4 \, dA = 4 \cdot \pi(3)^2 = 36\pi$$

In Examples 1 and 2 we found that the double integral was easier to evaluate than the line integral. (Try setting up the line integral in Example 2 and you'll soon be convinced!) But sometimes it's easier to evaluate the line integral, and Green's Theorem is used in the reverse direction. For instance, if it is known that $P(x, y) = Q(x, y) = 0$ on the curve C , then Green's Theorem gives

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int_C P \, dx + Q \, dy = 0$$

no matter what values P and Q assume in the region D .

Another application of the reverse direction of Green's Theorem is in computing areas. Since the area of D is $\iint_D 1 \, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\frac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \frac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D :

5

$$A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx$$

EXAMPLE 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION The ellipse has parametric equations $x = a \cos t$ and $y = b \sin t$, where $0 \leq t \leq 2\pi$. Using the third formula in Equation 5, we have

$$\begin{aligned} A &= \frac{1}{2} \int_C x \, dy - y \, dx \\ &= \frac{1}{2} \int_0^{2\pi} (a \cos t)(b \cos t) \, dt - (b \sin t)(-a \sin t) \, dt \\ &= \frac{ab}{2} \int_0^{2\pi} dt = \pi ab \end{aligned}$$

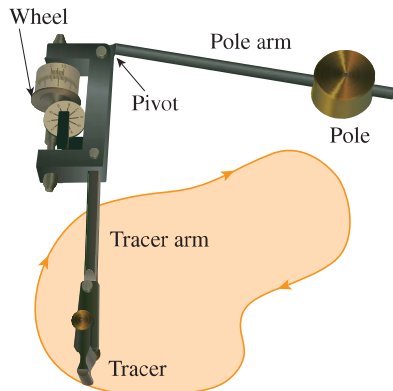


FIGURE 5
A Keuffel and Esser polar planimeter

Formula 5 can be used to explain how planimeters work. A **planimeter** is a mechanical instrument used for measuring the area of a region by tracing its boundary curve. These devices are useful in all the sciences: in biology for measuring the area of leaves or wings, in medicine for measuring the size of cross-sections of organs or tumors, in forestry for estimating the size of forested regions from photographs.

Figure 5 shows the operation of a polar planimeter: the pole is fixed and, as the tracer is moved along the boundary curve of the region, the wheel partly slides and partly rolls perpendicular to the tracer arm. The planimeter measures the distance that the wheel rolls and this is proportional to the area of the enclosed region. The explanation as a consequence of Formula 5 can be found in the following articles:

- R. W. Gatterman, "The planimeter as an example of Green's Theorem" *Amer. Math. Monthly*, Vol. 88 (1981), pp. 701–4.
- Tanya Leise, "As the planimeter wheel turns" *College Math. Journal*, Vol. 38 (2007), pp. 24–31.

Extended Versions of Green's Theorem

Although we have proved Green's Theorem only for the case where D is simple, we can now extend it to the case where D is a finite union of simple regions. For example, if D is the region shown in Figure 6, then we can write $D = D_1 \cup D_2$, where D_1 and D_2 are both simple. The boundary of D_1 is $C_1 \cup C_3$ and the boundary of D_2 is $C_2 \cup (-C_3)$ so, applying Green's Theorem to D_1 and D_2 separately, we get

$$\int_{C_1 \cup C_3} P \, dx + Q \, dy = \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\int_{C_2 \cup (-C_3)} P \, dx + Q \, dy = \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

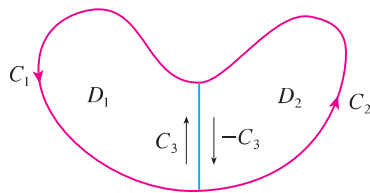


FIGURE 6

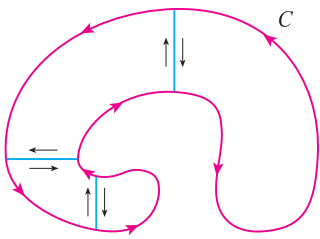


FIGURE 7

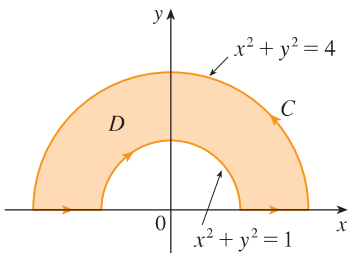


FIGURE 8

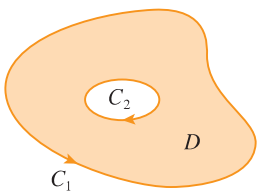


FIGURE 9

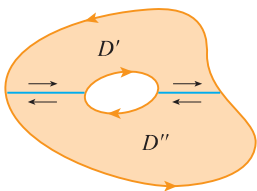


FIGURE 10

If we add these two equations, the line integrals along C_3 and $-C_3$ cancel, so we get

$$\int_{C_1 \cup C_2} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

which is Green's Theorem for $D = D_1 \cup D_2$, since its boundary is $C = C_1 \cup C_2$.

