

38. Find a formula for $\iint_S \mathbf{F} \cdot d\mathbf{S}$ similar to Formula 10 for the case where S is given by $x = k(y, z)$ and \mathbf{n} is the unit normal that points forward (that is, toward the viewer when the axes are drawn in the usual way).
39. Find the center of mass of the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, if it has constant density.
40. Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = 10 - z$.
41. (a) Give an integral expression for the moment of inertia I_z about the z -axis of a thin sheet in the shape of a surface S if the density function is ρ .
 (b) Find the moment of inertia about the z -axis of the funnel in Exercise 40.
42. Let S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies above the plane $z = 4$. If S has constant density k , find (a) the center of mass and (b) the moment of inertia about the z -axis.
43. A fluid has density 870 kg/m^3 and flows with velocity $\mathbf{v} = z \mathbf{i} + y^2 \mathbf{j} + x^2 \mathbf{k}$, where x, y , and z are measured in meters and the components of \mathbf{v} in meters per second. Find the rate of flow outward through the cylinder $x^2 + y^2 = 4$, $0 \leq z \leq 1$.
44. Seawater has density 1025 kg/m^3 and flows in a velocity field $\mathbf{v} = y \mathbf{i} + x \mathbf{j}$, where x, y , and z are measured in meters and the components of \mathbf{v} in meters per second. Find the rate of flow outward through the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$.
45. Use Gauss's Law to find the charge contained in the solid hemisphere $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$, if the electric field is $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + 2z \mathbf{k}$
46. Use Gauss's Law to find the charge enclosed by the cube with vertices $(\pm 1, \pm 1, \pm 1)$ if the electric field is $\mathbf{E}(x, y, z) = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$
47. The temperature at the point (x, y, z) in a substance with conductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$, $0 \leq x \leq 4$.
48. The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.
49. Let \mathbf{F} be an inverse square field, that is, $\mathbf{F}(\mathbf{r}) = c\mathbf{r}/|\mathbf{r}|^3$ for some constant c , where $\mathbf{r} = x \mathbf{i} + y \mathbf{j} + z \mathbf{k}$. Show that the flux of \mathbf{F} across a sphere S with center the origin is independent of the radius of S .

16.8 Stokes' Theorem

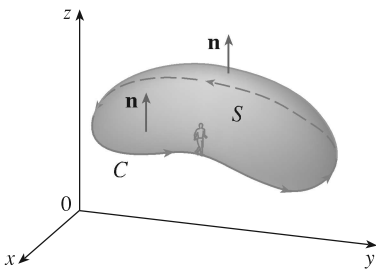


FIGURE 1

Stokes' Theorem can be regarded as a higher-dimensional version of Green's Theorem. Whereas Green's Theorem relates a double integral over a plane region D to a line integral around its plane boundary curve, Stokes' Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (which is a space curve). Figure 1 shows an oriented surface with unit normal vector \mathbf{n} . The orientation of S induces the **positive orientation of the boundary curve** C shown in the figure. This means that if you walk in the positive direction around C with your head pointing in the direction of \mathbf{n} , then the surface will always be on your left.

Stokes' Theorem Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let \mathbf{F} be a vector field whose components have continuous partial derivatives on an open region in \mathbb{R}^3 that contains S . Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds \quad \text{and} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} \, dS$$

George Stokes

Stokes' Theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903). Stokes was a professor at Cambridge University (in fact he held the same position as Newton, Lucasian Professor of Mathematics) and was especially noted for his studies of fluid flow and light. What we call Stokes' Theorem was actually discovered by the Scottish physicist Sir William Thomson (1824–1907, known as Lord Kelvin). Stokes learned of this theorem in a letter from Thomson in 1850 and asked students to prove it on an examination at Cambridge University in 1854. We don't know if any of those students was able to do so.

Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of \mathbf{F} is equal to the surface integral over S of the normal component of the curl of \mathbf{F} .

The positively oriented boundary curve of the oriented surface S is often written as ∂S , so Stokes' Theorem can be expressed as

$$\boxed{1} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

There is an analogy among Stokes' Theorem, Green's Theorem, and the Fundamental Theorem of Calculus. As before, there is an integral involving derivatives on the left side of Equation 1 (recall that $\text{curl } \mathbf{F}$ is a sort of derivative of \mathbf{F}) and the right side involves the values of \mathbf{F} only on the *boundary* of S .

In fact, in the special case where the surface S is flat and lies in the xy -plane with upward orientation, the unit normal is \mathbf{k} , the surface integral becomes a double integral, and Stokes' Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{k} \, dA$$

This is precisely the vector form of Green's Theorem given in Equation 16.5.12. Thus we see that Green's Theorem is really a special case of Stokes' Theorem.

