when $x = 0 \Rightarrow y = -\log_e(0) = -1$

$x = \infty \Rightarrow y = -\log_e(\infty) = \infty$?

$\acute{S}2$

$$|J| = \left|\frac{dx}{dy}\right| = |e^{-y}(-1)| = e^{-y}$$

$$g(y,\theta) = \theta(e^{-y})^{\theta-1} \times e^{-y} = \theta e^{-y(\theta-1)-y} = \theta e^{-y\theta}, \quad \theta>0,\ y>0$$

$\therefore g(y,\theta)$ is an exponential dist

let $T^* = -\frac{1}{n}\sum_{i=1}^{n} \log_e x_i = \frac{\sum_i Y_i}{n}$

$E(T^*) = \frac{1}{n}\sum_i^n E(Y_i) = \frac{n}{n}\cdot\frac{1}{\theta} = \frac{1}{\theta}, \quad Y_i \sim \exp(\theta)$

$V(T^*) = \frac{1}{n^2}\sum_i^n V(Y_i) = \frac{n}{n^2}\cdot\frac{1}{\theta^2} = \frac{1}{n\theta^2}$ $= \boxed{V(T^*)}$

As $\frac{\partial l}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log x_i$ $\quad = \frac{1}{n}\sum \log x_i + \frac{n}{\theta}$

$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2} \Rightarrow -\frac{\partial^2 l}{\partial \theta^2} = \frac{n}{\theta^2}$

$\Rightarrow -E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = \frac{n}{\theta^2}$

As $\tau(\theta) = 1/\theta$, $\boxed{\tau'(\theta) = -1/\theta^2}$

By CRLB,

$$Var(T^*) \geq \frac{(\tau'(\theta))^2}{-E\left[\frac{\partial^2 l}{\partial \theta^2}\right]} = \frac{(-1/\theta^2)^2}{n/\theta^2} = \frac{1}{n\theta^2}$$

which is same as $V(T^*) = \frac{1}{n\theta^2}$, as above

(ii) $f_x(x;\theta) = \frac{\log_e(\theta)\,\theta^x}{\theta-1}, \quad 0<x<1, \quad \theta>0$

$$L(\theta;\underline{x}) = \frac{(\log_e(\theta))^n\,\theta^{\sum_i^n x_i}}{(\theta-1)^n}$$

$$l = \ln L(\theta;\underline{x}) = n\log_e(\log_e\theta) + \sum_{i=1}^{n} x_i \log_e\theta - n\log_e(\theta-1)$$

$$\frac{\partial l}{\partial \theta} = n\frac{\partial}{\partial \theta}\ln(\ln\theta) + \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n}{\theta-1}$$

$$= n\left[\frac{1}{\ln\theta}\frac{\partial}{\partial\theta}\ln\theta\right] + \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n}{\theta-1}$$

$$= \frac{n}{\theta \ln\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{n}{\theta - 1}$$

$$\frac{\partial l}{\partial \theta} = \frac{\sum_1 x_i}{\theta} + \frac{n}{\theta \ln\theta} - \frac{n}{\theta - 1}$$

$$= \frac{n\bar{x}}{\theta} + \frac{n}{\theta \ln\theta} - \frac{n}{\theta - 1}$$

$$= \frac{n}{\theta}\left[\bar{x} + \frac{1}{\ln\theta} - \frac{1}{(\theta-1)/\theta}\right]$$

$$= \frac{n}{\theta}\left[\bar{x} - \left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)\right]$$

$$\underbrace{\left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)}_{\tau(\theta)}$$

Comparing with,

$$\frac{\partial l}{\partial \theta} = k(n;\theta)\left(\hat{\theta} - \tau(\theta)\right)$$

hence $\bar{x}$ is an unbiased estimator of $\left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)$ i.e

rification

Sey

alience

$$E(\bar{x}) = E(x) = \int_0^1 x\, f_X(x;\theta)\,dx$$

$$= \int_0^1 x \cdot \frac{\ln\theta}{\theta-1}\, \theta^x\, dx$$

$$= \frac{\ln\theta}{\theta-1} \int_0^1 x\,\theta^x\, dx \qquad \text{let } A = \frac{\ln\theta}{\theta-1}$$