The same sort of argument allows us to establish Green's Theorem for any finite union of nonoverlapping simple regions (see Figure 7).

EXAMPLE 4 Evaluate $\oint_C y^2 dx + 3xy dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

SOLUTION Notice that although D is not simple, the y -axis divides it into two simple regions (see Figure 8). In polar coordinates we can write

$$D = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$$

Therefore Green's Theorem gives

$$\begin{aligned} \oint_C y^2 dx + 3xy dy &= \iint_D \left[\frac{\partial}{\partial x} (3xy) - \frac{\partial}{\partial y} (y^2) \right] dA \\ &= \iint_D y dA = \int_0^\pi \int_1^2 (r \sin \theta) r dr d\theta \\ &= \int_0^\pi \sin \theta d\theta \int_1^2 r^2 dr = [-\cos \theta]_0^\pi \left[\frac{1}{3} r^3 \right]_1^2 = \frac{14}{3} \end{aligned}$$

Green's Theorem can be extended to apply to regions with holes, that is, regions that are not simply-connected. Observe that the boundary C of the region D in Figure 9 consists of two simple closed curves C_1 and C_2 . We assume that these boundary curves are oriented so that the region D is always on the left as the curve C is traversed. Thus the positive direction is counterclockwise for the outer curve C_1 but clockwise for the inner curve C_2 . If we divide D into two regions D' and D'' by means of the lines shown in Figure 10 and then apply Green's Theorem to each of D' and D'' , we get

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA &= \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \int_{\partial D'} P dx + Q dy + \int_{\partial D''} P dx + Q dy \end{aligned}$$

Since the line integrals along the common boundary lines are in opposite directions, they cancel and we get

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_{C_1} P dx + Q dy + \int_{C_2} P dx + Q dy = \int_C P dx + Q dy$$

which is Green's Theorem for the region D .

EXAMPLE 5 If $\mathbf{F}(x, y) = (-y \mathbf{i} + x \mathbf{j}) / (x^2 + y^2)$, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi$ for every positively oriented simple closed path that encloses the origin.

SOLUTION Since C is an arbitrary closed path that encloses the origin, it's difficult to compute the given integral directly. So let's consider a counterclockwise-oriented circle C'

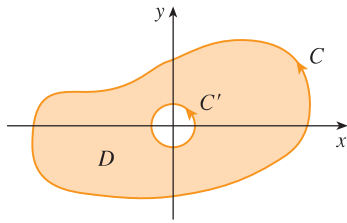


FIGURE 11

with center the origin and radius a , where a is chosen to be small enough that C' lies inside C . (See Figure 11.) Let D be the region bounded by C and C' . Then its positively oriented boundary is $C \cup (-C')$ and so the general version of Green's Theorem gives

$$\begin{aligned} \int_C P dx + Q dy + \int_{-C'} P dx + Q dy &= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \\ &= \iint_D \left[\frac{y^2 - x^2}{(x^2 + y^2)^2} - \frac{y^2 - x^2}{(x^2 + y^2)^2} \right] dA = 0 \end{aligned}$$

Therefore
$$\int_C P dx + Q dy = \int_{C'} P dx + Q dy$$

that is,
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r}$$

We now easily compute this last integral using the parametrization given by $\mathbf{r}(t) = a \cos t \mathbf{i} + a \sin t \mathbf{j}$, $0 \leq t \leq 2\pi$. Thus

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt \\ &= \int_0^{2\pi} \frac{(-a \sin t)(-a \sin t) + (a \cos t)(a \cos t)}{a^2 \cos^2 t + a^2 \sin^2 t} dt = \int_0^{2\pi} dt = 2\pi \quad \blacksquare \end{aligned}$$

We end this section by using Green's Theorem to discuss a result that was stated in the preceding section.

SKETCH OF PROOF OF THEOREM 16.3.6 We're assuming that $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ is a vector field on an open simply-connected region D , that P and Q have continuous first-order partial derivatives, and that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

If C is any simple closed path in D and R is the region that C encloses, then Green's Theorem gives

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_R 0 dA = 0$$

A curve that is not simple crosses itself at one or more points and can be broken up into a number of simple curves. We have shown that the line integrals of \mathbf{F} around these simple curves are all 0 and, adding these integrals, we see that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C . Therefore $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D by Theorem 16.3.3. It follows that \mathbf{F} is a conservative vector field. ■

16.4 EXERCISES

1–4 Evaluate the line integral by two methods: (a) directly and (b) using Green's Theorem.

