For verification of dist of $T = \sum_{i=1}^{n} x_i$, we can use m.g.f Technique.

Note that $X_i \sim$ Gamma(α, β) then

$E(X) = \alpha/\beta$, $V(X) = \frac{\alpha}{\beta^2}$ for Gamma I

$E(X) = \alpha\beta$, $V(X) = \alpha\beta^2$ for Gamma II i.e

$$f(x) = \frac{1}{\Gamma\alpha\, \beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

Now

$$E(e^{tT}) = E\left(e^{t\sum_{i=1}^{n} x_i}\right) = E\left(e^{tx_1 + tx_2 + \dots + tx_n}\right)$$

$$= E(e^{tx_1})\, E(e^{tx_2}) \dots E(e^{tx_n}) \quad \text{due to iid's}$$

$$= (1-\beta t)^{-\alpha} (1-\beta t)^{-\alpha} \dots (1-\beta t)^{-\alpha}$$

$$= \left((1-\beta t)^{-\alpha}\right)^n$$

$$= (1-\beta t)^{-n\alpha}$$

i.e $T = \sum_{i=1}^{n} x_i \sim$ Gamma$(n\alpha, \beta)$.

M-I: Fisher Information Through Sufficient Statistics:

$$g(t, n\alpha, \beta) = \frac{1}{\Gamma n\alpha} \beta^{n\alpha} t^{n\alpha-1} e^{-t\beta} \quad \text{i.e } E(t) = n\alpha/\beta, \; V(t) = n\alpha/\beta^2$$

$$\log_e g(t, n\alpha, \beta) = n\alpha \log_e \beta + (n\alpha - 1)\log_e t - t\beta \log_e e - \log_e \Gamma n\alpha$$

$$\frac{\partial}{\partial\beta} \log_e g(t; n\alpha, \beta) = \frac{n\alpha}{\beta} - t$$

$$E\left(\frac{\partial}{\partial\beta} \log_e g(t; n\alpha, \beta)\right)^2 = E\left(\frac{n\alpha}{\beta} - t\right)^2$$

$$= E\left[t - \frac{n\alpha}{\beta}\right]^2 = E[t - E(t)]^2 = V(t)$$

$$I_T(\theta) = \frac{n\alpha}{\beta^2}$$

M-II: Fisher Information Through likelihood fn:-

$$f(x) = \frac{1}{\Gamma\alpha} \beta^\alpha x^{\alpha-1} e^{-x\beta}, \quad x > 0$$

$$2\frac{\partial}{\partial\theta}\log_e h(\underline{x}/T=t;\theta)\, E\left[\frac{\partial}{\partial\theta}\log_e g(t;\theta)\right] =$$

$$2\frac{\partial}{\partial\theta}\log_e h(\underline{x}/T=t;\theta)\int\frac{\partial}{\partial\theta}\log_e g(t;\theta)\, g(t,\theta)\,dt$$

$$'' \quad \int\frac{1}{g(t;\theta)}\frac{\partial}{\partial\theta} g(t;\theta)\, g(t;\theta)\,dt$$

$$'' \quad \frac{\partial}{\partial\theta}\int g(t;\theta)\,dt \longrightarrow \text{Total area under the curve is 1}$$

$$2\frac{\partial}{\partial\theta}\log_e h(\underline{x}/T=t;\theta)\,\frac{\partial}{\partial\theta}(1) \qquad \because \frac{\partial}{\partial\theta}(1)=0$$

$$= \text{zero.}$$

So equ (1) can be written as

$$E\left[\frac{\partial}{\partial\theta}\log_e L(\theta;\underline{x})\right]^2 = E\left[\frac{\partial}{\partial\theta}\log_e g(t;\theta)\right]^2 + E\left[\frac{\partial}{\partial\theta}\log_e h(\underline{x}/T=t;\theta)\right]^2.$$

$$\underset{\sim}{I}_{\underline{x}}(\theta) = \underset{\sim}{I}_T(\theta) + \underset{\sim}{I}_{\underline{x}/T}(\theta) \quad \text{for all } \theta\in\Phi$$

This implies that $I_{\underline{x}}(\theta) \geq I_T(\theta)$ for all $\theta\in\Phi$

Hence $\underset{\sim}{I}_{\underline{x}}(\theta) = \underset{\sim}{I}_T(\theta)$ iff $\underset{\sim}{I}_{x/T}(\theta)=0$,

that is $h(\underline{x}/T=t;\theta)$ must be free from θ. It then follows from the definition of sufficiency that T is sufficient for θ.