Integrating by parts,

$$= A\left[x \int_0^1 \theta^x\, dx - \int_0^1 \left(\left(\int \theta^x dx\right) \frac{d}{dx} x\right) dx\right]$$

$$= A\left[x \cdot \frac{\theta^x}{\ln\theta}\Big|_0^1 - \int_0^1 \frac{\theta^x}{\ln\theta} \cdot dx\right]$$

$$= A\left[\frac{\theta}{\ln\theta} - 0 - \frac{1}{\ln\theta}\int_0^1 \theta^x\, dx\right]$$

$$= A\left[\frac{\theta}{\ln\theta} - \frac{1}{\ln\theta} \cdot \frac{\theta^x}{\ln\theta}\Big|_0^1\right]$$

$$E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = -\frac{n}{\theta^2} E(\bar{x}) - \frac{n(1+\ln\theta)}{(\theta \ln\theta)^2} + \frac{n}{(\theta-1)^2}$$

$$= -\frac{n}{\theta^2}\left[\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right] - \frac{n(1+\ln\theta)}{(\theta\ln\theta)^2} + \frac{n}{(\theta-1)^2}$$

$$= -\frac{n}{\theta^2}\left(\frac{\theta\ln\theta - (\theta-1)}{(\theta-1)\ln\theta}\right) - \text{''} + \text{''}$$

By CRLB,

$$V(\bar{x}) \geq \frac{(\tau'(\theta))^2}{-E\left[\frac{\partial^2 l}{\partial\theta^2}\right]} = \frac{\left[-1/(\theta-1)^2 + \frac{1}{\theta(\ln\theta)^2}\right]^2}{-\left[\frac{-n}{\theta^2}\left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right) - \frac{n(1+\ln\theta)}{(\theta\ln\theta)^2} + \frac{n}{(\theta-1)^2}\right]}$$

$$\cancel{\geq \frac{\frac{1}{(\theta-1)^4} + \frac{1}{\theta^2(\ln\theta)^4} - \frac{2}{\theta(\theta-1)^2(\ln\theta)^2}}{\frac{n}{\theta^2}\left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right) + \frac{n(1+\ln\theta)}{(\theta\ln\theta)^2} - \frac{n}{(\theta-1)^2}}} \rightarrow (1)$$

Solving numerator, we get

$$\frac{\left(-\theta(\ln\theta)^2 + (\theta-1)^2\right)^2}{\theta^2(\theta-1)^4(\ln\theta)^4}$$

$$= \left(\frac{-\theta(\ln\theta)^2 + (\theta-1)^2}{\theta(\ln\theta)^2(\theta-1)^2}\right)^2$$

By solving denominator, we get

$$= n\left(\frac{-\theta(\ln\theta)^2 + (\theta-1)^2}{\theta^2(\theta-1)^2(\ln\theta)^2}\right)$$

$$\left(\frac{-n\theta}{\theta^2(\theta-1)} + \frac{n}{(\ln\theta)\theta^2} - \frac{n(1+\ln\theta)}{(\theta\ln\theta)^2} + \frac{n}{(\theta-1)^2}\right)$$

hence,

$$V(\bar{x}) \geq \frac{\left[-\theta(\ln\theta)^2 + (\theta-1)^2\right]^{\cancel{2}}}{\cancel{\theta^2}(\theta-1)^{\cancel{4}}(\ln\theta)^{\cancel{4}}} \times \frac{\cancel{\theta^2}(\theta-1)^{\cancel{2}}(\ln\theta)^{\cancel{2}}}{n\cancel{\left[-\theta(\ln\theta)^2 + (\theta-1)^2\right]}}$$

$$V(\bar{x}) \geq \frac{-\theta(\ln\theta)^2 + (\theta-1)^2}{n(\theta-1)^2(\ln\theta)^2}.$$

$$V(\bar{x}) \geq \frac{1}{n}\left[\frac{-\theta}{(\theta-1)^2} + \frac{1}{(\ln\theta)^2}\right]$$

which is same as above (direct method)

=>. Rao Blackwell theorem uses sufficient and unbiased statistic:

$E[u/b / MSS] = f(MSS) \rightarrow$ minimal sufficient stats

unbiased

$\Downarrow$ which is u/b estimator of θ and its variance is less than the var of u/b estimator

$\frac{\partial l}{\partial \theta} = I_\theta(\hat{\theta} - \theta)$, $\frac{\partial l}{\partial \theta} = k(n,\theta)(\hat{\theta} - \tau(\theta))$ ss

Attainment condition of CRLB.