1. $\oint_C y^2 dx + x^2 y dy$,
 C is the rectangle with vertices $(0, 0)$, $(5, 0)$, $(5, 4)$, and $(0, 4)$

2. $\oint_C y dx - x dy$,
 C is the circle with center the origin and radius 4

3. $\oint_C xy dx + x^2 y^3 dy$,
 C is the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$

4. $\oint_C x^2y^2 dx + xy dy$, C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$

5–10 Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

5. $\int_C ye^x dx + 2e^x dy$,
 C is the rectangle with vertices $(0, 0)$, $(3, 0)$, $(3, 4)$, and $(0, 4)$
6. $\int_C (x^2 + y^2) dx + (x^2 - y^2) dy$,
 C is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$
7. $\int_C (y + e^{\sqrt{x}}) dx + (2x + \cos y^2) dy$,
 C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$
8. $\int_C y^4 dx + 2xy^3 dy$, C is the ellipse $x^2 + 2y^2 = 2$
9. $\int_C y^3 dx - x^3 dy$, C is the circle $x^2 + y^2 = 4$
10. $\int_C (1 - y^3) dx + (x^3 + e^{y^2}) dy$, C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$

11–14 Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

11. $\mathbf{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$,
 C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$
12. $\mathbf{F}(x, y) = \langle e^{-x} + y^2, e^{-y} + x^2 \rangle$,
 C consists of the arc of the curve $y = \cos x$ from $(-\pi/2, 0)$ to $(\pi/2, 0)$ and the line segment from $(\pi/2, 0)$ to $(-\pi/2, 0)$
13. $\mathbf{F}(x, y) = \langle y - \cos y, x \sin y \rangle$,
 C is the circle $(x - 3)^2 + (y + 4)^2 = 4$ oriented clockwise
14. $\mathbf{F}(x, y) = \langle \sqrt{x^2 + 1}, \tan^{-1}x \rangle$, C is the triangle from $(0, 0)$ to $(1, 1)$ to $(0, 1)$ to $(0, 0)$

CAS 15–16 Verify Green's Theorem by using a computer algebra system to evaluate both the line integral and the double integral.

15. $P(x, y) = x^3y^4$, $Q(x, y) = x^5y^4$,
 C consists of the line segment from $(-\pi/2, 0)$ to $(\pi/2, 0)$ followed by the arc of the curve $y = \cos x$ from $(\pi/2, 0)$ to $(-\pi/2, 0)$
16. $P(x, y) = 2x - x^3y^5$, $Q(x, y) = x^3y^8$,
 C is the ellipse $4x^2 + y^2 = 4$

17. Use Green's Theorem to find the work done by the force $\mathbf{F}(x, y) = x(x + y) \mathbf{i} + xy^2 \mathbf{j}$ in moving a particle from the origin along the x -axis to $(1, 0)$, then along the line segment to $(0, 1)$, and then back to the origin along the y -axis.
18. A particle starts at the origin, moves along the x -axis to $(5, 0)$, then along the quarter-circle $x^2 + y^2 = 25$, $x \geq 0$, $y \geq 0$ to the point $(0, 5)$, and then down the y -axis back to the origin. Use Green's Theorem to find

the work done on this particle by the force field $\mathbf{F}(x, y) = \langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \rangle$.

19. Use one of the formulas in (5) to find the area under one arch of the cycloid $x = t - \sin t$, $y = 1 - \cos t$.

20. If a circle C with radius 1 rolls along the outside of the circle $x^2 + y^2 = 16$, a fixed point P on C traces out a curve called an *epicycloid*, with parametric equations $x = 5 \cos t - \cos 5t$, $y = 5 \sin t - \sin 5t$. Graph the epicycloid and use (5) to find the area it encloses.

21. (a) If C is the line segment connecting the point (x_1, y_1) to the point (x_2, y_2) , show that

$$\int_C x dy - y dx = x_1y_2 - x_2y_1$$

- (b) If the vertices of a polygon, in counterclockwise order, are (x_1, y_1) , (x_2, y_2) , \dots , (x_n, y_n) , show that the area of the polygon is

$$A = \frac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + \dots + (x_{n-1}y_n - x_ny_{n-1}) + (x_ny_1 - x_1y_n)]$$

- (c) Find the area of the pentagon with vertices $(0, 0)$, $(2, 1)$, $(1, 3)$, $(0, 2)$, and $(-1, 1)$.

22. Let D be a region bounded by a simple closed path C in the xy -plane. Use Green's Theorem to prove that the coordinates of the centroid (\bar{x}, \bar{y}) of D are

$$\bar{x} = \frac{1}{2A} \oint_C x^2 dy \quad \bar{y} = -\frac{1}{2A} \oint_C y^2 dx$$

where A is the area of D .

