

39. We have seen that all vector fields of the form $\mathbf{F} = \nabla g$ satisfy the equation $\text{curl } \mathbf{F} = \mathbf{0}$ and that all vector fields of the form $\mathbf{F} = \text{curl } \mathbf{G}$ satisfy the equation $\text{div } \mathbf{F} = 0$ (assuming continuity of the appropriate partial derivatives). This suggests the question: are there any equations that all functions of the

form $f = \text{div } \mathbf{G}$ must satisfy? Show that the answer to this question is “No” by proving that *every* continuous function f on \mathbb{R}^3 is the divergence of some vector field.

[Hint: Let $\mathbf{G}(x, y, z) = \langle g(x, y, z), 0, 0 \rangle$, where $g(x, y, z) = \int_0^x f(t, y, z) dt$.]

16.6 Parametric Surfaces and Their Areas

So far we have considered special types of surfaces: cylinders, quadric surfaces, graphs of functions of two variables, and level surfaces of functions of three variables. Here we use vector functions to describe more general surfaces, called *parametric surfaces*, and compute their areas. Then we take the general surface area formula and see how it applies to special surfaces.

Parametric Surfaces

In much the same way that we describe a space curve by a vector function $\mathbf{r}(t)$ of a single parameter t , we can describe a surface by a vector function $\mathbf{r}(u, v)$ of two parameters u and v . We suppose that

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

is a vector-valued function defined on a region D in the uv -plane. So x , y , and z , the component functions of \mathbf{r} , are functions of the two variables u and v with domain D . The set of all points (x, y, z) in \mathbb{R}^3 such that

$$x = x(u, v) \quad y = y(u, v) \quad z = z(u, v)$$

and (u, v) varies throughout D , is called a **parametric surface** S and Equations 2 are called **parametric equations** of S . Each choice of u and v gives a point on S ; by making all choices, we get all of S . In other words, the surface S is traced out by the tip of the position vector $\mathbf{r}(u, v)$ as (u, v) moves throughout the region D . (See Figure 1.)

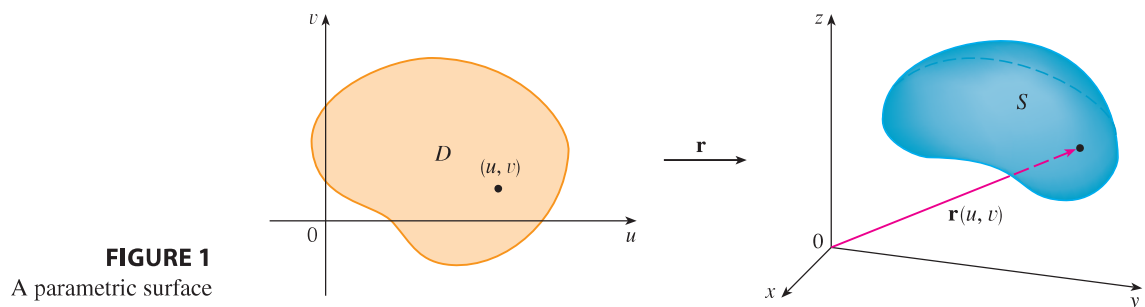


FIGURE 1
A parametric surface

EXAMPLE 1 Identify and sketch the surface with vector equation

$$\mathbf{r}(u, v) = 2 \cos u \mathbf{i} + v \mathbf{j} + 2 \sin u \mathbf{k}$$

SOLUTION The parametric equations for this surface are

$$x = 2 \cos u \quad y = v \quad z = 2 \sin u$$

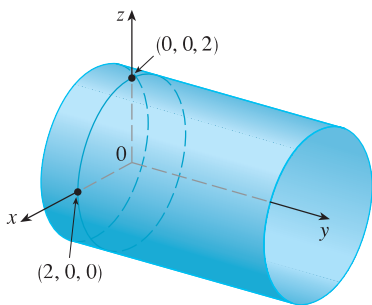


FIGURE 2

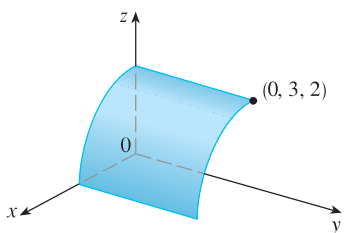


FIGURE 3

TEC Visual 16.6 shows animated versions of Figures 4 and 5, with moving grid curves, for several parametric surfaces.

So for any point (x, y, z) on the surface, we have

$$x^2 + z^2 = 4 \cos^2 u + 4 \sin^2 u = 4$$

This means that vertical cross-sections parallel to the xz -plane (that is, with y constant) are all circles with radius 2. Since $y = v$ and no restriction is placed on v , the surface is a circular cylinder with radius 2 whose axis is the y -axis (see Figure 2). ■

In Example 1 we placed no restrictions on the parameters u and v and so we obtained the entire cylinder. If, for instance, we restrict u and v by writing the parameter domain as

$$0 \leq u \leq \pi/2 \quad 0 \leq v \leq 3$$

then $x \geq 0, z \geq 0, 0 \leq y \leq 3$, and we get the quarter-cylinder with length 3 illustrated in Figure 3.

If a parametric surface S is given by a vector function $\mathbf{r}(u, v)$, then there are two useful families of curves that lie on S , one family with u constant and the other with v constant. These families correspond to vertical and horizontal lines in the uv -plane. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a curve C_1 lying on S . (See Figure 4.)

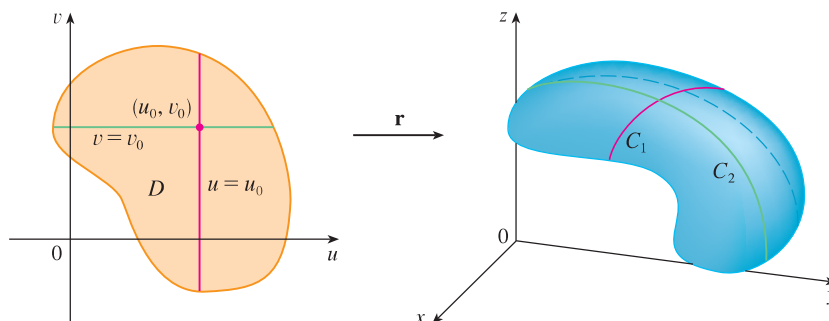


FIGURE 4

Similarly, if we keep v constant by putting $v = v_0$, we get a curve C_2 given by $\mathbf{r}(u, v_0)$ that lies on S . We call these curves **grid curves**. (In Example 1, for instance, the grid curves obtained by letting u be constant are horizontal lines whereas the grid curves with v constant are circles.) In fact, when a computer graphs a parametric surface, it usually depicts the surface by plotting these grid curves, as we see in the following example.

EXAMPLE 2 Use a computer algebra system to graph the surface

$$\mathbf{r}(u, v) = \langle (2 + \sin v) \cos u, (2 + \sin v) \sin u, u + \cos v \rangle$$

Which grid curves have u constant? Which have v constant?

SOLUTION We graph the portion of the surface with parameter domain $0 \leq u \leq 4\pi, 0 \leq v \leq 2\pi$ in Figure 5. It has the appearance of a spiral tube. To identify the grid curves, we write the corresponding parametric equations:

$$x = (2 + \sin v) \cos u \quad y = (2 + \sin v) \sin u \quad z = u + \cos v$$

If v is constant, then $\sin v$ and $\cos v$ are constant, so the parametric equations resemble those of the helix in Example 13.1.4. Thus the grid curves with v constant are the spiral curves in Figure 5. We deduce that the grid curves with u constant must be the curves

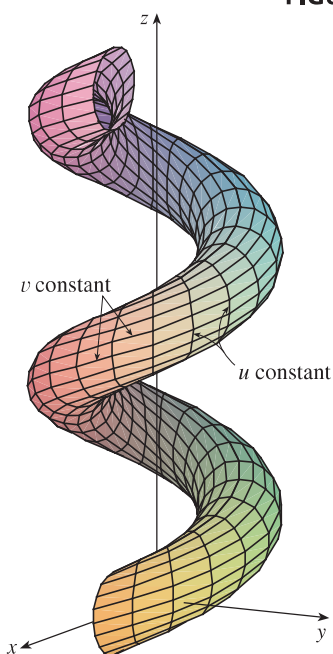


FIGURE 5

that look like circles in the figure. Further evidence for this assertion is that if u is kept constant, $u = u_0$, then the equation $z = u_0 + \cos v$ shows that the z -values vary from $u_0 - 1$ to $u_0 + 1$. ■

In Examples 1 and 2 we were given a vector equation and asked to graph the corresponding parametric surface. In the following examples, however, we are given the more challenging problem of finding a vector function to represent a given surface. In the rest of this chapter we will often need to do exactly that.

EXAMPLE 3 Find a vector function that represents the plane that passes through the point P_0 with position vector \mathbf{r}_0 and that contains two nonparallel vectors \mathbf{a} and \mathbf{b} .

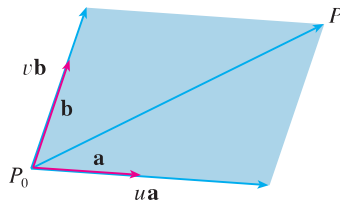


FIGURE 6

SOLUTION If P is any point in the plane, we can get from P_0 to P by moving a certain distance in the direction of \mathbf{a} and another distance in the direction of \mathbf{b} . So there are scalars u and v such that $\overrightarrow{P_0P} = u\mathbf{a} + v\mathbf{b}$. (Figure 6 illustrates how this works, by means of the Parallelogram Law, for the case where u and v are positive. See also Exercise 12.2.46.) If \mathbf{r} is the position vector of P , then

$$\mathbf{r} = \overrightarrow{OP_0} + \overrightarrow{P_0P} = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

So the vector equation of the plane can be written as

$$\mathbf{r}(u, v) = \mathbf{r}_0 + u\mathbf{a} + v\mathbf{b}$$

where u and v are real numbers.