Although Stokes' Theorem is too difficult for us to prove in its full generality, we can give a proof when S is a graph and \mathbf{F} , S , and C are well behaved.

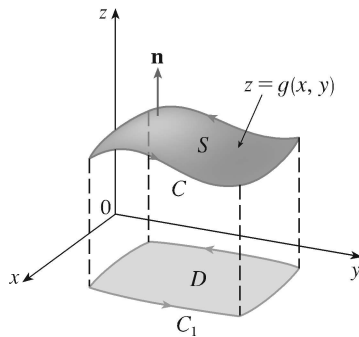


FIGURE 2

PROOF OF A SPECIAL CASE OF STOKES' THEOREM We assume that the equation of S is $z = g(x, y)$, $(x, y) \in D$, where g has continuous second-order partial derivatives and D is a simple plane region whose boundary curve C_1 corresponds to C . If the orientation of S is upward, then the positive orientation of C corresponds to the positive orientation of C_1 . (See Figure 2.) We are also given that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where the partial derivatives of P , Q , and R are continuous.

Since S is a graph of a function, we can apply Formula 16.7.10 with \mathbf{F} replaced by $\text{curl } \mathbf{F}$. The result is

$$\boxed{2} \quad \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \frac{\partial z}{\partial y} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \right] dA$$

where the partial derivatives of P , Q , and R are evaluated at $(x, y, g(x, y))$. If

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

is a parametric representation of C_1 , then a parametric representation of C is

$$x = x(t) \quad y = y(t) \quad z = g(x(t), y(t)) \quad a \leq t \leq b$$

This allows us, with the aid of the Chain Rule, to evaluate the line integral as follows:

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt \\ &= \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \left(\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right) \right] dt \\ &= \int_a^b \left[\left(P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt \\ &= \int_{C_1} \left(P + R \frac{\partial z}{\partial x} \right) dx + \left(Q + R \frac{\partial z}{\partial y} \right) dy \\ &= \iint_D \left[\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) \right] dA \end{aligned}$$

where we have used Green’s Theorem in the last step. Then, using the Chain Rule again and remembering that $P, Q,$ and R are functions of $x, y,$ and z and that z is itself a function of x and $y,$ we get

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_D \left[\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial x \partial y} \right) \right. \\ &\quad \left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial y \partial x} \right) \right] dA \end{aligned}$$

Four of the terms in this double integral cancel and the remaining six terms can be arranged to coincide with the right side of Equation 2. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} \quad \blacksquare$$

EXAMPLE 1 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r},$ where $\mathbf{F}(x, y, z) = -y^2 \mathbf{i} + x \mathbf{j} + z^2 \mathbf{k}$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1.$ (Orient C to be counterclockwise when viewed from above.)

SOLUTION The curve C (an ellipse) is shown in Figure 3. Although $\int_C \mathbf{F} \cdot d\mathbf{r}$ could be evaluated directly, it’s easier to use Stokes’ Theorem. We first compute

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1 + 2y) \mathbf{k}$$

Although there are many surfaces with boundary $C,$ the most convenient choice is the elliptical region S in the plane $y + z = 2$ that is bounded by $C.$ If we orient S upward, then C has the induced positive orientation. The projection D of S onto the xy -plane is

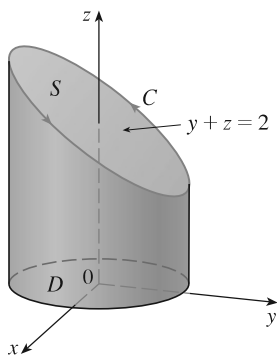


FIGURE 3

the disk $x^2 + y^2 \leq 1$ and so using Equation 16.7.10 with $z = g(x, y) = 2 - y$, we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y) \, dA \\ &= \int_0^{2\pi} \int_0^1 (1 + 2r \sin \theta) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{r^2}{2} + 2 \frac{r^3}{3} \sin \theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{2}{3} \sin \theta \right) d\theta \\ &= \frac{1}{2}(2\pi) + 0 = \pi \end{aligned}$$

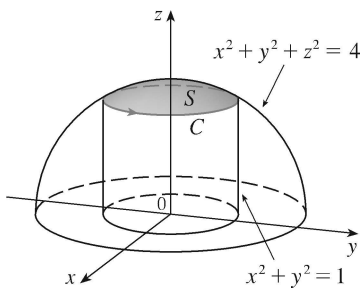


FIGURE 4

EXAMPLE 2 Use Stokes' Theorem to compute the integral $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = xz \mathbf{i} + yz \mathbf{j} + xy \mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy -plane. (See Figure 4.)

