that $\hat{\theta}$ attach Cramer Rao Lower Bound."

Lemma:

Under the same regularity conditions as above, there exists an unbiased estimator $\hat{\theta}$ which attains the CRLB iff

$$\frac{\partial l}{\partial \theta} = I_\theta (\hat{\theta} - \theta)$$

$\Rightarrow$ $y = mx + c$ i.e equation of straight line

Proof:

In deriving the bound, we used the inequality
$(Cov(u,v))^2 \leq Var(u) Var(v)$
and the bound will be attained iff equality is achieved here i.e $\left((Cov(u,v))^2 = Var(u) V(v) \right)$
However, equality occurs iff there is an exact linear relationship b/w u and v (corresponding to a correlation of ± 1 b/w u and v)
Remember that $u = \hat{\theta}$ and $v = \frac{\partial l}{\partial \theta}$. So we have,

$$\frac{\partial l}{\partial \theta} = c + d\hat{\theta}$$, where 'c' and 'd' are constants

$$E\left(\frac{\partial l}{\partial \theta}\right) = E(c + d\hat{\theta}) = c + d E(\hat{\theta})$$

$$\frac{dl}{d\theta} \int \frac{1}{L} \frac{dL}{d\theta} L\, dx$$

$$\frac{d}{d\theta} \int L\, dx$$

$$\frac{d}{d\theta}(1) = 0$$

$$\text{So } E\left(\frac{\partial l}{\partial \theta}\right) = 0$$

As $E\left(\frac{\partial l}{\partial \theta}\right) = 0 \Rightarrow c + d E(\hat{\theta}) = 0$

$$c + d\theta = 0 \qquad \therefore E(\hat{\theta}) = \theta$$

$$c = -d\theta$$

So

$$\frac{\partial l}{\partial \theta} = -d\theta + d\hat{\theta} = d(\hat{\theta} - \theta) \rightarrow (i)$$

Multiplying both sides by $\partial l / \partial \theta$, we get

$$\frac{\partial l}{\partial \theta}\left(\frac{\partial l}{\partial \theta}\right) = \frac{\partial l}{\partial \theta} d(\hat{\theta} - \theta)$$

Let $u = \hat{\theta}$, $v = \frac{\partial l}{\partial \theta}$

Now
$$E(v) = E\left(\frac{\partial l}{\partial \theta}\right) = \int \frac{\partial l}{\partial \theta} L \, d\underline{x} = \int \frac{\partial L}{\partial \theta} d\underline{x}$$
$$= \frac{\partial}{\partial \theta} \int L \, d\underline{x} = \frac{\partial}{\partial \theta}(1) = 0$$

Hence $Cov(u, v) = E(uv) - E(u)E(v)$
$$= E(uv) - 0$$
$$Cov(u, v) = 1 + \frac{\partial b}{\partial \theta} \qquad \because E(u, v) = 1 + \frac{\partial b}{\partial \theta}$$

Also $V(v) = E(v^2) - (E(v))^2$
$$= E\left(\frac{\partial l}{\partial \theta}\right)^2 = E\left[\frac{\partial}{\partial \theta} \ln L\right]^2 = I_\theta$$

As $Var(u) \geq \frac{(Cov(u, v))^2}{Var(v)}$

$$Var(u) = Var(\hat{\theta}) \geq \frac{(1 + \partial b / \partial \theta)^2}{I_\theta}$$

$$Var(\hat{\theta}) \geq \left(1 + \frac{\partial b}{\partial \theta}\right)^2 I_\theta^{-1}$$

good.

(ii) A lower bound for unbiased estimators of a function, $g(\theta)$, of θ is
$$Var(\hat{\theta}) \geq \left(\frac{\partial g}{\partial \theta}\right)^2 I_\theta^{-1}$$

$\because E(\hat{\theta}) = g(\theta)$

$\frac{\partial}{\partial \theta} E(\hat{\theta}) = g'(\theta)$

Proof :-
$$\therefore E(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n) L(\theta; \underline{x}) d\underline{x} = g(\theta)$$
$$\therefore g(\theta) = \int \hat{\theta}(x_1, \ldots x_n) L(\theta; \underline{x}) d\underline{x}.$$