23. Use Exercise 22 to find the centroid of a quarter-circular region of radius a .
24. Use Exercise 22 to find the centroid of the triangle with vertices $(0, 0)$, $(a, 0)$, and (a, b) , where $a > 0$ and $b > 0$.
25. A plane lamina with constant density $\rho(x, y) = \rho$ occupies a region in the xy -plane bounded by a simple closed path C . Show that its moments of inertia about the axes are

$$I_x = -\frac{\rho}{3} \oint_C y^3 dx \quad I_y = \frac{\rho}{3} \oint_C x^3 dy$$

26. Use Exercise 25 to find the moment of inertia of a circular disk of radius a with constant density ρ about a diameter. (Compare with Example 15.4.4.)

27. Use the method of Example 5 to calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y) = \frac{2xy \mathbf{i} + (y^2 - x^2) \mathbf{j}}{(x^2 + y^2)^2}$$

and C is any positively oriented simple closed curve that encloses the origin.

28. Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

29. If \mathbf{F} is the vector field of Example 5, show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path that does not pass through or enclose the origin.

30. Complete the proof of the special case of Green's Theorem by proving Equation 3.
31. Use Green's Theorem to prove the change of variables formula for a double integral (Formula 15.9.9) for the case where $f(x, y) = 1$:

$$\iint_R dx \, dy = \iint_S \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du \, dv$$

Here R is the region in the xy -plane that corresponds to the region S in the uv -plane under the transformation given by $x = g(u, v)$, $y = h(u, v)$.

[Hint: Note that the left side is $A(R)$ and apply the first part of Equation 5. Convert the line integral over ∂R to a line integral over ∂S and apply Green's Theorem in the uv -plane.]

16.5 Curl and Divergence

In this section we define two operations that can be performed on vector fields and that play a basic role in the applications of vector calculus to fluid flow and electricity and magnetism. Each operation resembles differentiation, but one produces a vector field whereas the other produces a scalar field.

Curl

If $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$ is a vector field on \mathbb{R}^3 and the partial derivatives of P , Q , and R all exist, then the curl of \mathbf{F} is the vector field on \mathbb{R}^3 defined by

$$\mathbf{1} \quad \text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

As an aid to our memory, let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}$$

It has meaning when it operates on a scalar function to produce the gradient of f :

$$\nabla f = \mathbf{i} \frac{\partial f}{\partial x} + \mathbf{j} \frac{\partial f}{\partial y} + \mathbf{k} \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field \mathbf{F} as follows:

$$\begin{aligned} \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= \text{curl } \mathbf{F} \end{aligned}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

2

$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F}$$

EXAMPLE 1 If $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$, find $\text{curl } \mathbf{F}$.

SOLUTION Using Equation 2, we have

$$\begin{aligned} \text{curl } \mathbf{F} &= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz) \right] \mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz) \right] \mathbf{j} \\ &\quad + \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz) \right] \mathbf{k} \\ &= (-2y - xy) \mathbf{i} - (0 - x) \mathbf{j} + (yz - 0) \mathbf{k} \\ &= -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k} \end{aligned}$$

CAS Most computer algebra systems have commands that compute the curl and divergence of vector fields. If you have access to a CAS, use these commands to check the answers to the examples and exercises in this section.

Recall that the gradient of a function f of three variables is a vector field on \mathbb{R}^3 and so we can compute its curl. The following theorem says that the curl of a gradient vector field is $\mathbf{0}$.

3 Theorem If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

PROOF We have

$$\begin{aligned} \text{curl}(\nabla f) &= \nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} \\ &= \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right) \mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right) \mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) \mathbf{k} \\ &= 0 \mathbf{i} + 0 \mathbf{j} + 0 \mathbf{k} = \mathbf{0} \end{aligned}$$

by Clairaut's Theorem. ■

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

Compare this with Exercise 16.3.29.

If \mathbf{F} is conservative, then $\text{curl } \mathbf{F} = \mathbf{0}$.

This gives us a way of verifying that a vector field is not conservative.

EXAMPLE 2 Show that the vector field $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$ is not conservative.

SOLUTION In Example 1 we showed that

$$\operatorname{curl} \mathbf{F} = -y(2 + x) \mathbf{i} + x \mathbf{j} + yz \mathbf{k}$$

This shows that $\operatorname{curl} \mathbf{F} \neq \mathbf{0}$ and so, by the remarks preceding this example, \mathbf{F} is not conservative. ■

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if \mathbf{F} is defined everywhere. (More generally it is true if the domain is simply-connected, that is, “has no hole.”) Theorem 4 is the three-dimensional version of Theorem 16.3.6. Its proof requires Stokes’ Theorem and is sketched at the end of Section 16.8.