SOLUTION To find the boundary curve C we solve the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. Subtracting, we get $z^2 = 3$ and so $z = \sqrt{3}$ (since $z > 0$). Thus C is the circle given by the equations $x^2 + y^2 = 1, z = \sqrt{3}$. A vector equation of C is

$$\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + \sqrt{3} \mathbf{k} \quad 0 \leq t \leq 2\pi$$

so

$$\mathbf{r}'(t) = -\sin t \mathbf{i} + \cos t \mathbf{j}$$

Also, we have

$$\mathbf{F}(\mathbf{r}(t)) = \sqrt{3} \cos t \mathbf{i} + \sqrt{3} \sin t \mathbf{j} + \cos t \sin t \mathbf{k}$$

Therefore, by Stokes' Theorem,

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \\ &= \int_0^{2\pi} (-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t) \, dt \\ &= \sqrt{3} \int_0^{2\pi} 0 \, dt = 0 \end{aligned}$$

Note that in Example 2 we computed a surface integral simply by knowing the values of \mathbf{F} on the boundary curve C . This means that if we have another oriented surface with the same boundary curve C , then we get exactly the same value for the surface integral!

In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then

$$\boxed{3} \quad \iint_{S_1} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \iint_{S_2} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

We now use Stokes' Theorem to throw some light on the meaning of the curl vector. Suppose that C is an oriented closed curve and \mathbf{v} represents the velocity field in fluid flow. Consider the line integral

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot \mathbf{T} \, ds$$

and recall that $\mathbf{v} \cdot \mathbf{T}$ is the component of \mathbf{v} in the direction of the unit tangent vector \mathbf{T} . This means that the closer the direction of \mathbf{v} is to the direction of \mathbf{T} , the larger the value of $\mathbf{v} \cdot \mathbf{T}$. Thus $\int_C \mathbf{v} \cdot d\mathbf{r}$ is a measure of the tendency of the fluid to move around C and is called the **circulation** of \mathbf{v} around C . (See Figure 5.)

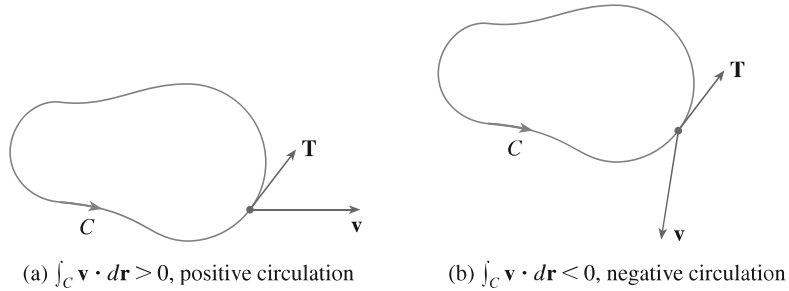


FIGURE 5

Now let $P_0(x_0, y_0, z_0)$ be a point in the fluid and let S_a be a small disk with radius a and center P_0 . Then $(\text{curl } \mathbf{F})(P) \approx (\text{curl } \mathbf{F})(P_0)$ for all points P on S_a because $\text{curl } \mathbf{F}$ is continuous. Thus, by Stokes' Theorem, we get the following approximation to the circulation around the boundary circle C_a :

$$\begin{aligned} \int_{C_a} \mathbf{v} \cdot d\mathbf{r} &= \iint_{S_a} \text{curl } \mathbf{v} \cdot d\mathbf{S} = \iint_{S_a} \text{curl } \mathbf{v} \cdot \mathbf{n} \, dS \\ &\approx \iint_{S_a} \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \, dS = \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) \pi a^2 \end{aligned}$$

Imagine a tiny paddle wheel placed in the fluid at a point P , as in Figure 6; the paddle wheel rotates fastest when its axis is parallel to $\text{curl } \mathbf{v}$.

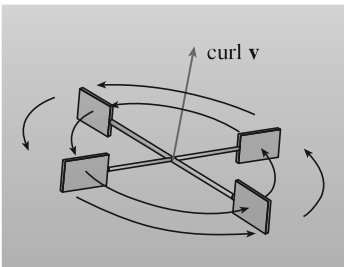


FIGURE 6

This approximation becomes better as $a \rightarrow 0$ and we have

$$\boxed{4} \quad \text{curl } \mathbf{v}(P_0) \cdot \mathbf{n}(P_0) = \lim_{a \rightarrow 0} \frac{1}{\pi a^2} \int_{C_a} \mathbf{v} \cdot d\mathbf{r}$$

Equation 4 gives the relationship between the curl and the circulation. It shows that $\text{curl } \mathbf{v} \cdot \mathbf{n}$ is a measure of the rotating effect of the fluid about the axis \mathbf{n} . The curling effect is greatest about the axis parallel to $\text{curl } \mathbf{v}$.