Hence T^* is an u/b estimator of $1/\theta$ which attains CRLB.

$$V(T^*) = \frac{[\tau'(\theta)]^2}{I_\theta}$$

$$\frac{\partial^2 l}{\partial \theta^2} = \frac{-n}{\theta^2}$$

$$-E\left[\frac{\partial^2 l}{\partial \theta^2}\right] = \frac{n}{\theta^2}$$

$$I_\theta = n/\theta^2$$

$$Var(T^*) \geq \frac{(-1/\theta^2)^2}{(n/\theta^2)}$$

$$Var(T^*) \geq \frac{1}{n\theta^2}$$

$$Var(\bar{X}) = \frac{1}{n} Var(X)$$

$$= \frac{1}{n}\frac{1}{\theta^2}$$

$$Var(\bar{X}) = \frac{1}{n\theta^2}$$

So it attains CRLB.

$V(\bar{x}) \geq V(T^*)$

exponentiated exponen...

$\theta^n e^{-\sum x} \theta (1-e^{-x})^{\theta}$

(c) $f_X(x;\theta) = \theta e^{-x}(1-e^{-x})^{\theta-1}, \quad x>0,\ \theta>0$

$L(\theta;\underline{x}) = \theta^n e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (1-e^{-x_i})^{\theta-1}$

$l = \ln L(\theta;\underline{x}) = n\ln\theta - \sum x_i \ln e + (\theta-1)\sum_{i=1}^{n} \ln(1-e^{-x_i})$

$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \ln(1-e^{-x_i})$

$\frac{\partial l}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} \log(1-e^{-x})$

$\frac{\partial l}{\partial\theta} = -n\left[-\frac{\sum}{n}\log(1-e^{-x}) - \frac{1}{\theta}\right]$

let $y = -\log(1-e^{-x})$ — $y = e^{-x}$; when $x=0$, $y=-\log(0)$

$-y = \log(1-e^{-x})$ — $\ln y$; when $x=\infty$, $y=-\log(1)$

$e^{-y} = 1-e^{-x}$

$e^{-x} = 1-e^{-y}$

$-x = \log(1-e^{-y})$

$x = -\log(1-e^{-y})$

$\frac{\partial x}{\partial y} = \frac{-1}{(1-e^{-y})} (+)e^{-y}$

$\frac{\partial x}{\partial y} = -\frac{e^{-y}}{1-e^{-y}}, \quad \left|\frac{\partial x}{\partial y}\right| = \left|\frac{-e^{-y}}{1-e^{-y}}\right|$

$g(y;\theta) = \theta(1-e^{-y})(e^{-y})^{\theta-1} \cdot \frac{e^{-y}}{1-e^{-y}}$

$g(y;\theta) = \theta e^{-y\theta}$

$T^* = -\frac{\sum}{n}\log(1-e^{-x})$

$E(T^*) = \frac{\sum}{n} E[-\log(1-e^{-x})]$

$= \frac{\sum}{n} E(y)$

$E(T^*) = 1/\theta$

Method 2

$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda}(\bar{x} - \lambda)$ comparing with,

$\frac{\partial l}{\partial \theta} = I_\theta (\hat{\theta} - \theta)$ $\therefore E(\bar{x}) = \lambda$

$V(\bar{X}) = \lambda / n = V(\hat{\theta})$

- So $\bar{x}$ attains the lower bound.

=> In exam " Find such estimator which verify CRLB and verify results"

Q) Question:-

For each of the following distributions. Let $X_1, \ldots, X_n$ be a r. sample, is there a function of θ, say $g(\theta)$? For which there exists an unbiased estimator, whose variance attain the CRLB? If so, find it; if not then show why not?

Power Dist^n

(a) $f_X(x;\theta) = \theta x^{\theta-1}, \quad 0 < x < 1, \quad \theta > 0$

$L(\theta; \underline{x}) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}$

$\ln L(\theta; \underline{x}) = n \ln \theta + (\theta - 1) \sum_{i=1}^{n} \log_e x_i$

$\frac{\partial}{\partial \theta} \ln L(\cdot) = \frac{n}{\theta} + \sum_{i=1}^{n} \log_e x_i$

$= -n \left[-\frac{1}{n} \sum_{i=1}^{n} \log_e x_i - \frac{1}{\theta} \right]$

$\hat{\theta}$

$\therefore$ Comparing with

$\frac{\partial}{\partial \theta} \ln L(\cdot) = \underbrace{k(\theta, x)}_{I_\theta} \left(t(x_1, \ldots x_n) - \tau(\theta) \right)$