Differentiate w.r.t θ,
$$\frac{\partial}{\partial \theta} g(\theta) = \int \hat{\theta}(x_1, \ldots x_n) \frac{\partial}{\partial \theta} L(\theta; \underline{x}) d\underline{x}$$
$$\frac{\partial g}{\partial \theta} = \int \hat{\theta}(x_1, \ldots x_n) \frac{\partial l}{\partial \theta} L \, d\underline{x} \qquad \because \frac{\partial L}{\partial \theta} = \left(\frac{\partial l}{\partial \theta}\right) L$$

Hence, $Cov(u,v) = E(uv) - E(u)E(v)$

$$cov(u,v) = E(uv) - 0$$
$$= 1 \quad \because E\left(\hat{\theta} \frac{\partial l}{\partial \theta}\right) = 1$$

Also
$$V(v) = E(v^2) - (E(v))^2$$
$$= E(v^2) - 0$$
$$= E\left(\frac{\partial l}{\partial \theta}\right)^2 = I_\theta$$

$Cov(U,V) \le Var(U)Var(V)$

$\frac{Cov(U,V)}{Var(V)} \le Var(U)$

$Var(U) \geq \frac{Cov(U,V)}{Var(V)}$

$$V(U) = V(\hat{\theta}) \geq \frac{1}{I_\theta} = I_\theta^{-1} \qquad \because V(u) \geq \frac{(Cov(u,v))}{V(v)}$$

$\therefore V(\hat{\theta}) \geq I_\theta^{-1}$, as required

Extensions To The Cramor Rao Inequality:

(i) If $\hat{\theta}$ is an estimator of θ with bias$(\hat{\theta})$, denoted by $b(\theta)$, then $V(\hat{\theta}) \geq \left(1 + \frac{\partial b}{\partial \theta}\right)^2 I_\theta^{-1}$

Proof:

$$E(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n) \, L(\theta; \underline{x}) d\underline{x}$$

$\hat{\theta}$ is a biased estimator of θ, so.

$$E(\hat{\theta}) = \theta + b(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n) \, L(\theta; \underline{x}) d\underline{x}$$

Differentiate w.r.t θ.

$$\frac{\partial}{\partial \theta} E(\hat{\theta}) = 1 + \frac{\partial b}{\partial \theta} = \int \hat{\theta}(x_1, \ldots x_n) \frac{\partial}{\partial \theta} L(\theta; \underline{x}) d\underline{x}$$

Now $= \frac{\partial l}{\partial \theta} = \frac{\partial}{\partial \theta}(\ln L) = \frac{1}{L} \frac{\partial L}{\partial \theta}$

$$\frac{\partial L}{\partial \theta} = L \frac{\partial l}{\partial \theta}$$

$$\Rightarrow 1 + \frac{\partial b}{\partial \theta} = \int \hat{\theta} \frac{\partial l}{\partial \theta} L \, d\underline{x}$$

$$1 + \frac{\partial b}{\partial \theta} = E\left(\hat{\theta} \frac{\partial l}{\partial \theta}\right) \quad \rightarrow (i)$$

For any two r.v's, u and v

$$(Cov(u,v))^2 \leq Var(u) Var(v)$$

$E(\hat{\theta}) = \theta$

Proof :- $E(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n)\, L(\theta; \underline{x})\, d\underline{x}$.

(A multiple integral w.r.t $x_1, \ldots x_n$; $L(\theta; \underline{x})$ is the joint p.d.f of the x_i's as well as the likelihood fn.)

$\hat{\theta}$ is an unbiased estimator of θ, so

$E(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n)\, L(\theta; \underline{x})\, d\underline{x} = \theta$.