4 Theorem If \mathbf{F} is a vector field defined on all of \mathbb{R}^3 whose component functions have continuous partial derivatives and $\operatorname{curl} \mathbf{F} = \mathbf{0}$, then \mathbf{F} is a conservative vector field.

EXAMPLE 3

(a) Show that

$$\mathbf{F}(x, y, z) = y^2z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2z^2 \mathbf{k}$$

is a conservative vector field.

(b) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION

(a) We compute the curl of \mathbf{F} :

$$\begin{aligned} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} \\ &= (6xyz^2 - 6xyz^2) \mathbf{i} - (3y^2z^2 - 3y^2z^2) \mathbf{j} + (2yz^3 - 2yz^3) \mathbf{k} \\ &= \mathbf{0} \end{aligned}$$

Since $\operatorname{curl} \mathbf{F} = \mathbf{0}$ and the domain of \mathbf{F} is \mathbb{R}^3 , \mathbf{F} is a conservative vector field by Theorem 4.

(b) The technique for finding f was given in Section 16.3. We have

$$\text{5} \quad f_x(x, y, z) = y^2z^3$$

$$\text{6} \quad f_y(x, y, z) = 2xyz^3$$

$$\text{7} \quad f_z(x, y, z) = 3xy^2z^2$$

Integrating (5) with respect to x , we obtain

$$\text{8} \quad f(x, y, z) = xy^2z^3 + g(y, z)$$

Differentiating (8) with respect to y , we get $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$, so comparison with (6) gives $g_y(y, z) = 0$. Thus $g(y, z) = h(z)$ and

$$f_z(x, y, z) = 3xy^2z^2 + h'(z)$$

Then (7) gives $h'(z) = 0$. Therefore

$$f(x, y, z) = xy^2z^3 + K \quad \blacksquare$$

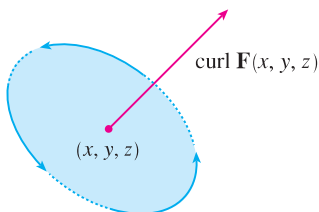


FIGURE 1

The reason for the name *curl* is that the curl vector is associated with rotations. One connection is explained in Exercise 37. Another occurs when \mathbf{F} represents the velocity field in fluid flow (see Example 16.1.3). Particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of $\text{curl } \mathbf{F}(x, y, z)$, and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1). If $\text{curl } \mathbf{F} = \mathbf{0}$ at a point P , then the fluid is free from rotations at P and \mathbf{F} is called **irrotational** at P . In other words, there is no whirlpool or eddy at P . If $\text{curl } \mathbf{F} = \mathbf{0}$, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis. If $\text{curl } \mathbf{F} \neq \mathbf{0}$, the paddle wheel rotates about its axis. We give a more detailed explanation in Section 16.8 as a consequence of Stokes' Theorem.

■ Divergence

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on \mathbb{R}^3 and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of \mathbf{F}** is the function of three variables defined by

9

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that $\text{curl } \mathbf{F}$ is a vector field but $\text{div } \mathbf{F}$ is a scalar field. In terms of the gradient operator $\nabla = (\partial/\partial x)\mathbf{i} + (\partial/\partial y)\mathbf{j} + (\partial/\partial z)\mathbf{k}$, the divergence of \mathbf{F} can be written symbolically as the dot product of ∇ and \mathbf{F} :

10

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

EXAMPLE 4 If $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$, find $\text{div } \mathbf{F}$.

SOLUTION By the definition of divergence (Equation 9 or 10) we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz \quad \blacksquare$$

If \mathbf{F} is a vector field on \mathbb{R}^3 , then $\text{curl } \mathbf{F}$ is also a vector field on \mathbb{R}^3 . As such, we can compute its divergence. The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on \mathbb{R}^3 and P , Q , and R have continuous second-order partial derivatives, then

$$\text{div } \text{curl } \mathbf{F} = 0$$

PROOF Using the definitions of divergence and curl, we have

Note the analogy with the scalar triple product: $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

$$\begin{aligned}\operatorname{div} \operatorname{curl} \mathbf{F} &= \nabla \cdot (\nabla \times \mathbf{F}) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\ &= \frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 Q}{\partial x \partial z} + \frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 R}{\partial y \partial x} + \frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 P}{\partial z \partial y} \\ &= 0\end{aligned}$$

because the terms cancel in pairs by Clairaut's Theorem. ■

EXAMPLE 5 Show that the vector field $\mathbf{F}(x, y, z) = xz \mathbf{i} + xyz \mathbf{j} - y^2 \mathbf{k}$ can't be written as the curl of another vector field, that is, $\mathbf{F} \neq \operatorname{curl} \mathbf{G}$.