If we write $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then we can write the parametric equations of the plane through the point (x_0, y_0, z_0) as follows:

$$x = x_0 + ua_1 + vb_1 \quad y = y_0 + ua_2 + vb_2 \quad z = z_0 + ua_3 + vb_3 \quad \blacksquare$$

EXAMPLE 4 Find a parametric representation of the sphere

$$x^2 + y^2 + z^2 = a^2$$

SOLUTION The sphere has a simple representation $\rho = a$ in spherical coordinates, so let's choose the angles ϕ and θ in spherical coordinates as the parameters (see Section 15.8). Then, putting $\rho = a$ in the equations for conversion from spherical to rectangular coordinates (Equations 15.8.1), we obtain

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

as the parametric equations of the sphere. The corresponding vector equation is

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

We have $0 \leq \phi \leq \pi$ and $0 \leq \theta \leq 2\pi$, so the parameter domain is the rectangle $D = [0, \pi] \times [0, 2\pi]$. The grid curves with ϕ constant are the circles of constant latitude (including the equator). The grid curves with θ constant are the meridians (semi-circles), which connect the north and south poles (see Figure 7). ■

NOTE We saw in Example 4 that the grid curves for a sphere are curves of constant latitude or constant longitude. For a general parametric surface we are really making a map and the grid curves are similar to lines of latitude and longitude. Describing a point on a parametric surface (like the one in Figure 5) by giving specific values of u and v is like giving the latitude and longitude of a point.

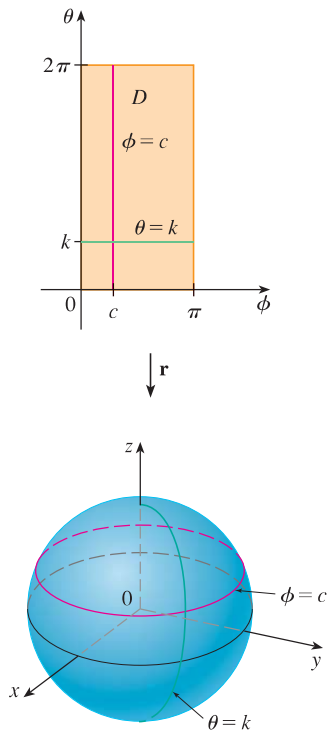


FIGURE 7

One of the uses of parametric surfaces is in computer graphics. Figure 8 shows the result of trying to graph the sphere $x^2 + y^2 + z^2 = 1$ by solving the equation for z and graphing the top and bottom hemispheres separately. Part of the sphere appears to be missing because of the rectangular grid system used by the computer. The much better picture in Figure 9 was produced by a computer using the parametric equations found in Example 4.

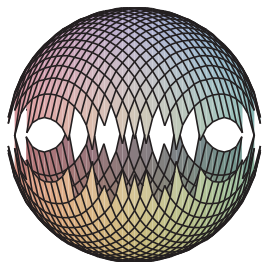


FIGURE 8

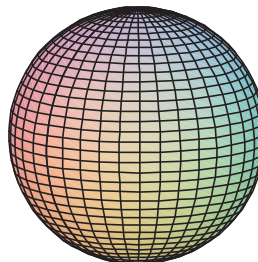


FIGURE 9

EXAMPLE 5 Find a parametric representation for the cylinder

$$x^2 + y^2 = 4 \quad 0 \leq z \leq 1$$

SOLUTION The cylinder has a simple representation $r = 2$ in cylindrical coordinates, so we choose as parameters θ and z in cylindrical coordinates. Then the parametric equations of the cylinder are

$$x = 2 \cos \theta \quad y = 2 \sin \theta \quad z = z$$

where $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 1$. ■

EXAMPLE 6 Find a vector function that represents the elliptic paraboloid $z = x^2 + 2y^2$.

SOLUTION If we regard x and y as parameters, then the parametric equations are simply

$$x = x \quad y = y \quad z = x^2 + 2y^2$$

and the vector equation is

$$\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + (x^2 + 2y^2) \mathbf{k} \quad \blacksquare$$

TEC In Module 16.6 you can investigate several families of parametric surfaces.

In general, a surface given as the graph of a function of x and y , that is, with an equation of the form $z = f(x, y)$, can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equations as

$$x = x \quad y = y \quad z = f(x, y)$$

Parametric representations (also called parametrizations) of surfaces are not unique. The next example shows two ways to parametrize a cone.

EXAMPLE 7 Find a parametric representation for the surface $z = 2\sqrt{x^2 + y^2}$, that is, the top half of the cone $z^2 = 4x^2 + 4y^2$.

SOLUTION 1 One possible representation is obtained by choosing x and y as parameters:

$$x = x \quad y = y \quad z = 2\sqrt{x^2 + y^2}$$

So the vector equation is

$$\mathbf{r}(x, y) = x \mathbf{i} + y \mathbf{j} + 2\sqrt{x^2 + y^2} \mathbf{k}$$

SOLUTION 2 Another representation results from choosing as parameters the polar coordinates r and θ . A point (x, y, z) on the cone satisfies $x = r \cos \theta$, $y = r \sin \theta$, and

For some purposes the parametric representations in Solutions 1 and 2 are equally good, but Solution 2 might be preferable in certain situations. If we are interested only in the part of the cone that lies below the plane $z = 1$, for instance, all we have to do in Solution 2 is change the parameter domain to

$$0 \leq r \leq \frac{1}{2} \quad 0 \leq \theta \leq 2\pi$$

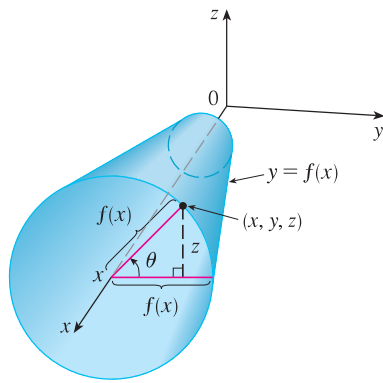


FIGURE 10

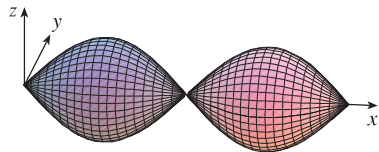


FIGURE 11

$z = 2\sqrt{x^2 + y^2} = 2r$. So a vector equation for the cone is

$$\mathbf{r}(r, \theta) = r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j} + 2r \mathbf{k}$$

where $r \geq 0$ and $0 \leq \theta \leq 2\pi$. ■

Surfaces of Revolution

Surfaces of revolution can be represented parametrically and thus graphed using a computer. For instance, let's consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x -axis, where $f(x) \geq 0$. Let θ be the angle of rotation as shown in Figure 10. If (x, y, z) is a point on S , then

$$\boxed{3} \quad x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta$$

Therefore we take x and θ as parameters and regard Equations 3 as parametric equations of S . The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

EXAMPLE 8 Find parametric equations for the surface generated by rotating the curve $y = \sin x$, $0 \leq x \leq 2\pi$, about the x -axis. Use these equations to graph the surface of revolution.

SOLUTION From Equations 3, the parametric equations are

$$x = x \quad y = \sin x \cos \theta \quad z = \sin x \sin \theta$$

and the parameter domain is $0 \leq x \leq 2\pi$, $0 \leq \theta \leq 2\pi$. Using a computer to plot these equations and then rotating the image, we obtain the graph in Figure 11. ■

We can adapt Equations 3 to represent a surface obtained through revolution about the y - or z -axis (see Exercise 30).

Tangent Planes

We now find the tangent plane to a parametric surface S traced out by a vector function

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k}$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$. If we keep u constant by putting $u = u_0$, then $\mathbf{r}(u_0, v)$ becomes a vector function of the single parameter v and defines a grid curve C_1 lying on S . (See Figure 12.) The tangent vector to C_1 at P_0 is obtained by taking the partial derivative of \mathbf{r} with respect to v :

$$\boxed{4} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0) \mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0) \mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0) \mathbf{k}$$

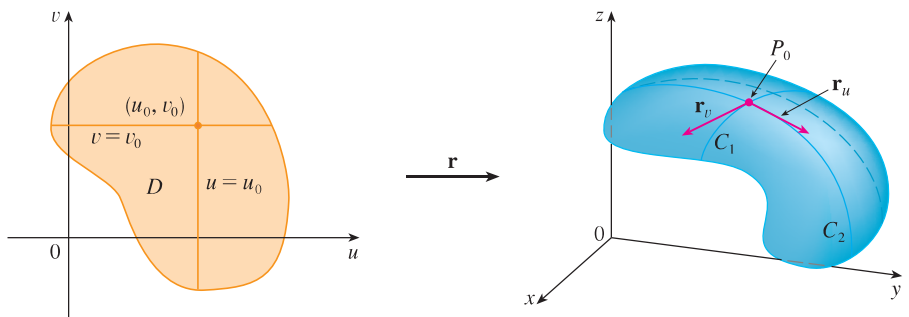


FIGURE 12

Similarly, if we keep u constant by putting $u = u_0$, we get a grid curve C_2 given by $\mathbf{r}(u_0, v)$ that lies on S , and its tangent vector at P_0 is

$$\mathbf{5} \quad \mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

If $\mathbf{r}_u \times \mathbf{r}_v$ is not $\mathbf{0}$, then the surface S is called **smooth** (it has no “corners”). For a smooth surface, the **tangent plane** is the plane that contains the tangent vectors \mathbf{r}_u and \mathbf{r}_v , and the vector $\mathbf{r}_u \times \mathbf{r}_v$ is a normal vector to the tangent plane.

Figure 13 shows the self-intersecting surface in Example 9 and its tangent plane at $(1, 1, 3)$.

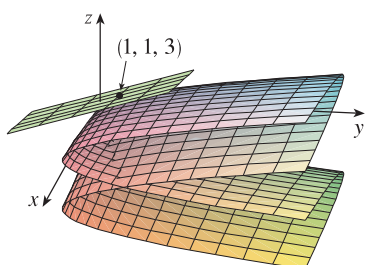


FIGURE 13

EXAMPLE 9 Find the tangent plane to the surface with parametric equations $x = u^2$, $y = v^2$, $z = u + 2v$ at the point $(1, 1, 3)$.