$\left(-1/n \sum_{i=1}^{n} \log_e x_i\right)$ is an unbiased estimator of $1/\theta$, so we can verify CRLB

Results verification:

$X \sim Power(\theta)$

Let $Y = -\log_e X \sim \theta e^{-x}$

$\log_e x = -y \Rightarrow x = e^{-y}$

$$V(\bar{x}) = \frac{1}{n}\left[\frac{\theta}{\theta-1} - \frac{2\theta}{(\theta-1)\ln\theta} + \frac{2}{(\ln\theta)^2} - \frac{\theta^2}{(\theta-1)^2} - \frac{1}{(\ln\theta)^2}\right]$$

$$= \frac{1}{n}\left[\frac{\theta}{\theta-1} - \frac{\theta^2}{(\theta-1)^2} + \frac{1}{(\ln\theta)^2}\right]$$

$$= \frac{1}{n}\left[\frac{\theta(\theta-1)-\theta^2}{(\theta-1)^2} + \frac{1}{(\ln\theta)^2}\right]$$

$$= \frac{1}{n}\left[\frac{\cancel{\theta^2}-\theta-\cancel{\theta^2}}{(\theta-1)^2} + \frac{1}{(\ln\theta)^2}\right]$$

$$V(\bar{x}) = \frac{1}{n}\left[\frac{-\theta}{(\theta-1)^2} + \frac{1}{(\ln\theta)^2}\right]$$

Now we can verify this variance by CRLB.

As $\tau(\theta) = \frac{\theta}{\theta-1} - \frac{1}{\ln\theta}$

$$\tau'(\theta) = \frac{d}{d\theta}\,\theta(\theta-1)^{-1} - \frac{d}{d\theta}(\ln\theta)^{-1}$$

$$= (\theta)(-1)(\theta-1)^{-2} + (\theta-1)^{-1} - (-1)(\ln\theta)^{-2}\cdot\frac{1}{\theta}$$

$$= \frac{-\theta}{(\theta-1)^2} + \frac{1}{(\theta-1)} + \frac{1}{\theta(\ln\theta)^2}$$

$$= \frac{-\theta+(\theta-1)}{(\theta-1)^2} + \frac{1}{\theta(\ln\theta)^2}$$

$$\tau'(\theta) = -\frac{1}{(\theta-1)^2} + \frac{1}{\theta(\ln\theta)^2}$$

As

$$\frac{\partial l}{\partial\theta} = \frac{n}{\theta}\bar{x} + \frac{n}{(\theta\ln\theta)} - \frac{n}{\theta-1}$$

$$\frac{\partial^2 l}{\partial\theta^2} = \frac{-n\bar{x}}{\theta^2} + n\frac{\partial}{\partial\theta}(\theta\ln\theta)^{-1} + \frac{n}{(\theta-1)^2}$$

$$\frac{\partial^2 l}{\partial\theta^2} = -\frac{n\bar{x}}{\theta^2} - \frac{n(1+\ln\theta)}{(\theta\ln\theta)^2} + \frac{n}{(\theta-1)^2}$$

Mean $= 1/\theta$; $V = 1/\theta^2$ S.I

$$V(T^*) \geq \frac{(-1/\theta^2)^2}{n/\theta^2}$$

$$V(T^*) \geq \frac{1/\theta^4}{n/\theta^2} = \frac{1}{n\theta^2}$$

$V(\bar{x}) = V(T)$ should be ✓

How $Var(\bar{x})$ is equal to CRLB?

$$V(\bar{x}) = \frac{V(X)}{n} = \frac{1/\theta^2}{n} = \frac{1}{n\theta^2}$$

$\because X \sim$ exponent.

$V\left(\frac{\sum x}{n}\right) = \frac{\sum V(x)}{n^2} = \frac{n}{n^2}$

2) Example :-

Let $X_1, X_2, \dots X_n$ be a random sample from Poisson dist with λ. Find the CRLB for the variance of unbiased estimator of λ.