Finally, we mention that Stokes' Theorem can be used to prove Theorem 16.5.4 (which states that if $\text{curl } \mathbf{F} = \mathbf{0}$ on all of \mathbb{R}^3 , then \mathbf{F} is conservative). From our previous work (Theorems 16.3.3 and 16.3.4), we know that \mathbf{F} is conservative if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C . Given C , suppose we can find an orientable surface S whose boundary is C . (This can be done, but the proof requires advanced techniques.) Then Stokes' Theorem gives

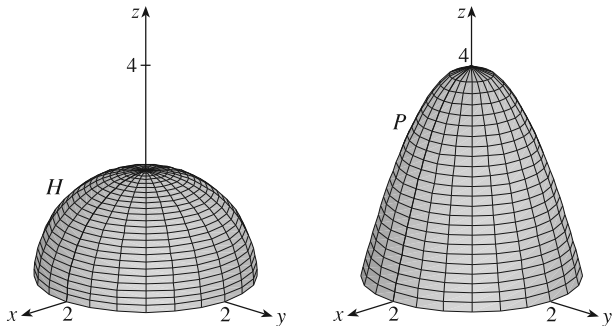
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{0} \cdot d\mathbf{S} = 0$$

A curve that is not simple can be broken into a number of simple curves, and the integrals around these simple curves are all 0. Adding these integrals, we obtain $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C .

16.8 EXERCISES

1. A hemisphere H and a portion P of a paraboloid are shown. Suppose \mathbf{F} is a vector field on \mathbb{R}^3 whose components have continuous partial derivatives. Explain why

$$\iint_H \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_P \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$



2–6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$.

- $\mathbf{F}(x, y, z) = x^2 \sin z \mathbf{i} + y^2 \mathbf{j} + xy \mathbf{k}$,
 S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy -plane, oriented upward
- $\mathbf{F}(x, y, z) = ze^y \mathbf{i} + x \cos y \mathbf{j} + xz \sin y \mathbf{k}$,
 S is the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geq 0$, oriented in the direction of the positive y -axis
- $\mathbf{F}(x, y, z) = \tan^{-1}(x^2 y z^2) \mathbf{i} + x^2 y \mathbf{j} + x^2 z^2 \mathbf{k}$,
 S is the cone $x = \sqrt{y^2 + z^2}$, $0 \leq x \leq 2$, oriented in the direction of the positive x -axis
- $\mathbf{F}(x, y, z) = xyz \mathbf{i} + xy \mathbf{j} + x^2 yz \mathbf{k}$,
 S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, oriented outward
- $\mathbf{F}(x, y, z) = e^{xy} \mathbf{i} + e^{xz} \mathbf{j} + x^2 z \mathbf{k}$,
 S is the half of the ellipsoid $4x^2 + y^2 + 4z^2 = 4$ that lies to the right of the xz -plane, oriented in the direction of the positive y -axis

7–10 Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

- $\mathbf{F}(x, y, z) = (x + y^2) \mathbf{i} + (y + z^2) \mathbf{j} + (z + x^2) \mathbf{k}$,
 C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$
- $\mathbf{F}(x, y, z) = \mathbf{i} + (x + yz) \mathbf{j} + (xy - \sqrt{z}) \mathbf{k}$,
 C is the boundary of the part of the plane $3x + 2y + z = 1$ in the first octant
- $\mathbf{F}(x, y, z) = xy \mathbf{i} + yz \mathbf{j} + zx \mathbf{k}$, C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ in the first octant

- $\mathbf{F}(x, y, z) = 2y \mathbf{i} + xz \mathbf{j} + (x + y) \mathbf{k}$, C is the curve of intersection of the plane $z = y + 2$ and the cylinder $x^2 + y^2 = 1$

11. (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = x^2 z \mathbf{i} + xy^2 \mathbf{j} + z^2 \mathbf{k}$$

and C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$, oriented counterclockwise as viewed from above.