Differentiate w.r.t θ, and interchange the order of differentiation and integration,

$\frac{\partial}{\partial\theta} E(\hat{\theta}) = \int \hat{\theta}(x_1, \ldots x_n) \frac{\partial}{\partial\theta} L(\theta; \underline{x})\, d\underline{x} = 1$

$\because \ell = \ln L(\cdot)$

Now $\frac{\partial \ell}{\partial\theta} = \frac{\partial}{\partial\theta}(\ln L) = \frac{1}{L}\frac{\partial L}{\partial\theta}$

so $\frac{\partial L}{\partial\theta} = L \frac{\partial \ell}{\partial\theta}$

Thus, $1 = \int \hat{\theta} \frac{\partial \ell}{\partial\theta} L\, d\underline{x}$

$1 = E\left(\hat{\theta} \frac{\partial \ell}{\partial\theta}\right) \longrightarrow (i)$

For any two r.v's, U and V

$(Cov(U,V))^2 \leq Var(U)\, Var(V)$ — By Cauchy-Shewartz inequality

$\Rightarrow \quad 0 \leq \rho^2 \leq 1$

where $\rho^2 = \frac{(Cov(X_1, X_2))^2}{Var(X_1)V(X_2)}$ $= \frac{(Cov(X_1, X_2))^2}{Var(X_1)\, Var(X_2)}$

$\rho^2 \leq 1 \Rightarrow \frac{(Cov(X_1, X_2))^2}{Var(X_1)V(X_2)} \leq 1 \Rightarrow Cov(X_1, X_2)^2 \leq V(X_1)V(X_2)$

Now let $U = \hat{\theta}$ and $V = \frac{\partial \ell}{\partial\theta}$

Now

$E(V) = E\left(\frac{\partial \ell}{\partial\theta}\right) = \int \frac{\partial \ell}{\partial\theta} L\, d\underline{x} = \int \frac{\partial L}{\partial\theta}\, d\underline{x}$ — interchang derivative & $\int$ — $\frac{\partial \ell L}{\partial\theta} = \frac{\partial L}{\partial\theta}$.

$= \frac{\partial}{\partial\theta}\underbrace{\int L\, d\underline{x}}_{1} = \frac{\partial}{\partial\theta}(1) = 0$

$$\left(\frac{\partial l}{\partial \theta}\right)^2 = d\left[\hat{\theta}\frac{\partial l}{\partial \theta} - \theta\frac{\partial l}{\partial \theta}\right]$$

$$= d\left[\hat{\theta}\frac{\partial l}{\partial \theta} - \theta\frac{\partial l}{\partial \theta}\right]$$

Applying expectation on both sides, if $E(\hat{\theta}) = \theta$, $= 1 = E(UV)$

$$E\left(\frac{\partial l}{\partial \theta}\right)^2 = d\left[E\left(\hat{\theta}\frac{\partial l}{\partial \theta}\right) - \theta \cdot E\left(\frac{\partial l}{\partial \theta}\right)\right]$$

$$= d\left[1 - \theta\,(\text{zero})\right] \qquad \because E\left(\hat{\theta}\frac{\partial l}{\partial \theta}\right) = 1, \quad E\left(\frac{\partial l}{\partial \theta}\right) = 0$$

$$E\left(\frac{\partial l}{\partial \theta}\right)^2 = d$$

$$\Rightarrow \quad d = E\left(\frac{\partial l}{\partial \theta}\right)^2 = I_\theta$$

Substitution in (i), we get

$$\frac{\partial l}{\partial \theta} = I_\theta(\hat{\theta} - \theta)$$

→ Problems in CR :

⇒ In CR, x range does not depend on θ, whereas Chapman Rao gives $0 < x \leq \theta$?

Merits of CR :

(i) It always gives unbiased estimator.

(ii) Unbiased estimator attains CR Lower Bound always.

→ The variance bound called the CRLB for unbiased estimator of $T(\theta)$ is appreciated, where one can,

(i) Derive an explicit expression of CRLB and

(ii) Easily locate an unbiased estimator of $T(\theta)$ whose variance coincides with Cramer Rao lower bound.