Solution :-

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x > 0, \lambda > 0$$

$$L(\lambda;\underline{x}) = \frac{e^{-n\lambda}\lambda^{\sum_i x_i}}{\prod x_i!}$$

$$l = \log_e L = -n\lambda + \sum_i^n x_i \log_e \lambda - \sum_{i=1}^n \log x_i!$$

$$\frac{\partial l}{\partial \lambda} = -n + \frac{\sum_{i=1}^n x_i}{\lambda}$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda}(\bar{x} - \lambda) = K(\underset{\sim}{x};\theta)(\hat{\theta} - \tau(\theta))$$

$$\frac{\partial l^2}{\partial \lambda^2} = -\sum_{i=1}^n \frac{x_i}{\lambda^2}$$

Applying expectation on both sides, we get

$$E\left(\frac{\partial l^2}{\partial \lambda^2}\right) = -\frac{1}{\lambda^2} E\left(\sum_{i=1}^n x_i\right)$$

$$I_\theta = -E\left(\frac{\partial l^2}{\partial \lambda^2}\right) = \frac{1}{\lambda^2}\sum_{i=1}^n E(x_i) = \frac{n}{\lambda^2}\cdot\lambda = \frac{n}{\lambda} = I_\theta$$

$$V(T) \geq I_\theta^{-1}$$

$$V(T) \geq \frac{1}{I_\theta} = \frac{1}{n/\lambda} = \frac{\lambda}{n}$$

$$V(T) \geq \frac{\lambda}{n}$$

As $\bar{X}$ attains the lower bound i.e

Example :

let $X_1, \ldots, X_n$ be random samples from $f_X(x;\theta) = \theta e^{-\theta x} I_{(0,\infty)}(x)$. Take $T(\theta) = \theta$ → ?
Find the CRLB for the variance of unbiased estimator of θ ie $Var(T) \geq I_\theta^{-1}$

Solution :

$$L(\theta; \underline{x}) = \theta^n e^{-\theta \sum_i^n x_i}$$

$$l = \log_e L(\theta; \underline{x}) = n \log_e \theta - \theta \sum_{i=1}^n x_i$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i = -n\left(\bar{x} - \frac{1}{\theta}\right)$$

$$-\left(\sum x - \frac{n}{\theta}\right) = -n\left(\bar{x} - \frac{1}{\theta}\right)$$

$$\frac{\partial^2 l}{\partial \theta^2} = -\frac{n}{\theta^2}$$

Applying expectation on both sides

$$-E\left(\frac{\partial^2 l}{\partial \theta^2}\right) = \frac{n}{\theta^2} = I_\theta$$

as

$$I_\theta = -E\left(\frac{\partial^2 l}{\partial \theta^2}\right)$$

CRLB

$\therefore V(T) \geq \frac{(\tau'(\theta))^2}{I_\theta}$

So $V(T) \geq I_\theta^{-1}$

$$V(T) \geq \frac{1}{I_\theta} = \frac{1}{n/\theta^2} = \theta^2/n$$

$$V(T) \geq \theta^2/n$$

$$\frac{\partial l}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n x_i = -n\left(\bar{x} - \frac{1}{\theta}\right)$$

⇒ Note that here θ is absent, so, $-n$ is not fisher information.

Here $\frac{\partial l}{\partial \theta}$ is a linear function of $\bar{x}$ and $\bar{x}$ is an unbiased estimator of $1/\theta$ and $V(\bar{x})$ is equal to CRLB

Method 1 → $\tau(\theta) = 1/\theta$, $\tau'(\theta) = -1/\theta^2$

$$V(T^*) \geq \frac{(\tau'(\theta))^2}{-E(\partial^2 l / \partial \theta^2)}$$

replace. The value of A, we get

$$E(\bar{x}) = \frac{\ln\theta}{\theta-1}\left[\frac{\theta}{\ln\theta} - \left(\frac{\theta}{(\ln\theta)^2} - \frac{1}{(\ln\theta)^2}\right)\right]$$

$$= \frac{\ln\theta}{\theta-1}\left[\frac{\theta}{\ln\theta} - \frac{(\theta-1)}{(\ln\theta)^2}\right]$$

$$E(\bar{x}) = \frac{\theta}{\theta-1} - \frac{1}{\ln\theta} \quad T(\theta) \checkmark \quad \text{verify}$$

$$E(x^2) = A\int_0^1 x^2\,\theta^x\,dx$$

$$= A\left[x^2\int_0^1 \theta^x dx - \int_0^1\left[\left(\int_0 \theta^x dx\right)\frac{d}{dx}x^2\right]dx\right]$$

$$= A\left[x^2.\frac{\theta^x}{\ln\theta}\Big|_0^1 - \int_0^1\left(\frac{\theta^x}{\ln\theta}.2x\right)dx\right]$$

$$= A\left[\frac{\theta}{\ln\theta} - \frac{2}{\ln\theta}\underbrace{\int_0^1 x\theta^x dx}_{\text{already derived}}\right]$$

$$= A\left[\frac{\theta}{\ln\theta} - \frac{2}{\ln\theta}\left(\frac{\theta}{\ln\theta} - \frac{(\theta-1)}{(\ln\theta)^2}\right)\right]$$