- Graph both the plane and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).
 - Find parametric equations for C and use them to graph C .
- (a) Use Stokes' Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = x^2 y \mathbf{i} + \frac{1}{3} x^3 \mathbf{j} + xy \mathbf{k}$ and C is the curve of intersection of the hyperbolic paraboloid $z = y^2 - x^2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise as viewed from above.
 - Graph both the hyperbolic paraboloid and the cylinder with domains chosen so that you can see the curve C and the surface that you used in part (a).
 - Find parametric equations for C and use them to graph C .

13–15 Verify that Stokes' Theorem is true for the given vector field \mathbf{F} and surface S .

- $\mathbf{F}(x, y, z) = -y \mathbf{i} + x \mathbf{j} - 2 \mathbf{k}$,
 S is the cone $z^2 = x^2 + y^2$, $0 \leq z \leq 4$, oriented downward
- $\mathbf{F}(x, y, z) = -2yz \mathbf{i} + y \mathbf{j} + 3x \mathbf{k}$,
 S is the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 1$, oriented upward
- $\mathbf{F}(x, y, z) = y \mathbf{i} + z \mathbf{j} + x \mathbf{k}$,
 S is the hemisphere $x^2 + y^2 + z^2 = 1$, $y \geq 0$, oriented in the direction of the positive y -axis

16. Let C be a simple closed smooth curve that lies in the plane $x + y + z = 1$. Show that the line integral

$$\int_C z \, dx - 2x \, dy + 3y \, dz$$

depends only on the area of the region enclosed by C and not on the shape of C or its location in the plane.

17. A particle moves along line segments from the origin to the points $(1, 0, 0)$, $(1, 2, 1)$, $(0, 2, 1)$, and back to the origin under the influence of the force field

$$\mathbf{F}(x, y, z) = z^2 \mathbf{i} + 2xy \mathbf{j} + 4y^2 \mathbf{k}$$

Find the work done.

18. Evaluate

$$\int_C (y + \sin x) dx + (z^2 + \cos y) dy + x^3 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \leq t \leq 2\pi$.

[Hint: Observe that C lies on the surface $z = 2xy$.]

19. If S is a sphere and \mathbf{F} satisfies the hypotheses of Stokes' Theorem, show that $\iint_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$.

20. Suppose S and C satisfy the hypotheses of Stokes' Theorem and f, g have continuous second-order partial derivatives. Use Exercises 24 and 26 in Section 16.5 to show the following.

$$(a) \int_C (f \nabla g) \cdot d\mathbf{r} = \iint_S (\nabla f \times \nabla g) \cdot d\mathbf{S}$$

$$(b) \int_C (f \nabla f) \cdot d\mathbf{r} = 0$$

$$(c) \int_C (f \nabla g + g \nabla f) \cdot d\mathbf{r} = 0$$

WRITING PROJECT

The photograph shows a stained-glass window at Cambridge University in honor of George Green.



Courtesy of the Masters and Fellows of Gonville and Caius College, Cambridge University, England

THREE MEN AND TWO THEOREMS

Although two of the most important theorems in vector calculus are named after George Green and George Stokes, a third man, William Thomson (also known as Lord Kelvin), played a large role in the formulation, dissemination, and application of both of these results. All three men were interested in how the two theorems could help to explain and predict physical phenomena in electricity and magnetism and fluid flow. The basic facts of the story are given in the margin notes on pages 1097 and 1135.

Write a report on the historical origins of Green's Theorem and Stokes' Theorem. Explain the similarities and relationship between the theorems. Discuss the roles that Green, Thomson, and Stokes played in discovering these theorems and making them widely known. Show how both theorems arose from the investigation of electricity and magnetism and were later used to study a variety of physical problems.

The dictionary edited by Gillispie [2] is a good source for both biographical and scientific information. The book by Hutchinson [5] gives an account of Stokes' life and the book by Thompson [8] is a biography of Lord Kelvin. The articles by Grattan-Guinness [3] and Gray [4] and the book by Cannell [1] give background on the extraordinary life and works of Green. Additional historical and mathematical information is found in the books by Katz [6] and Kline [7].

1. D. M. Cannell, *George Green, Mathematician and Physicist 1793–1841: The Background to His Life and Work* (Philadelphia: Society for Industrial and Applied Mathematics, 2001).
2. C. C. Gillispie, ed., *Dictionary of Scientific Biography* (New York: Scribner's, 1974). See the article on Green by P. J. Wallis in Volume XV and the articles on Thomson by Jed Buchwald and on Stokes by E. M. Parkinson in Volume XIII.
3. I. Grattan-Guinness, "Why Did George Green Write his Essay of 1828 on Electricity and Magnetism?" *Amer. Math. Monthly*, Vol. 102 (1995), pp. 387–96.
4. J. Gray, "There Was a Jolly Miller." *The New Scientist*, Vol. 139 (1993), pp. 24–27.
5. G. E. Hutchinson, *The Enchanted Voyage and Other Studies* (Westport, CT: Greenwood Press, 1978).
6. Victor Katz, *A History of Mathematics: An Introduction* (New York: HarperCollins, 1993), pp. 678–80.
7. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 683–85.
8. Sylvanus P. Thompson, *The Life of Lord Kelvin* (New York: Chelsea, 1976).