Example:.

1^{st} Method:. As we have shown for Poisson(λ),

$T = \sum_{i=1}^{n} X_i$ is sufficient for λ, then

$T = \sum_{i=1}^{n} X_i \sim$ Poisson$(n\lambda)$ and

$$g_T(\cdot;\lambda) = \frac{e^{-n\lambda}(n\lambda)^t}{t!}, \quad t = 0,1,\ldots\infty, \; \lambda > 0$$

$$\log g_T(\cdot;\lambda) = -n\lambda + t\log(n\lambda) - \log t!$$

$$\frac{\partial}{\partial\lambda}\log g_T(t,\lambda) = -n + \frac{t\,(\cancel{n})}{\cancel{n}\lambda}$$

$$E\left(\frac{\partial}{\partial\lambda}\log g_T(t,\lambda)\right)^2 = E\left(\frac{t - n\lambda}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}E(t - n\lambda)^2 \Rightarrow \frac{1}{\lambda^2}Var(t) = \frac{n\lambda}{\lambda^2}$$

$$= n/\lambda$$

2^{nd} Method:.

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,\ldots\infty$$

$$L(\lambda;\underline{x}) = \frac{e^{-n\lambda}\lambda^{\sum x_i}}{\prod_{i=1}^{n} x_i!}$$

$$\ln L(\lambda;\underline{x}) = -n\lambda + \sum_{i=1}^{n} x_i \ln\lambda - \sum_{i=1}^{n}\ln x_i!$$

$$\frac{\partial}{\partial\lambda}\ln L(\lambda;\underline{x}) = -n + \frac{\sum_{i=1}^{n} x_i}{\lambda}$$

$$\left(\frac{\partial}{\partial\lambda}\ln L(\lambda;\underline{x})\right)^2 = \left(-n + \frac{n\bar{x}}{\lambda}\right)^2 = \frac{n^2(\bar{x}-\lambda)^2}{\lambda^2}$$

$$E\left(\frac{\partial}{\partial\lambda}\ln L(\lambda;\underline{x})\right)^2 = \frac{n^2}{\lambda^2}E(\bar{x}-\lambda)^2 = \frac{n^{\cancel{2}}\cancel{\lambda}}{\cancel{n}\lambda^{\cancel{4}}} = \frac{n}{\lambda}.$$

$$\frac{\lambda}{n} = E(\bar{x}-\lambda)^2$$

OR

$$\left(\frac{\partial}{\partial\lambda}\ln L(\lambda;\underline{x})\right)^2 = \left(-n + \frac{\sum_{i=1}^{n} x_i}{\lambda}\right)^2$$

$$= \left(\frac{-n\lambda + \sum_{i=1}^{n} x_i}{\lambda}\right)^2$$

$$= \frac{\left(\sum_{i=1}^{n}(x_i - \lambda)\right)^2}{\lambda^2}$$

$$\left(\sum_{i}^{n} a_i\right)^2 = \sum_{i=1}^{n} a_i^2 + \sum\sum_{i \neq j} a_i a_j$$

Applying expectation on both sides,

$$E\left[\frac{\partial}{\partial\theta}\log_e L(\theta;\underline{x})\right]^2 = n\,E\left[\frac{\partial}{\partial\theta}\ln f(x_i;\theta)\right]^2$$

$$I_{\underline{X}}(\theta) = n\, I_{X}(\theta) \quad \text{ok} \quad X = X_1$$

for all $\theta \in \Phi$

Theorem:

Suppose $\underline{X}$ is the data and $T = T(\underline{X})$ is a statistic. Then $\boxed{I_{\underline{X}}(\theta) \geq I_T(\theta)}$ for all $\theta \in \Phi$

The equality holds for all θ iff T is sufficient for θ.