SOLUTION We first compute the tangent vectors:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} = 2u\mathbf{i} + \mathbf{k}$$

$$\mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} = 2v\mathbf{j} + 2\mathbf{k}$$

Thus a normal vector to the tangent plane is

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2u & 0 & 1 \\ 0 & 2v & 2 \end{vmatrix} = -2v\mathbf{i} - 4u\mathbf{j} + 4uv\mathbf{k}$$

Notice that the point $(1, 1, 3)$ corresponds to the parameter values $u = 1$ and $v = 1$, so the normal vector there is

$$-2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

Therefore an equation of the tangent plane at $(1, 1, 3)$ is

$$-2(x - 1) - 4(y - 1) + 4(z - 3) = 0$$

or
$$x + 2y - 2z + 3 = 0$$
 ■

■ **Surface Area**

Now we define the surface area of a general parametric surface given by Equation 1. For simplicity we start by considering a surface whose parameter domain D is a rectangle, and we divide it into subrectangles R_{ij} . Let’s choose (u_i^*, v_j^*) to be the lower left corner of R_{ij} . (See Figure 14.)

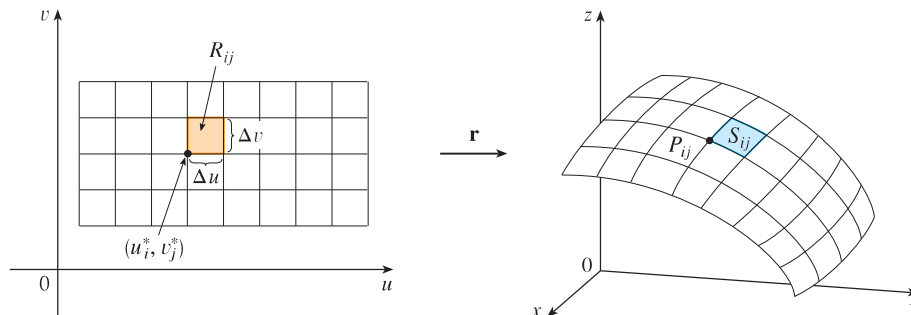


FIGURE 14
The image of the subrectangle R_{ij} is the patch S_{ij} .

The part S_{ij} of the surface S that corresponds to R_{ij} is called a *patch* and has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_u^* = \mathbf{r}_u(u_i^*, v_j^*) \quad \text{and} \quad \mathbf{r}_v^* = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij} as given by Equations 5 and 4.

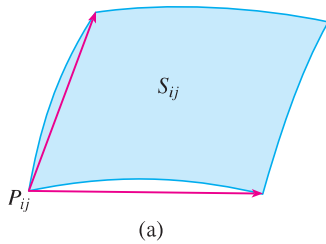
Figure 15(a) shows how the two edges of the patch that meet at P_{ij} can be approximated by vectors. These vectors, in turn, can be approximated by the vectors $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$ because partial derivatives can be approximated by difference quotients. So we approximate S_{ij} by the parallelogram determined by the vectors $\Delta u \mathbf{r}_u^*$ and $\Delta v \mathbf{r}_v^*$. This parallelogram is shown in Figure 15(b) and lies in the tangent plane to S at P_{ij} . The area of this parallelogram is

$$|(\Delta u \mathbf{r}_u^*) \times (\Delta v \mathbf{r}_v^*)| = |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

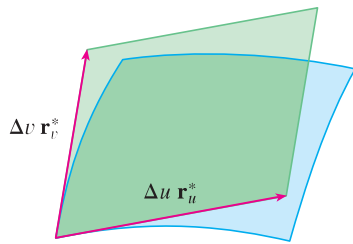
and so an approximation to the area of S is

$$\sum_{i=1}^m \sum_{j=1}^n |\mathbf{r}_u^* \times \mathbf{r}_v^*| \Delta u \Delta v$$

Our intuition tells us that this approximation gets better as we increase the number of subrectangles, and we recognize the double sum as a Riemann sum for the double integral $\iint_D |\mathbf{r}_u \times \mathbf{r}_v| du dv$. This motivates the following definition.



(a)



(b)

FIGURE 15
Approximating a patch by a parallelogram

6 Definition If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D , then the **surface area** of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

$$\text{where} \quad \mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

EXAMPLE 10 Find the surface area of a sphere of radius a .

SOLUTION In Example 4 we found the parametric representation

$$x = a \sin \phi \cos \theta \quad y = a \sin \phi \sin \theta \quad z = a \cos \phi$$

where the parameter domain is

$$D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

We first compute the cross product of the tangent vectors:

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos \phi \cos \theta & a \cos \phi \sin \theta & -a \sin \phi \\ -a \sin \phi \sin \theta & a \sin \phi \cos \theta & 0 \end{vmatrix} \\ &= a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k} \end{aligned}$$

Thus

$$\begin{aligned} |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= \sqrt{a^4 \sin^4 \phi \cos^2 \theta + a^4 \sin^4 \phi \sin^2 \theta + a^4 \sin^2 \phi \cos^2 \phi} \\ &= \sqrt{a^4 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi} = a^2 \sqrt{\sin^2 \phi} = a^2 \sin \phi \end{aligned}$$

since $\sin \phi \geq 0$ for $0 \leq \phi \leq \pi$. Therefore, by Definition 6, the area of the sphere is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA = \int_0^{2\pi} \int_0^\pi a^2 \sin \phi \, d\phi \, d\theta \\ &= a^2 \int_0^{2\pi} d\theta \int_0^\pi \sin \phi \, d\phi = a^2(2\pi)2 = 4\pi a^2 \end{aligned} \quad \blacksquare$$

■ Surface Area of the Graph of a Function

For the special case of a surface S with equation $z = f(x, y)$, where (x, y) lies in D and f has continuous partial derivatives, we take x and y as parameters. The parametric equations are

$$x = x \quad y = y \quad z = f(x, y)$$

so

$$\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial f}{\partial x} \right) \mathbf{k} \quad \mathbf{r}_y = \mathbf{j} + \left(\frac{\partial f}{\partial y} \right) \mathbf{k}$$

and

$$\boxed{7} \quad \mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k}$$

Thus we have

$$\boxed{8} \quad |\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 + 1} = \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2}$$

and the surface area formula in Definition 6 becomes

$$\boxed{9} \quad A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2} dA$$

Notice the similarity between the surface area formula in Equation 9 and the arc length formula

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

from Section 8.1.

EXAMPLE 11 Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

SOLUTION The plane intersects the paraboloid in the circle $x^2 + y^2 = 9$, $z = 9$. Therefore the given surface lies above the disk D with center the origin and radius 3. (See

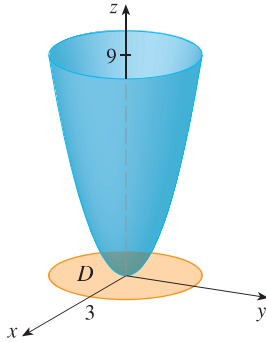


FIGURE 16

Figure 16.) Using Formula 9, we have

$$\begin{aligned} A &= \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dA \\ &= \iint_D \sqrt{1 + (2x)^2 + (2y)^2} dA \\ &= \iint_D \sqrt{1 + 4(x^2 + y^2)} dA \end{aligned}$$

Converting to polar coordinates, we obtain

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^3 \sqrt{1 + 4r^2} r dr d\theta = \int_0^{2\pi} d\theta \int_0^3 r\sqrt{1 + 4r^2} dr \\ &= 2\pi \left(\frac{1}{8}\right)^{\frac{2}{3}} (1 + 4r^2)^{3/2} \Big|_0^3 = \frac{\pi}{6} (37\sqrt{37} - 1) \end{aligned}$$

The question remains whether our definition of surface area (6) is consistent with the surface area formula from single-variable calculus (8.2.4).

We consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x -axis, where $f(x) \geq 0$ and f' is continuous. From Equations 3 we know that parametric equations of S are

$$x = x \quad y = f(x) \cos \theta \quad z = f(x) \sin \theta \quad a \leq x \leq b \quad 0 \leq \theta \leq 2\pi$$

To compute the surface area of S we need the tangent vectors

$$\mathbf{r}_x = \mathbf{i} + f'(x) \cos \theta \mathbf{j} + f'(x) \sin \theta \mathbf{k}$$

$$\mathbf{r}_\theta = -f(x) \sin \theta \mathbf{j} + f(x) \cos \theta \mathbf{k}$$

Thus

$$\begin{aligned} \mathbf{r}_x \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & f'(x) \cos \theta & f'(x) \sin \theta \\ 0 & -f(x) \sin \theta & f(x) \cos \theta \end{vmatrix} \\ &= f(x)f'(x) \mathbf{i} - f(x) \cos \theta \mathbf{j} - f(x) \sin \theta \mathbf{k} \end{aligned}$$

and so

$$\begin{aligned} |\mathbf{r}_x \times \mathbf{r}_\theta| &= \sqrt{[f(x)]^2 [f'(x)]^2 + [f(x)]^2 \cos^2 \theta + [f(x)]^2 \sin^2 \theta} \\ &= \sqrt{[f(x)]^2 [1 + [f'(x)]^2]} = f(x) \sqrt{1 + [f'(x)]^2} \end{aligned}$$

because $f(x) \geq 0$. Therefore the area of S is

$$\begin{aligned} A &= \iint_D |\mathbf{r}_x \times \mathbf{r}_\theta| dA \\ &= \int_0^{2\pi} \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx d\theta \\ &= 2\pi \int_a^b f(x) \sqrt{1 + [f'(x)]^2} dx \end{aligned}$$

This is precisely the formula that was used to define the area of a surface of revolution in single-variable calculus (8.2.4).

16.6 EXERCISES

1–2 Determine whether the points P and Q lie on the given surface.

1. $\mathbf{r}(u, v) = \langle u + v, u - 2v, 3 + u - v \rangle$
 $P(4, -5, 1), Q(0, 4, 6)$

2. $\mathbf{r}(u, v) = \langle 1 + u - v, u + v^2, u^2 - v^2 \rangle$
 $P(1, 2, 1), Q(2, 3, 3)$


3–6 Identify the surface with the given vector equation.