16.9 The Divergence Theorem

In Section 16.5 we rewrote Green's Theorem in a vector version as

$$\int_C \mathbf{F} \cdot \mathbf{n} \, ds = \iint_D \operatorname{div} \mathbf{F}(x, y) \, dA$$

where C is the positively oriented boundary curve of the plane region D . If we were seeking to extend this theorem to vector fields on \mathbb{R}^3 , we might make the guess that

$$\boxed{1} \quad \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_E \operatorname{div} \mathbf{F}(x, y, z) \, dV$$

where S is the boundary surface of the solid region E . It turns out that Equation 1 is true, under appropriate hypotheses, and is called the Divergence Theorem. Notice its similarity to Green's Theorem and Stokes' Theorem in that it relates the integral of a derivative of a function ($\operatorname{div} \mathbf{F}$ in this case) over a region to the integral of the original function \mathbf{F} over the boundary of the region.

At this stage you may wish to review the various types of regions over which we were able to evaluate triple integrals in Section 15.6. We state and prove the Divergence Theorem for regions E that are simultaneously of types 1, 2, and 3 and we call such regions **simple solid regions**. (For instance, regions bounded by ellipsoids or rectangular boxes are simple solid regions.) The boundary of E is a closed surface, and we use the convention, introduced in Section 16.7, that the positive orientation is outward; that is, the unit normal vector \mathbf{n} is directed outward from E .

The Divergence Theorem is sometimes called Gauss's Theorem after the great German mathematician Karl Friedrich Gauss (1777–1855), who discovered this theorem during his investigation of electrostatics. In Eastern Europe the Divergence Theorem is known as Ostrogradsky's Theorem after the Russian mathematician Mikhail Ostrogradsky (1801–1862), who published this result in 1826.

The Divergence Theorem Let E be a simple solid region and let S be the boundary surface of E , given with positive (outward) orientation. Let \mathbf{F} be a vector field whose component functions have continuous partial derivatives on an open region that contains E . Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F} \, dV$$

Thus the Divergence Theorem states that, under the given conditions, the flux of \mathbf{F} across the boundary surface of E is equal to the triple integral of the divergence of \mathbf{F} over E .

PROOF Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then

$$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{so} \quad \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E \frac{\partial P}{\partial x} \, dV + \iiint_E \frac{\partial Q}{\partial y} \, dV + \iiint_E \frac{\partial R}{\partial z} \, dV$$

If \mathbf{n} is the unit outward normal of S , then the surface integral on the left side of the Divergence Theorem is

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iint_S (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \mathbf{n} \, dS \\ &= \iint_S P\mathbf{i} \cdot \mathbf{n} \, dS + \iint_S Q\mathbf{j} \cdot \mathbf{n} \, dS + \iint_S R\mathbf{k} \cdot \mathbf{n} \, dS \end{aligned}$$

Therefore, to prove the Divergence Theorem, it suffices to prove the following three

equations:

$$\boxed{2} \quad \iint_S P \mathbf{i} \cdot \mathbf{n} \, dS = \iiint_E \frac{\partial P}{\partial x} \, dV$$

$$\boxed{3} \quad \iint_S Q \mathbf{j} \cdot \mathbf{n} \, dS = \iiint_E \frac{\partial Q}{\partial y} \, dV$$

$$\boxed{4} \quad \iint_S R \mathbf{k} \cdot \mathbf{n} \, dS = \iiint_E \frac{\partial R}{\partial z} \, dV$$

To prove Equation 4 we use the fact that E is a type 1 region:

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

where D is the projection of E onto the xy -plane. By Equation 15.6.6, we have

$$\iiint_E \frac{\partial R}{\partial z} \, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} \frac{\partial R}{\partial z}(x, y, z) \, dz \right] dA$$

and therefore, by the Fundamental Theorem of Calculus,

$$\boxed{5} \quad \iiint_E \frac{\partial R}{\partial z} \, dV = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))] \, dA$$