SOLUTION In Example 4 we showed that

$$\operatorname{div} \mathbf{F} = z + xz$$

and therefore $\operatorname{div} \mathbf{F} \neq 0$. If it were true that $\mathbf{F} = \operatorname{curl} \mathbf{G}$, then Theorem 11 would give

$$\operatorname{div} \mathbf{F} = \operatorname{div} \operatorname{curl} \mathbf{G} = 0$$

which contradicts $\operatorname{div} \mathbf{F} \neq 0$. Therefore \mathbf{F} is not the curl of another vector field. ■

The reason for this interpretation of $\operatorname{div} \mathbf{F}$ will be explained at the end of Section 16.9 as a consequence of the Divergence Theorem.

Again, the reason for the name *divergence* can be understood in the context of fluid flow. If $\mathbf{F}(x, y, z)$ is the velocity of a fluid (or gas), then $\operatorname{div} \mathbf{F}(x, y, z)$ represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume. In other words, $\operatorname{div} \mathbf{F}(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z) . If $\operatorname{div} \mathbf{F} = 0$, then \mathbf{F} is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f . If f is a function of three variables, we have

$$\operatorname{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$$

in terms of its components:

$$\nabla^2 \mathbf{F} = \nabla^2 P \mathbf{i} + \nabla^2 Q \mathbf{j} + \nabla^2 R \mathbf{k}$$

Vector Forms of Green’s Theorem

The curl and divergence operators allow us to rewrite Green’s Theorem in versions that will be useful in our later work. We suppose that the plane region D , its boundary curve C , and the functions P and Q satisfy the hypotheses of Green’s Theorem. Then we consider the vector field $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$. Its line integral is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P dx + Q dy$$

and, regarding \mathbf{F} as a vector field on \mathbb{R}^3 with third component 0, we have

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P(x, y) & Q(x, y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}$$

Therefore

$$(\text{curl } \mathbf{F}) \cdot \mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can now rewrite the equation in Green’s Theorem in the vector form

12 $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$

Equation 12 expresses the line integral of the tangential component of \mathbf{F} along C as the double integral of the vertical component of $\text{curl } \mathbf{F}$ over the region D enclosed by C . We now derive a similar formula involving the *normal* component of \mathbf{F} .

If C is given by the vector equation

$$\mathbf{r}(t) = x(t) \mathbf{i} + y(t) \mathbf{j} \quad a \leq t \leq b$$

then the unit tangent vector (see Section 13.2) is

$$\mathbf{T}(t) = \frac{x'(t)}{|\mathbf{r}'(t)|} \mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|} \mathbf{j}$$

You can verify that the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|} \mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|} \mathbf{j}$$

(See Figure 2.) Then, from Equation 16.2.3, we have

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{n} ds &= \int_a^b (\mathbf{F} \cdot \mathbf{n})(t) |\mathbf{r}'(t)| dt \\ &= \int_a^b \left[\frac{P(x(t), y(t)) y'(t)}{|\mathbf{r}'(t)|} - \frac{Q(x(t), y(t)) x'(t)}{|\mathbf{r}'(t)|} \right] |\mathbf{r}'(t)| dt \\ &= \int_a^b P(x(t), y(t)) y'(t) dt - Q(x(t), y(t)) x'(t) dt \\ &= \int_C P dy - Q dx = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA \end{aligned}$$

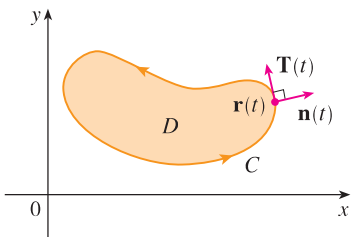


FIGURE 2

by Green's Theorem. But the integrand in this double integral is just the divergence of \mathbf{F} . So we have a second vector form of Green's Theorem.

13

$$\oint_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

This version says that the line integral of the normal component of \mathbf{F} along C is equal to the double integral of the divergence of \mathbf{F} over the region D enclosed by C .

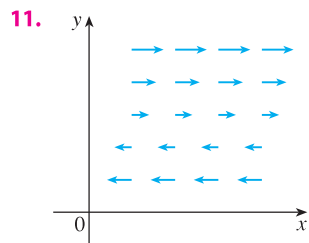
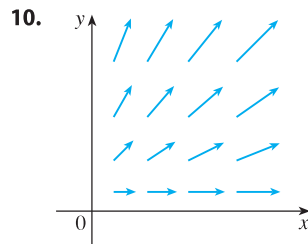
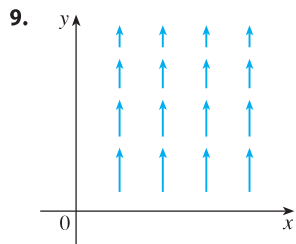
16.5 EXERCISES

1–8 Find (a) the curl and (b) the divergence of the vector field.