3. $\mathbf{r}(u, v) = (u + v)\mathbf{i} + (3 - v)\mathbf{j} + (1 + 4u + 5v)\mathbf{k}$

4. $\mathbf{r}(u, v) = u^2\mathbf{i} + u \cos v\mathbf{j} + u \sin v\mathbf{k}$

5. $\mathbf{r}(s, t) = \langle s \cos t, s \sin t, s \rangle$

6. $\mathbf{r}(s, t) = \langle 3 \cos t, s, \sin t \rangle, \quad -1 \leq s \leq 1$

 **7–12** Use a computer to graph the parametric surface. Get a printout and indicate on it which grid curves have u constant and which have v constant.

7. $\mathbf{r}(u, v) = \langle u^2, v^2, u + v \rangle,$
 $-1 \leq u \leq 1, -1 \leq v \leq 1$

8. $\mathbf{r}(u, v) = \langle u, v^3, -v \rangle,$
 $-2 \leq u \leq 2, -2 \leq v \leq 2$

9. $\mathbf{r}(u, v) = \langle u^3, u \sin v, u \cos v \rangle,$
 $-1 \leq u \leq 1, 0 \leq v \leq 2\pi$

10. $\mathbf{r}(u, v) = \langle u, \sin(u + v), \sin v \rangle,$
 $-\pi \leq u \leq \pi, -\pi \leq v \leq \pi$

11. $x = \sin v, \quad y = \cos u \sin 4v, \quad z = \sin 2u \sin 4v,$
 $0 \leq u \leq 2\pi, -\pi/2 \leq v \leq \pi/2$

12. $x = \cos u, \quad y = \sin u \sin v, \quad z = \cos v,$
 $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$

13–18 Match the equations with the graphs labeled I–VI and give reasons for your answers. Determine which families of grid curves have u constant and which have v constant.

13. $\mathbf{r}(u, v) = u \cos v\mathbf{i} + u \sin v\mathbf{j} + v\mathbf{k}$

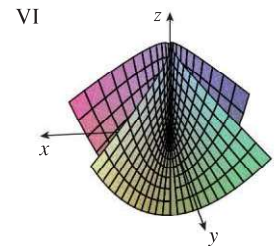
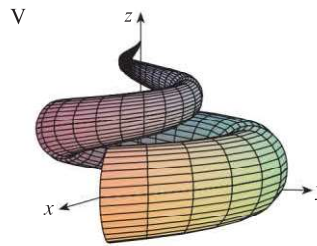
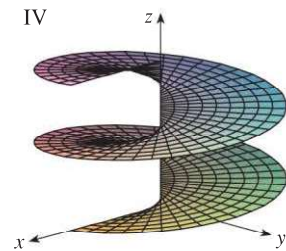
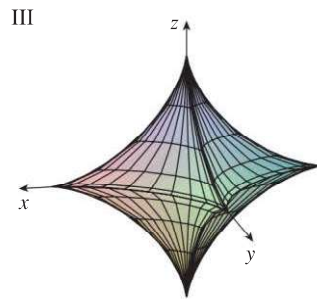
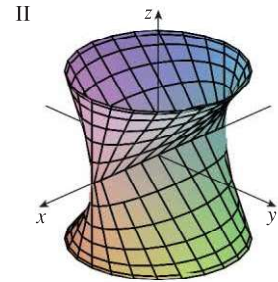
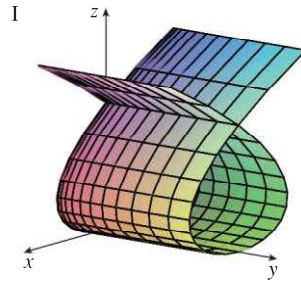
14. $\mathbf{r}(u, v) = uv^2\mathbf{i} + u^2v\mathbf{j} + (u^2 - v^2)\mathbf{k}$

15. $\mathbf{r}(u, v) = (u^3 - u)\mathbf{i} + v^2\mathbf{j} + u^2\mathbf{k}$

16. $x = (1 - u)(3 + \cos v) \cos 4\pi u,$
 $y = (1 - u)(3 + \cos v) \sin 4\pi u,$
 $z = 3u + (1 - u) \sin v$

17. $x = \cos^3 u \cos^3 v, \quad y = \sin^3 u \cos^3 v, \quad z = \sin^3 v$

18. $x = \sin u, \quad y = \cos u \sin v, \quad z = \sin v$



19–26 Find a parametric representation for the surface.

19. The plane through the origin that contains the vectors $\mathbf{i} - \mathbf{j}$ and $\mathbf{j} - \mathbf{k}$

20. The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4 \rangle$ and $\langle -3, 2, 5 \rangle$

21. The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in front of the yz -plane


22. The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz -plane

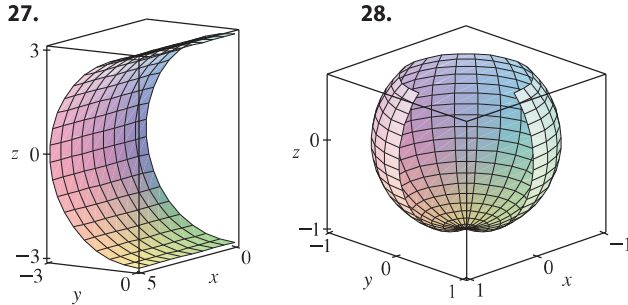
23. The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$




24. The part of the cylinder $x^2 + z^2 = 9$ that lies above the xy -plane and between the planes $y = -4$ and $y = 4$


25. The part of the sphere $x^2 + y^2 + z^2 = 36$ that lies between the planes $z = 0$ and $z = 3\sqrt{3}$

26. The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$

 27–28 Use a graphing device to produce a graph that looks like the given one.



27.  29. Find parametric equations for the surface obtained by rotating the curve $y = 1/(1 + x^2)$, $-2 \leq x \leq 2$, about the x -axis and use them to graph the surface.
30.  Find parametric equations for the surface obtained by rotating the curve $x = 1/y$, $y \geq 1$, about the y -axis and use them to graph the surface.
31.  (a) What happens to the spiral tube in Example 2 (see Figure 5) if we replace $\cos u$ by $\sin u$ and $\sin u$ by $\cos u$?
(b) What happens if we replace $\cos u$ by $\cos 2u$ and $\sin u$ by $\sin 2u$?

32.  The surface with parametric equations

$$x = 2 \cos \theta + r \cos(\theta/2)$$

$$y = 2 \sin \theta + r \cos(\theta/2)$$

$$z = r \sin(\theta/2)$$

where $-\frac{1}{2} \leq r \leq \frac{1}{2}$ and $0 \leq \theta \leq 2\pi$, is called a **Möbius strip**. Graph this surface with several viewpoints. What is unusual about it?


- 33–36 Find an equation of the tangent plane to the given parametric surface at the specified point.

33. $x = u + v$, $y = 3u^2$, $z = u - v$; $(2, 3, 0)$

34. $x = u^2 + 1$, $y = v^3 + 1$, $z = u + v$; $(5, 2, 3)$

35. $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$; $u = 1$, $v = \pi/3$

36. $\mathbf{r}(u, v) = \sin u \mathbf{i} + \cos u \sin v \mathbf{j} + \sin v \mathbf{k}$;
 $u = \pi/6$, $v = \pi/6$

 37–38 Find an equation of the tangent plane to the given parametric surface at the specified point. Graph the surface and the tangent plane.

37. $\mathbf{r}(u, v) = u^2 \mathbf{i} + 2u \sin v \mathbf{j} + u \cos v \mathbf{k}$; $u = 1$, $v = 0$

38. $\mathbf{r}(u, v) = (1 - u^2 - v^2) \mathbf{i} - v \mathbf{j} - u \mathbf{k}$; $(-1, -1, -1)$


- 39–50 Find the area of the surface.

39. The part of the plane $3x + 2y + z = 6$ that lies in the first octant
40. The part of the plane with vector equation $\mathbf{r}(u, v) = \langle u + v, 2 - 3u, 1 + u - v \rangle$ that is given by $0 \leq u \leq 2$, $-1 \leq v \leq 1$
41. The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$
42. The part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$
43. The surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$
44. The part of the surface $z = 4 - 2x^2 + y$ that lies above the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$
45. The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$
46. The part of the surface $x = z^2 + y$ that lies between the planes $y = 0$, $y = 2$, $z = 0$, and $z = 2$
47. The part of the paraboloid $y = x^2 + z^2$ that lies within the cylinder $x^2 + z^2 = 16$
48. The helicoid (or spiral ramp) with vector equation $\mathbf{r}(u, v) = u \cos v \mathbf{i} + u \sin v \mathbf{j} + v \mathbf{k}$, $0 \leq u \leq 1$, $0 \leq v \leq \pi$
49. The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \leq u \leq 1$, $0 \leq v \leq 2$
50. The part of the sphere $x^2 + y^2 + z^2 = b^2$ that lies inside the cylinder $x^2 + y^2 = a^2$, where $0 < a < b$


51. If the equation of a surface S is $z = f(x, y)$, where $x^2 + y^2 \leq R^2$, and you know that $|f_x| \leq 1$ and $|f_y| \leq 1$, what can you say about $A(S)$?




52–53 Find the area of the surface correct to four decimal places by expressing the area in terms of a single integral and using your calculator to estimate the integral.

52. The part of the surface $z = \cos(x^2 + y^2)$ that lies inside the cylinder $x^2 + y^2 = 1$
53. The part of the surface $z = \ln(x^2 + y^2 + 2)$ that lies above the disk $x^2 + y^2 \leq 1$

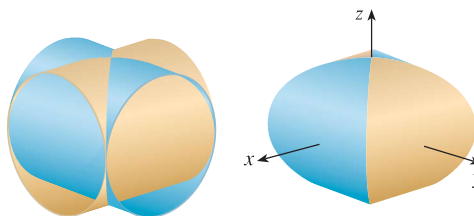
 54. Find, to four decimal places, the area of the part of the surface $z = (1 + x^2)/(1 + y^2)$ that lies above the square $|x| + |y| \leq 1$. Illustrate by graphing this part of the surface.