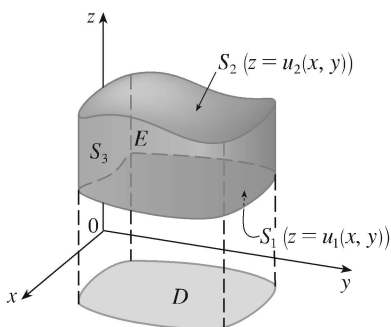


FIGURE 1

The boundary surface S consists of three pieces: the bottom surface S_1 , the top surface S_2 , and possibly a vertical surface S_3 , which lies above the boundary curve of D . (See Figure 1. It might happen that S_3 doesn't appear, as in the case of a sphere.) Notice that on S_3 we have $\mathbf{k} \cdot \mathbf{n} = 0$, because \mathbf{k} is vertical and \mathbf{n} is horizontal, and so

$$\iint_{S_3} R \mathbf{k} \cdot \mathbf{n} \, dS = \iint_{S_3} 0 \, dS = 0$$

Thus, regardless of whether there is a vertical surface, we can write

$$\boxed{6} \quad \iint_S R \mathbf{k} \cdot \mathbf{n} \, dS = \iint_{S_1} R \mathbf{k} \cdot \mathbf{n} \, dS + \iint_{S_2} R \mathbf{k} \cdot \mathbf{n} \, dS$$

The equation of S_2 is $z = u_2(x, y)$, $(x, y) \in D$, and the outward normal \mathbf{n} points upward, so from Equation 16.7.10 (with \mathbf{F} replaced by $R \mathbf{k}$) we have

$$\iint_{S_2} R \mathbf{k} \cdot \mathbf{n} \, dS = \iint_D R(x, y, u_2(x, y)) \, dA$$

On S_1 we have $z = u_1(x, y)$, but here the outward normal \mathbf{n} points downward, so we multiply by -1 :

$$\iint_{S_1} R \mathbf{k} \cdot \mathbf{n} \, dS = - \iint_D R(x, y, u_1(x, y)) \, dA$$

Therefore Equation 6 gives

$$\iint_S R \mathbf{k} \cdot \mathbf{n} \, dS = \iint_D [R(x, y, u_2(x, y)) - R(x, y, u_1(x, y))] \, dA$$

Comparison with Equation 5 shows that

$$\iint_S R \mathbf{k} \cdot \mathbf{n} \, dS = \iiint_E \frac{\partial R}{\partial z} \, dV$$

Notice that the method of proof of the Divergence Theorem is very similar to that of Green's Theorem.

Equations 2 and 3 are proved in a similar manner using the expressions for E as a type 2 or type 3 region, respectively. ■

EXAMPLE 1 Find the flux of the vector field $\mathbf{F}(x, y, z) = z \mathbf{i} + y \mathbf{j} + x \mathbf{k}$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

SOLUTION First we compute the divergence of \mathbf{F} :

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(x) = 1$$

The unit sphere S is the boundary of the unit ball B given by $x^2 + y^2 + z^2 \leq 1$. Thus the Divergence Theorem gives the flux as

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_B \operatorname{div} \mathbf{F} \, dV = \iiint_B 1 \, dV = V(B) = \frac{4}{3}\pi(1)^3 = \frac{4\pi}{3}$$

The solution in Example 1 should be compared with the solution in Example 16.7.4.

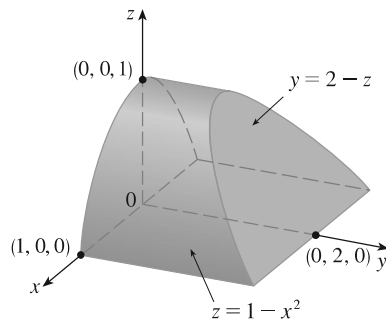


FIGURE 2

EXAMPLE 2 Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F}(x, y, z) = xy \mathbf{i} + (y^2 + e^{xz^2}) \mathbf{j} + \sin(xy) \mathbf{k}$$

and S is the surface of the region E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$, and $y + z = 2$. (See Figure 2.)