Proof:

The likelihood function can be written as,

$$L(\theta;\underline{x}) = g(t;\theta)\, h(\underline{x}\,|\,T=t;\theta)$$

Taking $\log_e$ on both sides,

$$\log_e L(\theta;\underline{x}) = \log_e g(t;\theta) + \log_e h(\underline{x}\,|\,T=t;\theta)$$

Differentiating w.r.t θ, we get

$$\frac{\partial}{\partial\theta}\log_e L(\theta;\underline{x}) = \frac{\partial}{\partial\theta}\log_e g(t;\theta) + \frac{\partial}{\partial\theta}\log_e h(\underline{x}\,|\,T=t;\theta)$$

Taking square and apply expectation,

$$E\left(\frac{\partial}{\partial\theta}\log_e L(\theta;\underline{x})\right)^2 = E\left[\frac{\partial}{\partial\theta}\log_e g(t;\theta)\right]^2 + E\left[\frac{\partial}{\partial\theta}\log_e h(\underline{x}\,|\,T=t;\theta)\right]^2 + 2\,\frac{\partial}{\partial\theta}\log_e h(\underline{x}\,|\,T=t;\theta)\; E\left[\frac{\partial}{\partial\theta}\log_e g(t;\theta)\right] \rightarrow (1)$$

Consider the last term of equ (1) as

$$L(\theta;\underline{x}) = \frac{H(x_{(1)} - x_{(n)} + 2\theta)}{(2\theta)^n} \cdot H(x_{(n)} - x_{(1)})$$

$$\qquad k_1(x_{(1)}, x_{(n)};\theta) \qquad k_2(\underline{x})$$

Hence by factorization theorem, $x_{(1)}$ and $x_{(n)}$ are jointly sufficient for θ.

Method 3rd :

$\Rightarrow \dfrac{L(\theta;\underline{x})}{g(t)}$ is independent of θ

$L(\theta;\underline{x}) \to$ likelihood fn

$g(t) \to$ dist of sufficient stats

then we have minimal sufficient statistics.

Here $x_{(1)}$ and $x_{(n)}$ are sufficient. Find their joint distribution.

The joint distribution of $x_{(i)}$ and $x_{(j)}$ is given by;

$$g(x_{(i)}, x_{(j)}) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} [1 - F(x_j)]^{n-j} f(x_i) f(x_j)$$

$$-\theta < x_{(i)} < x_{(j)} < \theta$$

$n-1=1 \qquad n-2$

Put $\boxed{i=1}$, $\boxed{j=n}$

$$g(x_{(1)}, x_{(n)}) = \frac{n!}{(n-2)!} [F(x_{(n)}) - F(x_{(1)})]^{n-2} f(x_{(1)}) f(x_{(n)})$$

$$= n(n-1) [F(x_{(n)}) - F(x_{(1)})]^{n-2} f(x_{(1)}) f(x_{(n)})$$

$$\text{As } F(x) = \int_{-\theta}^{x} f(x)\,dx = \int_{-\theta}^{x} \frac{1}{2\theta}\,dx = \frac{1}{2\theta}\, x \Big|_{-\theta}^{x} = \frac{x+\theta}{2\theta}$$

$$g(x_{(1)}, x_{(n)}) = n(n-1) \left[\frac{x_{(n)}+\theta}{2\theta} - \frac{x_{(1)}+\theta}{2\theta}\right]^{n-2} \left(\frac{1}{2\theta}\right)\left(\frac{1}{2\theta}\right)$$

$$= n(n-1) \frac{[x_{(n)} - x_{(1)}]^{n-2}}{(2\theta)^{n-2+2}}$$

$$= n(n-1) \frac{(x_{(n)} - x_{(1)})^{n-2}}{(2\theta)^n}$$

From definition of sufficiency, we have

$$\frac{L(\theta;\underline{x})}{g(x_{(1)}, x_{(n)};\theta)} = \frac{(1/2\theta)^n}{n(n-1)\dfrac{(x_{(n)} - x_{(1)})^{n-2}}{(2\theta)^n}} = \frac{1}{n(n-1)(x_{(n)} - x_{(1)})^{n-2}}$$

$T = \sum_{i=1} x_i$ is sufficient stat. $\quad E(x) = \alpha/\beta$

$E(T) = \sum_i E(x_i) = n \cdot \frac{1}{\theta} = \frac{n}{\theta}$

$V(T) = \sum_i V(x_i) = n/\theta^2 \quad V(x) = \alpha/\beta^2$

Now we find the p.d.f of T i.e. by m.g.f

$E(e^{tT}) = E(e^{t(\sum x_i)}) = E(e^{t(x_1 + tx_2 + \dots + tx_n)})$

$= E(e^{tx_1}) E(e^{tx_2}) \dots E(e^{tx_n})$, iid's

As
$E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx = \int_0^\infty e^{tx} e^{-\theta x}\theta\,dx$