By replacing A, we get

$$E(x^2) = \frac{\cancel{\ln\theta}}{\theta-1}\left[\frac{\theta}{\ln\theta} - \frac{2\theta}{(\ln\theta)^2} + \frac{2(\theta-1)}{(\ln\theta)^3}\right]$$

$$= \frac{\theta}{\theta-1} - \frac{2\theta}{(\theta-1)\ln\theta} + \frac{2}{(\ln\theta)^2}$$

hence

$$V(\bar{x}) = V\left(\frac{\sum_{i=1}^n x_i}{n}\right) = \frac{1}{n^2}\sum_{i=1}^n V(x_i)$$

$$= \frac{1}{n}\left[E(X^2) - (E(X))^2\right]$$

$$= \frac{1}{n}\left[\frac{\theta}{\theta-1} - \frac{2\theta}{(\theta-1)\ln\theta} + \frac{2}{(\ln\theta)^2} - \left(\frac{\theta}{\theta-1} - \frac{1}{\ln\theta}\right)^2\right]$$

⇒ Rao Black well Theorem :-

We will continue our search for UMVUE. Now we show that how sufficiency aid in this search. Loosely speaking, an UB estimator which is a fun of sufficient statistic has smaller variance than an UB estimator which is not based on sufficient statistics.

Let $f(\cdot;\theta)$ be the density from which we can sample and suppose that we want to estimate $\tau(\theta)$ and that $S = S(X_1, \ldots, X_n)$ is a sufficient statistic. It can be shown that another UB estimator, denoted by T' can be derived from T such that

(i) $T' = E(T/S)$. i.e. T' is a function of sufficient statistic

(ii) T' is UB estimator of $\tau(\theta)$, then

$Var(T') \leq Var(T)$.

Therefore, in our search for UMVUEs we need to consider only UB estimators that are fn of sufficient statistic.

We shall formalize the idea in following theorem:

⇒ Rao Black well Theorem:-

We will continue our search for UMVUE. Now we show that how sufficiency aid in this search. Loosely speaking, an UB estimator which is a fun of sufficient statistic has smaller variance than an UB estimator which is not based on sufficient statistics.

Let $f(\cdot;\theta)$ be the density from which we can sample and suppose that we want to estimate $\tau(\theta)$ and that $S = s(X_1, \dots, X_n)$ is a sufficient statistic. It can be shown that another UB estimator, denoted by T' can be derived from T such that

(i) $T' = E(T/S)$. i.e. T' is a function of sufficient statistic

(ii) T' is UB estimator of $\tau(\theta)$, then
$Var(T') \leq Var(T)$.

Therefore, in our search for UMVUEs we need to consider only UB estimators that are fn of sufficient statistic.

We shall formalize the idea in following theorem:

Gossip!

Day/Date

Example Let $X_1, \dots, X_n$ be a random sample from $f(x;\theta) = \theta^x (1-\theta)^{1-x}$, $x = 0,1$,

Let we have two estimators $T_* = X_1$ and $T^* = \frac{\Sigma X}{n}$, are U/B estimators of θ,

Show that T^* is a UMVUE.

Sol

We know that

$E(X) = \theta$.

, so $E(X^2) = \Sigma x^2 \theta^x (1-\theta)^{1-x}$ $x=0,1$

$= 0^2 \cdot \theta^0 (1-\theta)^{1-0} + 1^2 \theta^1 (1-\theta)^{1-1}$

$= \theta$

$Var(T^*) = Var\left(\frac{\Sigma x}{n}\right) = \frac{1}{n^2} Var(\Sigma X_i) = \frac{1}{n^2} \Sigma Var(X_i)$

$= \frac{n}{n^2} Var(X_i)$

$Var(T^*) = \frac{1}{n}\left[E(X^2) - E(X)^2\right]$

$= \frac{1}{n}\left[\theta - \theta^2\right] = \frac{\theta - \theta^2}{n}$ — (1)

Now $Var(T) = E(X^2) - \{E(X)\}^2$

$= \theta - \theta^2 = \theta(1-\theta)$ — (2)

from (1) & (2), obviously,

$Var(T^*) < Var(T)$

Therefore two conditions of UMVUE, satisfied.