55. (a) Use the Midpoint Rule for double integrals (see Section 15.1) with six squares to estimate the area of the surface $z = 1/(1 + x^2 + y^2)$, $0 \leq x \leq 6$, $0 \leq y \leq 4$.

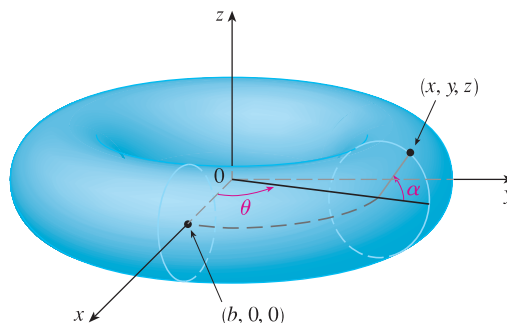
 (b) Use a computer algebra system to approximate the surface area in part (a) to four decimal places. Compare with the answer to part (a).

- CAS 56.** Find the area of the surface with vector equation $\mathbf{r}(u, v) = \langle \cos^3 u \cos^3 v, \sin^3 u \cos^3 v, \sin^3 v \rangle$, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$. State your answer correct to four decimal places.
- CAS 57.** Find the exact area of the surface $z = 1 + 2x + 3y + 4y^2$, $1 \leq x \leq 4$, $0 \leq y \leq 1$.
- 58.** (a) Set up, but do not evaluate, a double integral for the area of the surface with parametric equations $x = au \cos v$, $y = bu \sin v$, $z = u^2$, $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$.
 (b) Eliminate the parameters to show that the surface is an elliptic paraboloid and set up another double integral for the surface area.
-  (c) Use the parametric equations in part (a) with $a = 2$ and $b = 3$ to graph the surface.
- CAS** (d) For the case $a = 2$, $b = 3$, use a computer algebra system to find the surface area correct to four decimal places.
- 59.** (a) Show that the parametric equations $x = a \sin u \cos v$, $y = b \sin u \sin v$, $z = c \cos u$, $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, represent an ellipsoid.
 (b) Use the parametric equations in part (a) to graph the ellipsoid for the case $a = 1$, $b = 2$, $c = 3$.
 (c) Set up, but do not evaluate, a double integral for the surface area of the ellipsoid in part (b).
- 60.** (a) Show that the parametric equations $x = a \cosh u \cos v$, $y = b \cosh u \sin v$, $z = c \sinh u$, represent a hyperboloid of one sheet.
 (b) Use the parametric equations in part (a) to graph the hyperboloid for the case $a = 1$, $b = 2$, $c = 3$.
 (c) Set up, but do not evaluate, a double integral for the surface area of the part of the hyperboloid in part (b) that lies between the planes $z = -3$ and $z = 3$.

- 61.** Find the area of the part of the sphere $x^2 + y^2 + z^2 = 4z$ that lies inside the paraboloid $z = x^2 + y^2$.
- 62.** The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



- 63.** Find the area of the part of the sphere $x^2 + y^2 + z^2 = a^2$ that lies inside the cylinder $x^2 + y^2 = ax$.
- 64.** (a) Find a parametric representation for the torus obtained by rotating about the z -axis the circle in the xz -plane with center $(b, 0, 0)$ and radius $a < b$. [*Hint:* Take as parameters the angles θ and α shown in the figure.]
 (b) Use the parametric equations found in part (a) to graph the torus for several values of a and b .
 (c) Use the parametric representation from part (a) to find the surface area of the torus.



16.7 Surface Integrals

The relationship between surface integrals and surface area is much the same as the relationship between line integrals and arc length. Suppose f is a function of three variables whose domain includes a surface S . We will define the surface integral of f over S in such a way that, in the case where $f(x, y, z) = 1$, the value of the surface integral is equal to the surface area of S . We start with parametric surfaces and then deal with the special case where S is the graph of a function of two variables.

Parametric Surfaces

Suppose that a surface S has a vector equation

$$\mathbf{r}(u, v) = x(u, v) \mathbf{i} + y(u, v) \mathbf{j} + z(u, v) \mathbf{k} \quad (u, v) \in D$$

We first assume that the parameter domain D is a rectangle and we divide it into subrect-

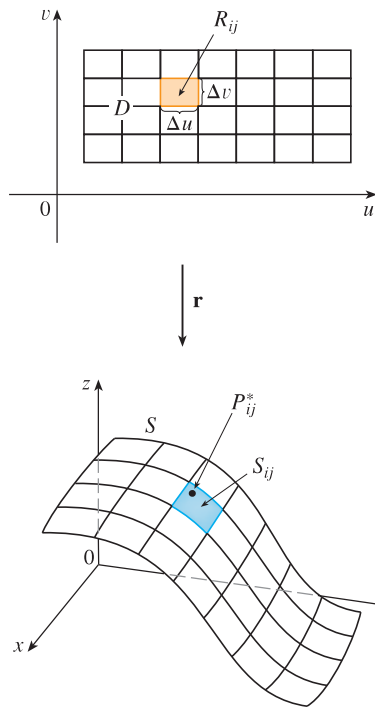


FIGURE 1

We assume that the surface is covered only once as (u, v) ranges throughout D . The value of the surface integral does not depend on the parametrization that is used.

angles R_{ij} with dimensions Δu and Δv . Then the surface S is divided into corresponding patches S_{ij} as in Figure 1. We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij} of the patch, and form the Riemann sum

$$\sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Then we take the limit as the number of patches increases and define the **surface integral of f over the surface S** as

1

$$\iint_S f(x, y, z) \, dS = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(P_{ij}^*) \Delta S_{ij}$$

Notice the analogy with the definition of a line integral (16.2.2) and also the analogy with the definition of a double integral (15.1.5).

To evaluate the surface integral in Equation 1 we approximate the patch area ΔS_{ij} by the area of an approximating parallelogram in the tangent plane. In our discussion of surface area in Section 16.6 we made the approximation

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k}$ $\mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$

are the tangent vectors at a corner of S_{ij} . If the components are continuous and \mathbf{r}_u and \mathbf{r}_v are nonzero and nonparallel in the interior of D , it can be shown from Definition 1, even when D is not a rectangle, that

2

$$\iint_S f(x, y, z) \, dS = \iint_D f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, dA$$

This should be compared with the formula for a line integral:

$$\int_C f(x, y, z) \, ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| \, dt$$

Observe also that

$$\iint_S 1 \, dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = A(S)$$

Formula 2 allows us to compute a surface integral by converting it into a double integral over the parameter domain D . When using this formula, remember that $f(\mathbf{r}(u, v))$ is evaluated by writing $x = x(u, v)$, $y = y(u, v)$, and $z = z(u, v)$ in the formula for $f(x, y, z)$.

EXAMPLE 1 Compute the surface integral $\iint_S x^2 \, dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 16.6.4, we use the parametric representation

$$x = \sin \phi \cos \theta \quad y = \sin \phi \sin \theta \quad z = \cos \phi \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

that is, $\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k}$

As in Example 16.6.10, we can compute that

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin \phi$$

Therefore, by Formula 2,

$$\begin{aligned} \iint_S x^2 dS &= \iint_D (\sin \phi \cos \theta)^2 |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA \\ &= \int_0^{2\pi} \int_0^\pi \sin^2 \phi \cos^2 \theta \sin \phi d\phi d\theta = \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\pi \sin^3 \phi d\phi \\ &= \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) d\theta \int_0^\pi (\sin \phi - \sin \phi \cos^2 \phi) d\phi \\ &= \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi} \left[-\cos \phi + \frac{1}{3} \cos^3 \phi \right]_0^\pi = \frac{4\pi}{3} \quad \blacksquare \end{aligned}$$

Here we use the identities

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin^2 \phi = 1 - \cos^2 \phi$$

Instead, we could use Formulas 64 and 67 in the Table of Integrals.

Surface integrals have applications similar to those for the integrals we have previously considered. For example, if a thin sheet (say, of aluminum foil) has the shape of a surface S and the density (mass per unit area) at the point (x, y, z) is $\rho(x, y, z)$, then the total **mass** of the sheet is

$$m = \iint_S \rho(x, y, z) dS$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m} \iint_S x \rho(x, y, z) dS \quad \bar{y} = \frac{1}{m} \iint_S y \rho(x, y, z) dS \quad \bar{z} = \frac{1}{m} \iint_S z \rho(x, y, z) dS$$

Moments of inertia can also be defined as before (see Exercise 41).

■ Graphs of Functions

Any surface S with equation $z = g(x, y)$ can be regarded as a parametric surface with parametric equations

$$x = x \quad y = y \quad z = g(x, y)$$

and so we have $\mathbf{r}_x = \mathbf{i} + \left(\frac{\partial g}{\partial x}\right) \mathbf{k}$ $\mathbf{r}_y = \mathbf{j} + \left(\frac{\partial g}{\partial y}\right) \mathbf{k}$

Thus

$$\boxed{3} \quad \mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, in this case, Formula 2 becomes

$$\boxed{4} \quad \iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \, dA$$

Similar formulas apply when it is more convenient to project S onto the yz -plane or xz -plane. For instance, if S is a surface with equation $y = h(x, z)$ and D is its projection onto the xz -plane, then

$$\iint_S f(x, y, z) \, dS = \iint_D f(x, h(x, z), z) \sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1} \, dA$$

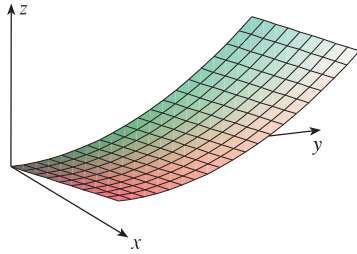


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S y \, dS$, where S is the surface $z = x + y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$. (See Figure 2.)