- $\mathbf{F}(x, y, z) = xy^2z^2 \mathbf{i} + x^2yz^2 \mathbf{j} + x^2y^2z \mathbf{k}$
- $\mathbf{F}(x, y, z) = x^3yz^2 \mathbf{j} + y^4z^3 \mathbf{k}$
- $\mathbf{F}(x, y, z) = xy e^z \mathbf{i} + yz e^x \mathbf{k}$
- $\mathbf{F}(x, y, z) = \sin yz \mathbf{i} + \sin zx \mathbf{j} + \sin xy \mathbf{k}$
- $\mathbf{F}(x, y, z) = \frac{\sqrt{x}}{1+z} \mathbf{i} + \frac{\sqrt{y}}{1+x} \mathbf{j} + \frac{\sqrt{z}}{1+y} \mathbf{k}$
- $\mathbf{F}(x, y, z) = \ln(2y + 3z) \mathbf{i} + \ln(x + 3z) \mathbf{j} + \ln(x + 2y) \mathbf{k}$
- $\mathbf{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$
- $\mathbf{F}(x, y, z) = \langle \arctan(xy), \arctan(yz), \arctan(zx) \rangle$

9–11 The vector field \mathbf{F} is shown in the xy -plane and looks the same in all other horizontal planes. (In other words, \mathbf{F} is independent of z and its z -component is 0.)

- Is $\operatorname{div} \mathbf{F}$ positive, negative, or zero? Explain.
- Determine whether $\operatorname{curl} \mathbf{F} = \mathbf{0}$. If not, in which direction does $\operatorname{curl} \mathbf{F}$ point?



12. Let f be a scalar field and \mathbf{F} a vector field. State whether each expression is meaningful. If not, explain why. If so, state whether it is a scalar field or a vector field.

- | | |
|--|--|
| (a) $\operatorname{curl} f$ | (b) $\operatorname{grad} f$ |
| (c) $\operatorname{div} \mathbf{F}$ | (d) $\operatorname{curl}(\operatorname{grad} f)$ |
| (e) $\operatorname{grad} \mathbf{F}$ | (f) $\operatorname{grad}(\operatorname{div} \mathbf{F})$ |
| (g) $\operatorname{div}(\operatorname{grad} f)$ | (h) $\operatorname{grad}(\operatorname{div} f)$ |
| (i) $\operatorname{curl}(\operatorname{curl} \mathbf{F})$ | (j) $\operatorname{div}(\operatorname{div} \mathbf{F})$ |
| (k) $(\operatorname{grad} f) \times (\operatorname{div} \mathbf{F})$ | (l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ |

13–18 Determine whether or not the vector field is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

- $\mathbf{F}(x, y, z) = y^2z^3 \mathbf{i} + 2xyz^3 \mathbf{j} + 3xy^2z^2 \mathbf{k}$
- $\mathbf{F}(x, y, z) = xyz^4 \mathbf{i} + x^2z^4 \mathbf{j} + 4x^2yz^3 \mathbf{k}$
- $\mathbf{F}(x, y, z) = z \cos y \mathbf{i} + xz \sin y \mathbf{j} + x \cos y \mathbf{k}$
- $\mathbf{F}(x, y, z) = \mathbf{i} + \sin z \mathbf{j} + y \cos z \mathbf{k}$
- $\mathbf{F}(x, y, z) = e^{yz} \mathbf{i} + xze^{yz} \mathbf{j} + xye^{yz} \mathbf{k}$
- $\mathbf{F}(x, y, z) = e^x \sin yz \mathbf{i} + ze^x \cos yz \mathbf{j} + ye^x \cos yz \mathbf{k}$

19. Is there a vector field \mathbf{G} on \mathbb{R}^3 such that $\operatorname{curl} \mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

20. Is there a vector field \mathbf{G} on \mathbb{R}^3 such that $\operatorname{curl} \mathbf{G} = \langle x, y, z \rangle$? Explain.

21. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x) \mathbf{i} + g(y) \mathbf{j} + h(z) \mathbf{k}$$

where f, g, h are differentiable functions, is irrotational.

22. Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z) \mathbf{i} + g(x, z) \mathbf{j} + h(x, y) \mathbf{k}$$

is incompressible.