SOLUTION It would be extremely difficult to evaluate the given surface integral directly. (We would have to evaluate four surface integrals corresponding to the four pieces of S .) Furthermore, the divergence of \mathbf{F} is much less complicated than \mathbf{F} itself:

$$\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(y^2 + e^{xz^2}) + \frac{\partial}{\partial z}(\sin xy) = y + 2y = 3y$$

Therefore we use the Divergence Theorem to transform the given surface integral into a triple integral. The easiest way to evaluate the triple integral is to express E as a type 3 region:

$$E = \{(x, y, z) \mid -1 \leq x \leq 1, 0 \leq z \leq 1 - x^2, 0 \leq y \leq 2 - z\}$$

Then we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div} \mathbf{F} \, dV = \iiint_E 3y \, dV \\ &= 3 \int_{-1}^1 \int_0^{1-x^2} \int_0^{2-z} y \, dy \, dz \, dx = 3 \int_{-1}^1 \int_0^{1-x^2} \frac{(2-z)^2}{2} \, dz \, dx \\ &= \frac{3}{2} \int_{-1}^1 \left[-\frac{(2-z)^3}{3} \right]_0^{1-x^2} dx = -\frac{1}{2} \int_{-1}^1 [(x^2 + 1)^3 - 8] \, dx \\ &= -\int_0^1 (x^6 + 3x^4 + 3x^2 - 7) \, dx = \frac{184}{35} \end{aligned}$$

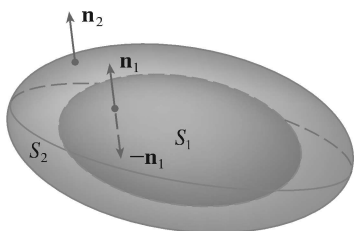


FIGURE 3

Although we have proved the Divergence Theorem only for simple solid regions, it can be proved for regions that are finite unions of simple solid regions. (The procedure is similar to the one we used in Section 16.4 to extend Green’s Theorem.)

For example, let’s consider the region E that lies between the closed surfaces S_1 and S_2 , where S_1 lies inside S_2 . Let \mathbf{n}_1 and \mathbf{n}_2 be outward normals of S_1 and S_2 . Then the boundary surface of E is $S = S_1 \cup S_2$ and its normal \mathbf{n} is given by $\mathbf{n} = -\mathbf{n}_1$ on S_1 and $\mathbf{n} = \mathbf{n}_2$ on S_2 . (See Figure 3.) Applying the Divergence Theorem to S , we get

$$\begin{aligned}
 \boxed{7} \quad \iiint_E \operatorname{div} \mathbf{F} \, dV &= \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\
 &= \iint_{S_1} \mathbf{F} \cdot (-\mathbf{n}_1) \, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n}_2 \, dS \\
 &= -\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}
 \end{aligned}$$

EXAMPLE 3 In Example 16.1.5 we considered the electric field

$$\mathbf{E}(\mathbf{x}) = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x}$$

where the electric charge Q is located at the origin and $\mathbf{x} = \langle x, y, z \rangle$ is a position vector. Use the Divergence Theorem to show that the electric flux of \mathbf{E} through any closed surface S_2 that encloses the origin is

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = 4\pi\varepsilon Q$$

SOLUTION The difficulty is that we don’t have an explicit equation for S_2 because it is *any* closed surface enclosing the origin. The simplest such surface would be a sphere, so we let S_1 be a small sphere with radius a and center the origin. You can verify that $\operatorname{div} \mathbf{E} = 0$. (See Exercise 23.) Therefore Equation 7 gives

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} + \iiint_E \operatorname{div} \mathbf{E} \, dV = \iint_{S_1} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS$$

The point of this calculation is that we can compute the surface integral over S_1 because S_1 is a sphere. The normal vector at \mathbf{x} is $\mathbf{x}/|\mathbf{x}|$. Therefore

$$\mathbf{E} \cdot \mathbf{n} = \frac{\varepsilon Q}{|\mathbf{x}|^3} \mathbf{x} \cdot \left(\frac{\mathbf{x}}{|\mathbf{x}|} \right) = \frac{\varepsilon Q}{|\mathbf{x}|^4} \mathbf{x} \cdot \mathbf{x} = \frac{\varepsilon Q}{|\mathbf{x}|^2} = \frac{\varepsilon Q}{a^2}$$

since the equation of S_1 is $|\mathbf{x}| = a$. Thus we have

$$\iint_{S_2} \mathbf{E} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{E} \cdot \mathbf{n} \, dS = \frac{\varepsilon Q}{a^2} \iint_{S_1} dS = \frac{\varepsilon Q}{a^2} A(S_1) = \frac{\varepsilon Q}{a^2} 4\pi a^2 = 4\pi\varepsilon Q$$

This shows that the electric flux of \mathbf{E} is $4\pi\varepsilon Q$ through *any* closed surface S_2 that contains the origin. [This is a special case of Gauss’s Law (Equation 16.7.11) for a single charge. The relationship between ε and ε_0 is $\varepsilon = 1/(4\pi\varepsilon_0)$.] ■