SOLUTION Since

$$\frac{\partial z}{\partial x} = 1 \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

Formula 4 gives

$$\begin{aligned} \iint_S y \, dS &= \iint_D y \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\ &= \int_0^1 \int_0^2 y \sqrt{1 + 1 + 4y^2} \, dy \, dx \\ &= \int_0^1 dx \sqrt{2} \int_0^2 y \sqrt{1 + 2y^2} \, dy \\ &= \sqrt{2} \left(\frac{1}{4}\right)^{2/3} (1 + 2y^2)^{3/2} \Big|_0^2 = \frac{13\sqrt{2}}{3} \end{aligned}$$

If S is a piecewise-smooth surface, that is, a finite union of smooth surfaces S_1, S_2, \dots, S_n that intersect only along their boundaries, then the surface integral of f over S is defined by

$$\iint_S f(x, y, z) \, dS = \iint_{S_1} f(x, y, z) \, dS + \dots + \iint_{S_n} f(x, y, z) \, dS$$

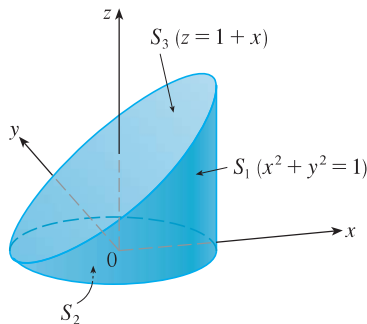


FIGURE 3

EXAMPLE 3 Evaluate $\iint_S z \, dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \leq 1$ in the plane $z = 0$, and whose top S_3 is the part of the plane $z = 1 + x$ that lies above S_2 .

SOLUTION The surface S is shown in Figure 3. (We have changed the usual position of the axes to get a better look at S .) For S_1 we use θ and z as parameters (see Example 16.6.5) and write its parametric equations as

$$x = \cos \theta \quad y = \sin \theta \quad z = z$$

where

$$0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq z \leq 1 + x = 1 + \cos \theta$$

Therefore

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$$

and $|\mathbf{r}_\theta \times \mathbf{r}_z| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

Thus the surface integral over S_1 is

$$\begin{aligned} \iint_{S_1} z \, dS &= \iint_D z |\mathbf{r}_\theta \times \mathbf{r}_z| \, dA \\ &= \int_0^{2\pi} \int_0^{1+\cos \theta} z \, dz \, d\theta = \int_0^{2\pi} \frac{1}{2} (1 + \cos \theta)^2 \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[1 + 2 \cos \theta + \frac{1}{2} (1 + \cos 2\theta) \right] \, d\theta \\ &= \frac{1}{2} \left[\frac{3}{2} \theta + 2 \sin \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi} = \frac{3\pi}{2} \end{aligned}$$

Since S_2 lies in the plane $z = 0$, we have

$$\iint_{S_2} z \, dS = \iint_{S_2} 0 \, dS = 0$$

The top surface S_3 lies above the unit disk D and is part of the plane $z = 1 + x$. So, taking $g(x, y) = 1 + x$ in Formula 4 and converting to polar coordinates, we have

$$\begin{aligned} \iint_{S_3} z \, dS &= \iint_D (1 + x) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA \\ &= \int_0^{2\pi} \int_0^1 (1 + r \cos \theta) \sqrt{1 + 1 + 0} \, r \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \int_0^1 (r + r^2 \cos \theta) \, dr \, d\theta \\ &= \sqrt{2} \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{3} \cos \theta \right) \, d\theta \\ &= \sqrt{2} \left[\frac{\theta}{2} + \frac{\sin \theta}{3} \right]_0^{2\pi} = \sqrt{2} \pi \end{aligned}$$

Therefore

$$\begin{aligned} \iint_S z \, dS &= \iint_{S_1} z \, dS + \iint_{S_2} z \, dS + \iint_{S_3} z \, dS \\ &= \frac{3\pi}{2} + 0 + \sqrt{2} \pi = \left(\frac{3}{2} + \sqrt{2} \right) \pi \end{aligned}$$

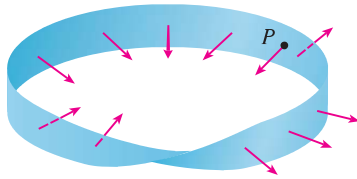


FIGURE 4
A Möbius strip

TEC Visual 16.7 shows a Möbius strip with a normal vector that can be moved along the surface.

■ Oriented Surfaces

To define surface integrals of vector fields, we need to rule out nonorientable surfaces such as the Möbius strip shown in Figure 4. [It is named after the German geometer August Möbius (1790–1868).] You can construct one for yourself by taking a long rectangular strip of paper, giving it a half-twist, and taping the short edges together as in Figure 5. If an ant were to crawl along the Möbius strip starting at a point P , it would end up on the “other side” of the strip (that is, with its upper side pointing in the opposite direction). Then, if the ant continued to crawl in the same direction, it would end up back at the same point P without ever having crossed an edge. (If you have constructed a Möbius strip, try drawing a pencil line down the middle.) Therefore a Möbius strip really has only one side. You can graph the Möbius strip using the parametric equations in Exercise 16.6.32.

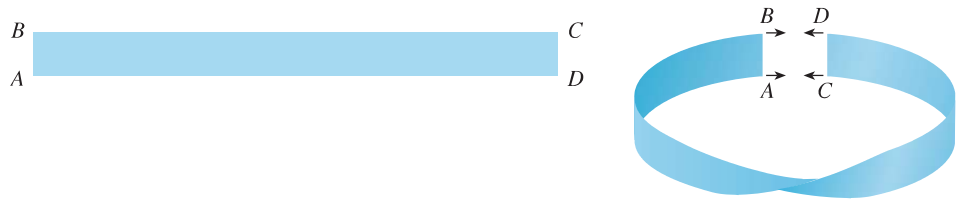


FIGURE 5
Constructing a Möbius strip

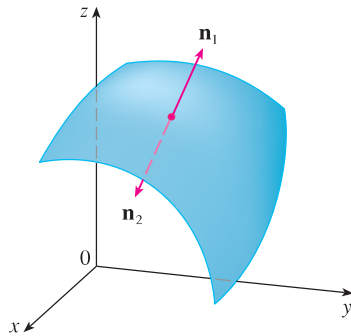


FIGURE 6

From now on we consider only orientable (two-sided) surfaces. We start with a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point). There are two unit normal vectors \mathbf{n}_1 and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z) . (See Figure 6.)

If it is possible to choose a unit normal vector \mathbf{n} at every such point (x, y, z) so that \mathbf{n} varies continuously over S , then S is called an **oriented surface** and the given choice of \mathbf{n} provides S with an **orientation**. There are two possible orientations for any orientable surface (see Figure 7).

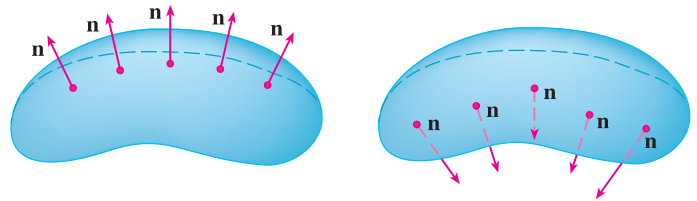


FIGURE 7
The two orientations of an orientable surface

For a surface $z = g(x, y)$ given as the graph of g , we use Equation 3 to associate with the surface a natural orientation given by the unit normal vector

$$\mathbf{n} = \frac{-\frac{\partial g}{\partial x} \mathbf{i} - \frac{\partial g}{\partial y} \mathbf{j} + \mathbf{k}}{\sqrt{1 + \left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2}} \quad (5)$$

Since the \mathbf{k} -component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then it is automatically supplied with the orientation of the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad (6)$$

and the opposite orientation is given by $-\mathbf{n}$. For instance, in Example 16.6.4 we found

the parametric representation

$$\mathbf{r}(\phi, \theta) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k}$$

for the sphere $x^2 + y^2 + z^2 = a^2$. Then in Example 16.6.10 we found that

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2 \phi \cos \theta \mathbf{i} + a^2 \sin^2 \phi \sin \theta \mathbf{j} + a^2 \sin \phi \cos \phi \mathbf{k}$$

and

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = a^2 \sin \phi$$

So the orientation induced by $\mathbf{r}(\phi, \theta)$ is defined by the unit normal vector

$$\mathbf{n} = \frac{\mathbf{r}_\phi \times \mathbf{r}_\theta}{|\mathbf{r}_\phi \times \mathbf{r}_\theta|} = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} = \frac{1}{a} \mathbf{r}(\phi, \theta)$$

Observe that \mathbf{n} points in the same direction as the position vector, that is, outward from the sphere (see Figure 8). The opposite (inward) orientation would have been obtained (see Figure 9) if we had reversed the order of the parameters because $\mathbf{r}_\theta \times \mathbf{r}_\phi = -\mathbf{r}_\phi \times \mathbf{r}_\theta$.

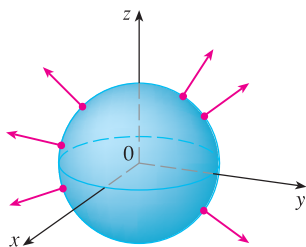


FIGURE 8
Positive orientation

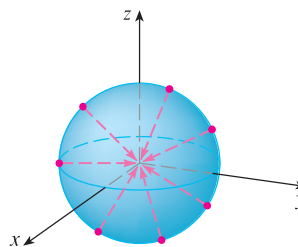


FIGURE 9
Negative orientation

For a **closed surface**, that is, a surface that is the boundary of a solid region E , the convention is that the **positive orientation** is the one for which the normal vectors point *outward* from E , and inward-pointing normals give the negative orientation (see Figures 8 and 9).