$= \theta \int_0^\infty e^{-(\theta - t)x}\,dx \quad e^{-(\theta-t)x}$

$= \theta \left. \frac{e^{-(\theta-t)x}}{t-\theta} \right|_0^\infty$

$= \frac{\theta}{t-\theta}(e^{-\infty} - e^{-0}) = \frac{\theta}{t-\theta}\left[\frac{1}{\infty} - 1\right] = \frac{-\theta}{t-\theta}$

$= \frac{\theta}{\theta - t}$

$E(e^{tT}) = \frac{\theta}{\theta-t} \cdot \frac{\theta}{\theta-t} \cdots \frac{\theta}{\theta-t}$

$= \left(\frac{\theta}{\theta-t}\right)^{n} \quad \text{Gamma} = \left(\frac{\beta}{\beta-t}\right)^\alpha$

Method I:

$T = \sum_i x_i \sim \text{Gamma}(n, \theta) \quad \left(1 - \frac{\beta}{t}\right)^\alpha$

$g(t, \theta) = \frac{1}{\Gamma n} \theta^n t^{n-1} e^{-t\theta} \quad \left(1 - \frac{t}{\beta}\right)$

$\ln g(\cdot) = n\log_e \theta + (n-1)\log_e t - t\theta \log_e - \log_e \Gamma n$

$\frac{\partial}{\partial\theta} \ln g(\cdot) = \frac{n}{\theta} - t$

$\left(\frac{\partial^2}{\partial\theta} \ln g(\cdot)\right)^2 = \left(\frac{n}{\theta} - t\right)^2$

$E\left(\frac{\partial}{\partial\theta}\ln g(\cdot)\right)^2 = E\left(t - \frac{n}{\theta}\right)^2 \quad \therefore E(t) = \alpha/\beta = n/\theta$

$V(t) = \alpha/\beta^2$

$= V(t) = n$

$$L(\alpha,\beta,\underline{x}) = \frac{1}{(\Gamma\alpha)^n}\beta^{n\alpha}\prod_{i=1}^{n} x_i^{\alpha-1} e^{-\beta\sum_{i=1}^{n} x_i}$$

$$\log_e L(\cdot) = n\alpha\log_e\beta + (\alpha-1)\sum_{i=1}^{n}\log_e x_i - \beta\sum_{i=1}^{n} x_i - n\ldots$$

$$\frac{\partial}{\partial\beta}\log_e L(\cdot) = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i$$

$$E\left(\frac{\partial}{\partial\beta}\log_e L(\cdot)\right)^2 = E\left[\frac{n\alpha}{\beta} - \sum_{i=1}^{n} x_i\right]^2$$

$$= E\left[\sum_{i=1}^{n} x_i - \frac{n\alpha}{\beta}\right]^2$$

$$= E\left[\sum_{i=1}^{n}\left(x_i - \alpha/\beta\right)\right]^2 \quad \because \left(\sum_i a_i\right)^2 = \sum_i a_i^2 + \sum\sum_{i\neq j}\ldots$$

$$= E\left[\sum_{i=1}^{n}\left(x_i - \frac{\alpha}{\beta}\right)^2 + \sum\sum_{i\neq j}\left(x_i - \alpha/\beta\right)\left(x_j - \alpha/\beta\right)\right]$$

$$= \sum_{i=1}^{n} E\left[x_i - \alpha/\beta\right]^2 + \text{zero (iid rs)}$$

$$= \sum_{i=1}^{n} V(x_i) = n\frac{\alpha}{\beta^2} = I_{\underline{x}}(\theta)$$

Hence

$$I_{T}(\theta) = I_{\underline{x}}(\theta) = \frac{n\alpha}{\beta^2}$$

(ii) Normal Distribution: (μ unknown & σ^2 known)

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$L(\cdot;\underline{x}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2+\mu^2-2x_i\mu)}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2)}}_{k(\underline{x})} \; \underbrace{e^{-\frac{1}{2\sigma^2}(n\mu^2-2\mu\sum x_i)}}_{k_1\left(\sum_{i=1}^{n} x_i,\,\mu\right)}$$

So $T = \sum_{i=1}^{n} x_i$ is sufficient statistics.