Gossip!

$E(X) = E_Y(E(X/Y))$

Day/Date

(ii) $E_\theta(T') = E_\theta(E(T/S))$

$= E_\theta(T) = \tau(\theta)$

(ii) $\mathrm{Var}(T) = E(T - E(T))^2$

$= E(T - \tau(\theta))^2$ $\quad E(T) = \tau(\theta)$

$\mathrm{Var}(T) = E(T - E(T^*))^2$ $\quad E(T^*) = \tau(\theta)$

$\mathrm{Var}(T) = E(T - T^* + T^* - E(T^*))^2$ $\quad T^*$ add & sub

$\mathrm{Var}(T) = E\left((T - T^*) + (T^* - E(T^*))\right)^2$

$= E(T - T^*)^2 + \mathrm{Var}(T^*) - 2E\left[(T - T^*)(T^* - E(T^*))\right]$ —— (A)

take cross product term

$E\left((T - T^*)(T^* - E(T^*))\right) = E\left(E\left((T - T^*)(T^* - E(T^*))/S\right)\right)$

$\because E(X) = E(E(X/Y))$

$= E(T^* - E(T^*))E\left(E\{(T - T^*)/S\}\right)$

$= E(T^* - E(T^*))\left[E(T/S) - E(T^*/S)\right]$

$= E(T^* - E(T^*))\left[T^* - T^*\right] = 0$

So from (A), we have

$\mathrm{Var}(T) = E(T - T^*)^2 + \mathrm{Var}(T^*)$

$\mathrm{Var}(T) \geq \mathrm{Var}(T^*)$ $\quad$, So

If $E(T - T^*) = 0$, then

$\mathrm{Var}(T) = \mathrm{Var}(T^*)$

Gossip!

So T' is UMVUE.

⇒ Lehmann-Scheffe' Theorem: There is only a single f^n for complete sufficient statistic, so it will be unique,

$E(CSS) = \tau(\theta)$

→ Rao Blackwell theorem did not talk about CSS but they only discuss minimal sufficient statistic.

What is Complete sufficient statistic:-

Let $X_1, \ldots, X_n$ denote a r.s from density $f(\cdot;\theta)$ with parameter space $\bar{\Theta}$ and let $T = t(X_1, \ldots, X_n)$ be a statistic. The family of densities of T is defined to be complete if and only if $E_\theta[z(T)] = 0$ for all $\theta \in \bar{\Theta}$ implies that

$P_\theta[z(T) = 0] = 1$ for all $\theta \in \bar{\Theta}$ where $z(T)$ is statistic.

Another way of stating that a statistic T is complete is the following:

T is complete if and only if U/B estimator of 0 that is a f^n of T is the statistic that is identically 0 with prob 1.

Day/Date

So T' is UMVUE.

⇒ Lehmann-Scheffé Theorem: There is only a single f^n for complete sufficient statistic, so it will be unique,

$E(CSS) = \tau(\theta)$

→ Rao Blackwell theorem did not talk about CSS but they only discuss minimal sufficient statistic.

What is complete sufficient statistic:-

Let $X_1, \ldots, X_n$ denote a r.s from density $f(\cdot;\theta)$ with parameter space Θ and let $T = t(X_1, \ldots, X_n)$ be a statistic. The family of densities of T is defined to be complete if and only if $E_\theta[z(T)] \equiv 0$ for all $\theta \in \Theta$ implies that

$P_\theta[z(T) = 0] = 1$ for all $\theta \in \Theta$ where $z(T)$ is statistic.

Another way of stating that a statistic T is complete is the following:

T is complete if and only if U/B estimator of 0 that is a f^n of T is the statistic that is identically 0 with prob 1.

Gossip!

Day/Date

Theorem Rao-Blackwell: Let $X_1, \dots, X_n$ be a random sample from the density $f(x; \theta)$ and let $S = s(X_1, \dots, X_n)$ is a SS (suffi stati)

~~Let~~

Let the statistics $T = t(X_1, \dots, X_n)$ is an UIB estimator of $\tau(\theta)$. Define another statistic T' which is derived from T such that

$$T' = E(T/S)$$

Then,

(i) T' is a statistic which is a f^n of SS.