23–29 Prove the identity, assuming that the appropriate partial derivatives exist and are continuous. If f is a scalar field and \mathbf{F}, \mathbf{G} are vector fields, then $f\mathbf{F}, \mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are defined by

$$\begin{aligned}(f\mathbf{F})(x, y, z) &= f(x, y, z)\mathbf{F}(x, y, z) \\ (\mathbf{F} \cdot \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z) \\ (\mathbf{F} \times \mathbf{G})(x, y, z) &= \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)\end{aligned}$$

- 23.** $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$
- 24.** $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$
- 25.** $\text{div}(f\mathbf{F}) = f \text{div } \mathbf{F} + \mathbf{F} \cdot \nabla f$
- 26.** $\text{curl}(f\mathbf{F}) = f \text{curl } \mathbf{F} + (\nabla f) \times \mathbf{F}$
- 27.** $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$
- 28.** $\text{div}(\nabla f \times \nabla g) = 0$
- 29.** $\text{curl}(\text{curl } \mathbf{F}) = \text{grad}(\text{div } \mathbf{F}) - \nabla^2 \mathbf{F}$

30–32 Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$.

- 30.** Verify each identity.
 - (a) $\nabla \cdot \mathbf{r} = 3$
 - (b) $\nabla \cdot (r\mathbf{r}) = 4r$
 - (c) $\nabla^2 r^3 = 12r$
- 31.** Verify each identity.
 - (a) $\nabla r = \mathbf{r}/r$
 - (b) $\nabla \times \mathbf{r} = \mathbf{0}$
 - (c) $\nabla(1/r) = -\mathbf{r}/r^3$
 - (d) $\nabla \ln r = \mathbf{r}/r^2$
- 32.** If $\mathbf{F} = \mathbf{r}/r^p$, find $\text{div } \mathbf{F}$. Is there a value of p for which $\text{div } \mathbf{F} = 0$?

33. Use Green's Theorem in the form of Equation 13 to prove **Green's first identity**:

$$\iint_D f \nabla^2 g \, dA = \oint_C f(\nabla g) \cdot \mathbf{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous. (The quantity $\nabla g \cdot \mathbf{n} = D_n g$ occurs in the line integral. This is the directional derivative in the direction of the normal vector \mathbf{n} and is called the **normal derivative** of g .)

34. Use Green's first identity (Exercise 33) to prove **Green's second identity**:

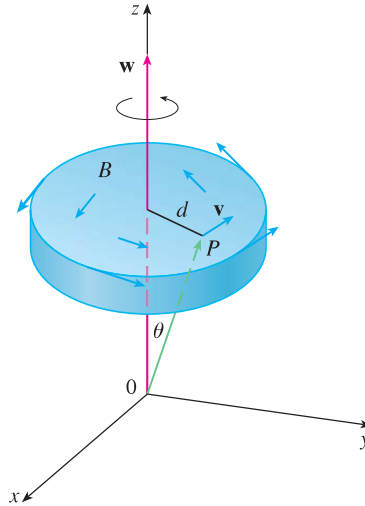
$$\iint_D (f \nabla^2 g - g \nabla^2 f) \, dA = \oint_C (f \nabla g - g \nabla f) \cdot \mathbf{n} \, ds$$

where D and C satisfy the hypotheses of Green's Theorem and the appropriate partial derivatives of f and g exist and are continuous.

35. Recall from Section 14.3 that a function g is called *harmonic* on D if it satisfies Laplace's equation, that is, $\nabla^2 g = 0$ on D . Use Green's first identity (with the same hypotheses as in

Exercise 33) to show that if g is harmonic on D , then $\oint_C D_n g \, ds = 0$. Here $D_n g$ is the normal derivative of g defined in Exercise 33.

- 36.** Use Green's first identity to show that if f is harmonic on D , and if $f(x, y) = 0$ on the boundary curve C , then $\iint_D |\nabla f|^2 \, dA = 0$. (Assume the same hypotheses as in Exercise 33.)
- 37.** This exercise demonstrates a connection between the curl vector and rotations. Let B be a rigid body rotating about the z -axis. The rotation can be described by the vector $\mathbf{w} = \omega \mathbf{k}$, where ω is the angular speed of B , that is, the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} = \langle x, y, z \rangle$ be the position vector of P .
 - (a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.
 - (b) Show that $\mathbf{v} = -\omega y \mathbf{i} + \omega x \mathbf{j}$.
 - (c) Show that $\text{curl } \mathbf{v} = 2\mathbf{w}$.



38. Maxwell's equations relating the electric field \mathbf{E} and magnetic field \mathbf{H} as they vary with time in a region containing no charge and no current can be stated as follows:

$$\begin{aligned}\text{div } \mathbf{E} &= 0 & \text{div } \mathbf{H} &= 0 \\ \text{curl } \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t} & \text{curl } \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}\end{aligned}$$

where c is the speed of light. Use these equations to prove the following:

- (a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$
- (b) $\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$
- (c) $\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2}$ [Hint: Use Exercise 29.]
- (d) $\nabla^2 \mathbf{H} = \frac{1}{c^2} \frac{\partial^2 \mathbf{H}}{\partial t^2}$