In these situations, one has then found the UMVUE of $T(\theta)$ that is

$$\frac{\partial l}{\partial \theta} = K(\theta, x)\left(t(x_1, \ldots, x_n) - T(\theta)\right)$$

$$\left(\frac{\partial l}{\partial \theta}\right)^2 = d\left[\hat{\theta}\frac{\partial l}{\partial \theta} - \theta\frac{\partial l}{\partial \theta}\right]$$

$$= d\left[\hat{\theta}\frac{\partial l}{\partial \theta} - \theta\frac{\partial l}{\partial \theta}\right]$$

Applying expectation on both sides, if $E(\hat{\theta}) = \theta$, $= 1 = E(UV)$

$$E\left(\frac{\partial l}{\partial \theta}\right)^2 = d\left[E\left(\hat{\theta}\frac{\partial l}{\partial \theta}\right) - \theta\, E\left(\frac{\partial l}{\partial \theta}\right)\right]$$

$$= d\left[1 - \theta\,(\text{zero})\right] \qquad \because E\left(\hat{\theta}\frac{\partial l}{\partial \theta}\right) = 1$$

$$E\left(\frac{\partial l}{\partial \theta}\right)^2 = d \qquad E\left(\frac{\partial l}{\partial \theta}\right) = 0$$

$$\Rightarrow \quad d = E\left(\frac{\partial l}{\partial \theta}\right)^2 = I_\theta$$

Substitution in (i), we get

$$\frac{\partial l}{\partial \theta} = I_\theta(\hat{\theta} - \theta)$$

Problems in CR:

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In these situations, one has then found the UMVUE of $T(\theta)$ that is

$$\frac{\partial l}{\partial \theta} = K(\theta, x)\left(t(x_1, \ldots, x_n) - T(\theta)\right).$$

Lemma :-

Under the same regularity conditions as for the CR inequality, $I_\theta = -E\left(\frac{\partial^2 l}{\partial\theta^2}\right)$ for.

Proof:-

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{\partial^2}{\partial\theta^2}(\ln L) \qquad \because l = \ln L(\cdot)$$

$$= \frac{\partial}{\partial\theta}\left(\frac{\partial \ln L}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(\frac{1}{L}\frac{\partial L}{\partial\theta}\right), \text{ By product theorem}$$

$$= -\frac{1}{L^2}\frac{\partial L}{\partial\theta}\cdot\frac{\partial L}{\partial\theta} + \frac{1}{L}\frac{\partial^2 L}{\partial\theta^2}$$

$$= -\frac{1}{L^2}\left(\frac{\partial L}{\partial\theta}\right)^2 + \frac{1}{L}\frac{\partial^2 L}{\partial\theta^2}$$

$$= -\left(\frac{1}{L}\frac{\partial l}{\partial\theta}\right)^2 + \frac{1}{L}\frac{\partial^2 L}{\partial\theta^2}$$

But

$$= -\left(\frac{\partial l}{\partial\theta}\right)^2 + \frac{1}{L}\frac{\partial^2 L}{\partial\theta^2} = 0 \quad \because \frac{\partial l}{\partial\theta} = \frac{1}{L}\frac{\partial L}{\partial\theta}$$

$$E\left(\frac{\partial^2 l}{\partial\theta^2}\right) = -E(\ ) + E(\ )$$

If we show that second term has zero expectation, then we have the required results.

But $E\left(\frac{1}{L}\frac{\partial^2 L}{\partial\theta^2}\right) = \int \frac{1}{L}\frac{\partial^2 L}{\partial\theta^2} L\,dx$ $\because$ By definition

Interchanging the order of integration & differenciation

$$E\left(\frac{1}{L}\frac{\partial^2 L}{\partial\theta^2}\right) = \int \frac{\partial^2 L}{\partial\theta^2}\,d\underline{x} = \frac{\partial^2}{\partial\theta^2}\int L\,d\underline{x} = \frac{\partial^2}{\partial\theta^2}(1) = 0$$

$\because$ here we used the regularity conditions to interchange the order of differenciation & integration.

$E\left(\frac{\partial^2 l}{\partial\theta^2}\right)$

$$\Rightarrow E\left(\frac{\partial^2 l}{\partial\theta^2}\right) = -E\left(\frac{\partial l}{\partial\theta}\right)^2 + \text{zero}$$

Imp

$$I_\theta = -E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \qquad \because I_\theta = E\left(\frac{\partial l}{\partial\theta}\right)^2$$

$\Rightarrow V(\hat\theta) \geq I_\theta^{-1}$, it will be equal i.e $V(\hat\theta) = I_\theta^{-1}$ when $\rho = 1$, as $0 \leq \rho^2 \leq 1$, $\rho = 1$ means (X, Y) have perfect linear relationship i.e u and v are perfectly correlated