■ Surface Integrals of Vector Fields

Suppose that S is an oriented surface with unit normal vector \mathbf{n} , and imagine a fluid with density $\rho(x, y, z)$ and velocity field $\mathbf{v}(x, y, z)$ flowing through S . (Think of S as an imaginary surface that doesn't impede the fluid flow, like a fishing net across a stream.) Then the rate of flow (mass per unit time) per unit area is $\rho\mathbf{v}$. If we divide S into small patches S_{ij} , as in Figure 10 (compare with Figure 1), then S_{ij} is nearly planar and so we can approximate the mass of fluid per unit time crossing S_{ij} in the direction of the normal \mathbf{n} by the quantity

$$(\rho\mathbf{v} \cdot \mathbf{n})A(S_{ij})$$

where ρ , \mathbf{v} , and \mathbf{n} are evaluated at some point on S_{ij} . (Recall that the component of the vector $\rho\mathbf{v}$ in the direction of the unit vector \mathbf{n} is $\rho\mathbf{v} \cdot \mathbf{n}$.) By summing these quantities and taking the limit we get, according to Definition 1, the surface integral of the function $\rho\mathbf{v} \cdot \mathbf{n}$ over S :

$$\boxed{7} \quad \iint_S \rho\mathbf{v} \cdot \mathbf{n} \, dS = \iint_S \rho(x, y, z)\mathbf{v}(x, y, z) \cdot \mathbf{n}(x, y, z) \, dS$$

and this is interpreted physically as the rate of flow through S .

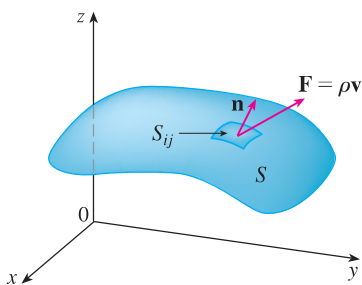


FIGURE 10

If we write $\mathbf{F} = \rho \mathbf{v}$, then \mathbf{F} is also a vector field on \mathbb{R}^3 and the integral in Equation 7 becomes

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

A surface integral of this form occurs frequently in physics, even when \mathbf{F} is not $\rho \mathbf{v}$, and is called the *surface integral* (or *flux integral*) of \mathbf{F} over S .

8 Definition If \mathbf{F} is a continuous vector field defined on an oriented surface S with unit normal vector \mathbf{n} , then the **surface integral of \mathbf{F} over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

This integral is also called the **flux** of \mathbf{F} across S .

In words, Definition 8 says that the surface integral of a vector field over S is equal to the surface integral of its normal component over S (as previously defined).

If S is given by a vector function $\mathbf{r}(u, v)$, then \mathbf{n} is given by Equation 6, and from Definition 8 and Equation 2 we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \, dS \\ &= \iint_D \left[\mathbf{F}(\mathbf{r}(u, v)) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \, dA \end{aligned}$$

where D is the parameter domain. Thus we have

Compare Equation 9 to the similar expression for evaluating line integrals of vector fields in Definition 16.2.13:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt$$

9

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA$$

Figure 11 shows the vector field \mathbf{F} in Example 4 at points on the unit sphere.

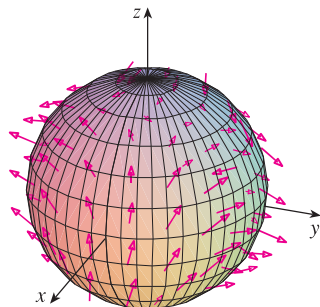


FIGURE 11

EXAMPLE 4 Find the flux of the vector field $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION As in Example 1, we use the parametric representation

$$\mathbf{r}(\phi, \theta) = \sin \phi \cos \theta \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \cos \phi \mathbf{k} \quad 0 \leq \phi \leq \pi \quad 0 \leq \theta \leq 2\pi$$

Then $\mathbf{F}(\mathbf{r}(\phi, \theta)) = \cos \phi \mathbf{i} + \sin \phi \sin \theta \mathbf{j} + \sin \phi \cos \theta \mathbf{k}$

and, from Example 16.6.10,

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \sin^2 \phi \cos \theta \mathbf{i} + \sin^2 \phi \sin \theta \mathbf{j} + \sin \phi \cos \phi \mathbf{k}$$

Therefore

$$\mathbf{F}(\mathbf{r}(\phi, \theta)) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = \cos \phi \sin^2 \phi \cos \theta + \sin^3 \phi \sin^2 \theta + \sin^2 \phi \cos \phi \cos \theta$$

and, by Formula 9, the flux is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) dA \\ &= \int_0^{2\pi} \int_0^\pi (2 \sin^2 \phi \cos \phi \cos \theta + \sin^3 \phi \sin^2 \theta) d\phi d\theta \\ &= 2 \int_0^\pi \sin^2 \phi \cos \phi d\phi \int_0^{2\pi} \cos \theta d\theta + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \\ &= 0 + \int_0^\pi \sin^3 \phi d\phi \int_0^{2\pi} \sin^2 \theta d\theta \quad \left(\text{since } \int_0^{2\pi} \cos \theta d\theta = 0\right) \\ &= \frac{4\pi}{3} \end{aligned}$$

by the same calculation as in Example 1. ■

If, for instance, the vector field in Example 4 is a velocity field describing the flow of a fluid with density 1, then the answer, $4\pi/3$, represents the rate of flow through the unit sphere in units of mass per unit time.

In the case of a surface S given by a graph $z = g(x, y)$, we can think of x and y as parameters and use Equation 3 to write

$$\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \left(-\frac{\partial g}{\partial x}\mathbf{i} - \frac{\partial g}{\partial y}\mathbf{j} + \mathbf{k}\right)$$

Thus Formula 9 becomes

10
$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA$$

This formula assumes the upward orientation of S ; for a downward orientation we multiply by -1 . Similar formulas can be worked out if S is given by $y = h(x, z)$ or $x = k(y, z)$. (See Exercises 37 and 38.)

EXAMPLE 5 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

SOLUTION S consists of a parabolic top surface S_1 and a circular bottom surface S_2 . (See Figure 12.) Since S is a closed surface, we use the convention of positive (outward) orientation. This means that S_1 is oriented upward and we can use Equation 10 with D being the projection of S_1 onto the xy -plane, namely, the disk $x^2 + y^2 \leq 1$. Since

$$P(x, y, z) = y \quad Q(x, y, z) = x \quad R(x, y, z) = z = 1 - x^2 - y^2$$

on S_1 and
$$\frac{\partial g}{\partial x} = -2x \quad \frac{\partial g}{\partial y} = -2y$$

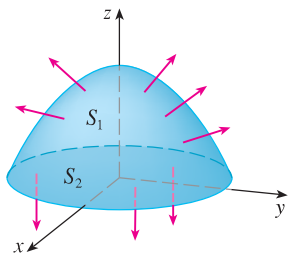


FIGURE 12

we have

$$\begin{aligned}
 \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R \right) dA \\
 &= \iint_D [-y(-2x) - x(-2y) + 1 - x^2 - y^2] dA \\
 &= \iint_D (1 + 4xy - x^2 - y^2) dA \\
 &= \int_0^{2\pi} \int_0^1 (1 + 4r^2 \cos \theta \sin \theta - r^2) r \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^1 (r - r^3 + 4r^3 \cos \theta \sin \theta) \, dr \, d\theta \\
 &= \int_0^{2\pi} \left(\frac{1}{4} + \cos \theta \sin \theta \right) d\theta = \frac{1}{4}(2\pi) + 0 = \frac{\pi}{2}
 \end{aligned}$$

The disk S_2 is oriented downward, so its unit normal vector is $\mathbf{n} = -\mathbf{k}$ and we have

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_2} \mathbf{F} \cdot (-\mathbf{k}) \, dS = \iint_D (-z) \, dA = \iint_D 0 \, dA = 0$$

since $z = 0$ on S_2 . Finally, we compute, by definition, $\iint_S \mathbf{F} \cdot d\mathbf{S}$ as the sum of the surface integrals of \mathbf{F} over the pieces S_1 and S_2 :

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \frac{\pi}{2} + 0 = \frac{\pi}{2} \quad \blacksquare$$

Although we motivated the surface integral of a vector field using the example of fluid flow, this concept also arises in other physical situations. For instance, if \mathbf{E} is an electric field (see Example 16.1.5), then the surface integral

$$\iint_S \mathbf{E} \cdot d\mathbf{S}$$

is called the **electric flux** of \mathbf{E} through the surface S . One of the important laws of electrostatics is **Gauss's Law**, which says that the net charge enclosed by a closed surface S is

$$\boxed{11} \quad Q = \varepsilon_0 \iint_S \mathbf{E} \cdot d\mathbf{S}$$

where ε_0 is a constant (called the permittivity of free space) that depends on the units used. (In the SI system, $\varepsilon_0 \approx 8.8542 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$.) Therefore, if the vector field \mathbf{F} in Example 4 represents an electric field, we can conclude that the charge enclosed by S is $Q = \frac{4}{3}\pi\varepsilon_0$.

Another application of surface integrals occurs in the study of heat flow. Suppose the temperature at a point (x, y, z) in a body is $u(x, y, z)$. Then the **heat flow** is defined as the vector field

$$\mathbf{F} = -K \nabla u$$

where K is an experimentally determined constant called the **conductivity** of the substance. The rate of heat flow across the surface S in the body is then given by the surface integral

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = -K \iint_S \nabla u \cdot d\mathbf{S}$$

EXAMPLE 6 The temperature u in a metal ball is proportional to the square of the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.