$E(T) = n\mu$, $V(T) = n\sigma^2$

$x_{(n-1)}$

$$\therefore E\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\right]^2 + E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right] = 0$$

$$E\left[\frac{\partial}{\partial\theta}\log_e f(x;\theta)\right]^2 = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log_e f(x;\theta)\right]$$

$$I_{\underset{\sim}{x}}(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log_e f(x;\theta)\right]$$

Example:.

Let $X_1, X_2, \ldots X_n$ be iid uniform $[-\theta, +\theta]$ where $0<\theta<\infty$ is an unknown parameter. Derive a minimal sufficient for θ

Solution:.

M-1:. $f(x;\theta) = \dfrac{1}{\theta-(-\theta)} = \dfrac{1}{\theta+\theta} = \dfrac{1}{2\theta}$, $-\theta<x<\theta$, $\theta>0$.

The likelihood fn is

$L(\theta;\underline{x}) = \dfrac{1}{(2\theta)^n}$, $-\theta < x_{(1)} < x_{(2)} < \ldots < x_{(n)} < \theta$

$= 0$, elsewhere

$-\theta < x_{(1)}$ $x_{(n)} < \theta$

$-\theta + x_{(n)} < x_{(1)} + \theta$

$\Rightarrow$ [$f(x;\theta) = \dfrac{1}{\theta}$, $0<x_i<\theta$

$0 \le x_{(1)} + \theta + \theta - x_{(n)}$

$0 < x_{(1)} - x_{(n)} + 2\theta$

The closest value of θ is sufficient like: $x_{(1)}$ and $x_{(n)}$ only "]

$x_{(1)} - x_{(n)} + 2\theta \ge 0$

if $0<x<\theta$ Then x_n is sufficient

$\left.\begin{matrix} -\theta < x_{(1)} \\ x_{(n)} < \theta \end{matrix}\right\}$ Adding

$x_{(n)} - \theta < x_{(1)} + \theta$

$0 < x_{(1)} + \theta + \theta - x_{(n)}$

$x_{(1)} - x_{(n)} + 2\theta > 0$ $- x_{(1)} \ge 0$

$H(z) = \begin{cases} 1 & \text{if } x_{(1)} - x_{(n)} + 2\theta \ge 0 \\ 0 & \text{if } x_{(1)} - x_{(n)} + 2\theta < 0 \end{cases}$ $H(z) = 1$ if $x_{(1)}$

or $H(z) = 1$, $x_n - x_1 \ge$; $= 0$, $x_n - x_1 <$

M-2:.

$x_{(n)} - x_{(1)} > 0$. [always greater than zero in order

Multiplying likelihood fn by $H(z)$ if $[-\theta < x < \theta$, then joint of ... last.

$$\left(\frac{\partial}{\partial \lambda} \ln L(\lambda; \underline{x})\right)^2 = \frac{1}{\lambda^2}\left[\sum_{i=1}^{n}(x_i-\lambda)^2 + \sum\sum_{i\neq j}(x_i-\lambda)(x_j-\lambda)\right]$$

Applying expectation on both sides,

$$E\left(\frac{\partial}{\partial \lambda} \ln L(\lambda; \underline{x})\right)^2 = \frac{1}{\lambda^2}\sum_{i=1}^{n} E(x_i-\lambda)^2 + \text{zero (iid)}$$

$$= \frac{1}{\lambda^2}\sum_{i=1}^{n} \lambda \qquad \because X_i \sim \text{Poisson}(\lambda)$$

$$= \frac{n\lambda}{\lambda^2} = \frac{n}{\lambda}$$

Fisher Information is same i.e

$$I_{\underline{X}}(\theta) = I_T(\theta)$$

likelihood fn $\leftarrow \underline{X}$; $T \rightarrow$ sufficient stats

$$n/\lambda = \frac{n}{\lambda}$$

Quesion:

Find Fisher Information for Gamma, Beta, normal, exponential and uniform dist^{n}s.