(ii) T' is an UIB estimator of $\tau(\theta)$

(iii) $Var(T') \leq Var(T)$ for every θ and $Var(T') < Var(T)$ for some θ unless T is equal to T' with probability 1.

Proof: (i) To derive statistic, conditional dist^n is required,

If S is sufficient statistic, so the conditional dist^n ~~that~~ of any particular statistic 'T' given S is independent of θ, hence

$$T' = E(T|S)$$ is independent of θ

and so T' is a statistic which is obviously a f^n of S.

14/10/2020 Wednesday
10:00 to 11:00

Malam Hafiza Memona

Statistical Inference I

Lecture 2

Statistical Inference is the process through which inferences about a population are made based on certain statistics calculated from a sample of data drawn from population.

There are two kind of inferences:

1) Inductive
2) Deductive

Introduction of Statistical Inference:-

The process of drawing inferences about a population on the basis of information contained in a sample taken from population is called Statistical ~~of~~ Inference.

Statistical Inference is traditionally divided into two major areas:

1) Estimation of parameters
   (Statistical Inference I)
2) Testing of Hypothesis
   (Statistical Inference II)

30-03-2015

The Cramer Rao Inequality (And Lower Bound)

Suppose that $X_1, \ldots, X_n$ from a random sample from a dist with pdf $f_X(x;\theta)$. Subject to certain regularity conditions on $f_X(x;\theta)$, we have that for any unbiased estimator $\hat{\theta}$ of θ, we have that

$V(\hat{\theta}) \geq I_\theta^{-1}$ where

due to iid's

$$I_\theta = E\left[\frac{\partial \ln L(x;\theta)}{\partial\theta}\right]^2 = E\left[\frac{\partial l}{\partial\theta}\right]^2 = nE_\theta\left[\frac{\partial}{\partial\theta}f_X(x;\theta)\right]^2 = nI_{X_1}(\theta)$$

here $L(\theta;\underline{x})$ is the likelihood fn and $l = \ln L(\theta;\underline{x})$

I_θ^{-1} is known as the Cramer Rao Lower Bound (CRLB) and the corresponding inequality is CR inequality

Comments:-

(i) I_θ is sometimes known as (Fisher) information about θ in the observations. Clearly, the lower is the attainable variance of $\hat{\theta}$, the more "information", in an intitutive sense, we have about θ, and the larger is I_θ. The inequality itself is sometimes known as the information inequality.

(ii) We can interchange the order of differenciation and integration. In general, it is not always tr. One particular condition is that, the range of the values of X must not depend on θ. So, e.g the result can not be applied when θ is the parameter in the uniform dist, $U[0,\theta]$.

(iii) The proof which follows goes through for discrete r.v's if we replace integrals by summations, and the regularity conditions will be some what less restrictive in this case (It is easier to interchange summation at differenciation).

$$\frac{\partial g}{\partial \theta} = \int \hat{\theta}\, \frac{\partial l}{\partial \theta}\, L\, d\underline{x}$$

$$\frac{\partial g}{\partial \theta} = E\left(\hat{\theta}\, \frac{\partial l}{\partial \theta}\right) = E(uv)$$

let $u = \hat{\theta}$, $v = \dfrac{\partial l}{\partial \theta}$

$$E(v) = E\left(\frac{\partial l}{\partial \theta}\right) = \int \frac{\partial l}{\partial \theta}\, L\, d\underline{x} = \int \frac{\partial L}{\partial \theta}\, d\underline{x}$$

$$= \frac{\partial}{\partial \theta} \int L\, d\underline{x} = \frac{\partial}{\partial \theta}(1) = 0$$

$$\operatorname{Cov}(u,v) = E(uv) - E(u)E(v)$$

$$\operatorname{Cov}(u,v) = \frac{\partial g}{\partial \theta} - 0$$

Also $V(v) = E(v^2) - (E(v))^2$

$$= E(v^2) - 0$$

$$= E\left(\frac{\partial l}{\partial \theta}\right)^2 = \tilde{I}_\theta$$

$$V(u) = V(\hat{\theta}) \geq \frac{(\operatorname{Cov}(u,v))^2}{\operatorname{Var}(v)}$$

$$V(\hat{\theta}) \geq \frac{(\partial g/\partial \theta)^2}{\underset{\sim}{I}_\theta}$$

$$V(\hat{\theta}) \geq \left(\frac{\partial g}{\partial \theta}\right)^2 \underset{\sim}{I}_\theta^{-1}$$

hence proved