SOLUTION Taking the center of the ball to be at the origin, we have

$$u(x, y, z) = C(x^2 + y^2 + z^2)$$

where C is the proportionality constant. Then the heat flow is

$$\mathbf{F}(x, y, z) = -K \nabla u = -KC(2x \mathbf{i} + 2y \mathbf{j} + 2z \mathbf{k})$$

where K is the conductivity of the metal. Instead of using the usual parametrization of the sphere as in Example 4, we observe that the outward unit normal to the sphere $x^2 + y^2 + z^2 = a^2$ at the point (x, y, z) is

$$\mathbf{n} = \frac{1}{a} (x \mathbf{i} + y \mathbf{j} + z \mathbf{k})$$

and so

$$\mathbf{F} \cdot \mathbf{n} = -\frac{2KC}{a} (x^2 + y^2 + z^2)$$

But on S we have $x^2 + y^2 + z^2 = a^2$, so $\mathbf{F} \cdot \mathbf{n} = -2aKC$. Therefore the rate of heat flow across S is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = -2aKC \iint_S dS \\ &= -2aKC A(S) = -2aKC(4\pi a^2) = -8KC\pi a^3 \end{aligned}$$

16.7 EXERCISES

- Let S be the surface of the box enclosed by the planes $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Approximate $\iint_S \cos(x + 2y + 3z) \, dS$ by using a Riemann sum as in Definition 1, taking the patches S_{ij} to be the squares that are the faces of the box S and the points P_{ij}^* to be the centers of the squares.
 - A surface S consists of the cylinder $x^2 + y^2 = 1$, $-1 \leq z \leq 1$, together with its top and bottom disks. Suppose you know that f is a continuous function with

$$f(\pm 1, 0, 0) = 2 \quad f(0, \pm 1, 0) = 3 \quad f(0, 0, \pm 1) = 4$$
 Estimate the value of $\iint_S f(x, y, z) \, dS$ by using a Riemann sum, taking the patches S_{ij} to be four quarter-cylinders and the top and bottom disks.
 - Let H be the hemisphere $x^2 + y^2 + z^2 = 50$, $z \geq 0$, and suppose f is a continuous function with $f(3, 4, 5) = 7$, $f(3, -4, 5) = 8$, $f(-3, 4, 5) = 9$, and $f(-3, -4, 5) = 12$. By dividing H into four patches, estimate the value of $\iint_H f(x, y, z) \, dS$.
 - Suppose that $f(x, y, z) = g(\sqrt{x^2 + y^2 + z^2})$, where g is a function of one variable such that $g(2) = -5$. Evaluate $\iint_S f(x, y, z) \, dS$, where S is the sphere $x^2 + y^2 + z^2 = 4$.
- 5–20** Evaluate the surface integral.
- $\iint_S (x + y + z) \, dS$,
 S is the parallelogram with parametric equations $x = u + v$,
 $y = u - v$, $z = 1 + 2u + v$, $0 \leq u \leq 2$, $0 \leq v \leq 1$

6. $\iint_S xyz \, dS$,
 S is the cone with parametric equations $x = u \cos v$,
 $y = u \sin v$, $z = u$, $0 \leq u \leq 1$, $0 \leq v \leq \pi/2$
7. $\iint_S y \, dS$, S is the helicoid with vector equation
 $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle$, $0 \leq u \leq 1$, $0 \leq v \leq \pi$
8. $\iint_S (x^2 + y^2) \, dS$,
 S is the surface with vector equation
 $\mathbf{r}(u, v) = \langle 2uv, u^2 - v^2, u^2 + v^2 \rangle$, $u^2 + v^2 \leq 1$
9. $\iint_S x^2 yz \, dS$,
 S is the part of the plane $z = 1 + 2x + 3y$ that lies above
the rectangle $[0, 3] \times [0, 2]$
10. $\iint_S xz \, dS$,
 S is the part of the plane $2x + 2y + z = 4$ that lies in the
first octant
11. $\iint_S x \, dS$,
 S is the triangular region with vertices $(1, 0, 0)$, $(0, -2, 0)$,
and $(0, 0, 4)$
12. $\iint_S y \, dS$,
 S is the surface $z = \frac{2}{3}(x^{3/2} + y^{3/2})$, $0 \leq x \leq 1$, $0 \leq y \leq 1$
13. $\iint_S z^2 \, dS$,
 S is the part of the paraboloid $x = y^2 + z^2$ given by
 $0 \leq x \leq 1$
14. $\iint_S y^2 z^2 \, dS$,
 S is the part of the cone $y = \sqrt{x^2 + z^2}$ given by $0 \leq y \leq 5$
15. $\iint_S x \, dS$,
 S is the surface $y = x^2 + 4z$, $0 \leq x \leq 1$, $0 \leq z \leq 1$
16. $\iint_S y^2 \, dS$,
 S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above
the cone $z = \sqrt{x^2 + y^2}$
17. $\iint_S (x^2 z + y^2 z) \, dS$,
 S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$
18. $\iint_S (x + y + z) \, dS$,
 S is the part of the half-cylinder $x^2 + z^2 = 1$, $z \geq 0$, that
lies between the planes $y = 0$ and $y = 2$
19. $\iint_S xz \, dS$,
 S is the boundary of the region enclosed by the cylinder
 $y^2 + z^2 = 9$ and the planes $x = 0$ and $x + y = 5$
20. $\iint_S (x^2 + y^2 + z^2) \, dS$,
 S is the part of the cylinder $x^2 + y^2 = 9$ between the planes
 $z = 0$ and $z = 2$, together with its top and bottom disks
21. $\mathbf{F}(x, y, z) = ze^{xy} \mathbf{i} - 3ze^{xy} \mathbf{j} + xy \mathbf{k}$,
 S is the parallelogram of Exercise 5 with upward orientation
22. $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$,
 S is the helicoid of Exercise 7 with upward orientation
23. $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$, S is the part of the
paraboloid $z = 4 - x^2 - y^2$ that lies above the square
 $0 \leq x \leq 1$, $0 \leq y \leq 1$, and has upward orientation
24. $\mathbf{F}(x, y, z) = -x \mathbf{i} - y \mathbf{j} + z^3 \mathbf{k}$, S is the part of the cone
 $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 3$ with
downward orientation
25. $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z^2 \mathbf{k}$, S is the sphere with radius 1
and center the origin
26. $\mathbf{F}(x, y, z) = y \mathbf{i} - x \mathbf{j} + 2z \mathbf{k}$, S is the hemisphere
 $x^2 + y^2 + z^2 = 4$, $z \geq 0$, oriented downward
27. $\mathbf{F}(x, y, z) = y \mathbf{j} - z \mathbf{k}$,
 S consists of the paraboloid $y = x^2 + z^2$, $0 \leq y \leq 1$,
and the disk $x^2 + z^2 \leq 1$, $y = 1$
28. $\mathbf{F}(x, y, z) = yz \mathbf{i} + zx \mathbf{j} + xy \mathbf{k}$,
 S is the surface $z = x \sin y$, $0 \leq x \leq 2$, $0 \leq y \leq \pi$, with
upward orientation
29. $\mathbf{F}(x, y, z) = x \mathbf{i} + 2y \mathbf{j} + 3z \mathbf{k}$,
 S is the cube with vertices $(\pm 1, \pm 1, \pm 1)$
30. $\mathbf{F}(x, y, z) = x \mathbf{i} + y \mathbf{j} + 5 \mathbf{k}$, S is the boundary of the
region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes
 $y = 0$ and $x + y = 2$
31. $\mathbf{F}(x, y, z) = x^2 \mathbf{i} + y^2 \mathbf{j} + z^2 \mathbf{k}$, S is the boundary of the
solid half-cylinder $0 \leq z \leq \sqrt{1 - y^2}$, $0 \leq x \leq 2$
32. $\mathbf{F}(x, y, z) = y \mathbf{i} + (z - y) \mathbf{j} + x \mathbf{k}$,
 S is the surface of the tetrahedron with vertices $(0, 0, 0)$,
 $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$

- CAS 33. Evaluate $\iint_S (x^2 + y^2 + z^2) \, dS$ correct to four deci-
mal places, where S is the surface $z = xe^y$, $0 \leq x \leq 1$,
 $0 \leq y \leq 1$.
- CAS 34. Find the exact value of $\iiint_S xyz \, dS$, where S is the surface
 $z = x^2 y^2$, $0 \leq x \leq 1$, $0 \leq y \leq 2$.
- CAS 35. Find the value of $\iint_S x^2 y^2 z^2 \, dS$ correct to four deci-
mal places, where S is the part of the paraboloid
 $z = 3 - 2x^2 - y^2$ that lies above the xy -plane.
- CAS 36. Find the flux of

$$\mathbf{F}(x, y, z) = \sin(xyz) \mathbf{i} + x^2 y \mathbf{j} + z^2 e^{x/5} \mathbf{k}$$

across the part of the cylinder $4y^2 + z^2 = 4$ that lies above
the xy -plane and between the planes $x = -2$ and $x = 2$
with upward orientation. Illustrate by using a computer
algebra system to draw the cylinder and the vector field on
the same screen.

37. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for
the case where S is given by $y = h(x, z)$ and \mathbf{n} is the unit
normal that points toward the left.

21–32 Evaluate the surface integral $\iint_S \mathbf{F} \cdot d\mathbf{S}$ for the given
vector field \mathbf{F} and the oriented surface S . In other words, find
the flux of \mathbf{F} across S . For closed surfaces, use the positive
(outward) orientation.

21. $\mathbf{F}(x, y, z) = ze^{xy} \mathbf{i} - 3ze^{xy} \mathbf{j} + xy \mathbf{k}$,
 S is the parallelogram of Exercise 5 with upward orientation

38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and \mathbf{n} is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).
39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if it has constant density.
40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = 10 - z$.
41. (a) Give an integral expression for the moment of inertia I_z about the z -axis of a thin sheet in the shape of a surface S if the density function is ρ .
 (b) Find the moment of inertia about the z -axis of the funnel in Exercise 40.
42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k , find (a) the center of mass and (b) the moment of inertia about the z -axis.
43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z \mathbf{i} + y^2 \mathbf{j} + x^2 \mathbf{k}$, where x, y , and z are measured in meters and the components of \mathbf{v} in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 1$.
44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y \mathbf{i} + x \mathbf{j}$, where x, y , and z are measured in meters and the components of \mathbf{v} in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$.
45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$, if the electric field is $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + 2z \mathbf{k}$
46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$
47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leq x \leq 4$.
48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.
49. Let \mathbf{F} be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c , where $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$. Show that the flux of \mathbf{F} across a sphere S with center the origin is independent of the radius of S .

16.8 Stokes' Theorem

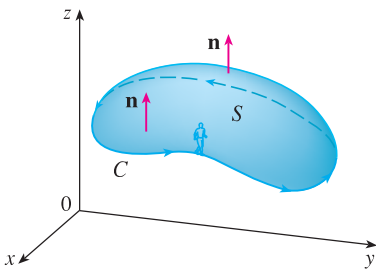


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector \mathbf{n} . The orientation of S induces the **positive orientation of the boundary curve** C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of \mathbf{n} , then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds \quad \text{and} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$$