Solution:

Gamma → $f(x;\alpha,\beta) = \frac{1}{\Gamma\alpha}\beta^\alpha x^{\alpha-1} e^{-x\beta}$, $x>0$

First we find the sufficient statistics in Gamma dist, where $\underline{\alpha \text{ is known}}$ and $\underline{\beta \text{ is unknown}}$

$$L(\alpha,\beta,\underline{x}) = \frac{1}{\Gamma\alpha^n}(\beta^\alpha)^n \prod_{i=1}^{n} x_i^{\alpha-1} e^{-\beta\sum_{i=1}^{n} x_i}$$

$$= \underbrace{\frac{1}{\Gamma\alpha^n}(\beta^\alpha)^n e^{-\beta\sum_{i=1}^{n} x_i}}_{k_1(\sum_{i=1}^{n} x_i,\ \beta)} \cdot \underbrace{\prod_{i=1}^{n} x_i^{\alpha-1}}_{k_2(\underline{x})}$$

Since $\sum_{i=1}^{n} x_i$ is a sufficient statistics for β.

Let $T = \sum_{i=1}^{n} x_i \sim \text{Gamma}(n\alpha, \beta)$

M-1 :.

$$g(t;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-\frac{1}{2n\sigma^2}(t-n\mu)^2}$$

$$\ln g(t;\mu,\sigma^2) = -\frac{1}{2n\sigma^2}(t-n\mu)^2 - \ln(2\pi n\sigma^2)^{+\frac{1}{2}}$$

$$\frac{\partial \ln g(\cdot)}{\partial \mu} = -\frac{\cancel{2}}{\cancel{2}\cancel{n}\sigma^2}(t-n\mu)(-\cancel{n}) = \left(\frac{t-n\mu}{\sigma^2}\right)$$

$$E\left(\frac{\partial \ln g(\cdot)}{\partial \mu}\right)^2 = E\left[\frac{t-n\mu}{\sigma^2 \cancel{n}^2}\right]^2$$

$$= \frac{1}{\sigma^4} E(t-n\mu)^2 = \frac{1}{\sigma^4} V(t) = \frac{n\sigma^{\cancel{4}}}{\sigma^{\cancel{4}}} = \frac{n}{\sigma^2}.$$

M-2 :.

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$L(\cdot) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}$$

$$\ln L(\cdot) = -\frac{1}{2}\sum_{i}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 - \frac{n}{2}\ln 2\pi\sigma^2$$

$$\frac{\partial}{\partial \mu}\ln L(\cdot) = -\frac{\cancel{2}}{\cancel{2}\sigma^2}\sum_{i}^{n}(x_i-\mu)(-1) - 0$$

$$\left(\frac{\partial \ln L(\cdot)}{\partial \mu}\right)^2 = \left(\frac{1}{\sigma^2}\sum_{i}(x_i-\mu)\right)^2 = \frac{1}{\sigma^4}\left(\sum_{i=1}^{n}(x_i-\mu)\right)^2$$

$$= \frac{1}{\sigma^4}\left[\sum_{i=1}^{n}(x_i-\mu)^2 + \sum_{i\neq j}\sum(x_i-\mu)(x_j-\mu)\right]$$

$$E\left[\frac{\partial l}{\partial \mu}\right]^2 = \frac{1}{\sigma^4}\left[\sum_{i}^{n} E(x_i-\mu)^2\right] + \frac{1}{\sigma^4}\sum_{i\neq j}\sum \underbrace{E(x_i-\mu)(x_j-\mu)}_{\text{iid r.s. (zero)}}$$

$$= \frac{1}{\sigma^4}\sum_{i}^{n} V(x_i) = \frac{n\sigma^{\cancel{4}}}{\sigma^{\cancel{4}}} = \frac{n}{\sigma^2}.$$

(iii) Exponential dist :.

$$f_X(x;\theta) = \theta e^{-x\theta}.$$

$$L(\cdot) = \theta^n e^{-\theta\sum_{i}^{n} x_i} \cdot 1$$

$$\downarrow k_1\left(\sum_{i}^{n} x_i, \theta\right) \quad \downarrow k_2(\underline{x})$$

Method II :-

$$f_X(x;\theta) = \theta e^{-x\theta}$$
$$L(\cdot) = \theta^n e^{-\theta\sum_i^n x_i}$$
$$\ln L(\cdot) = n\log_e\theta - \theta\sum_i^n x_i$$
$$\frac{\partial}{\partial\theta}\ln L(\cdot) = n/\theta - \sum_i^n x_i \Rightarrow \left(\frac{\partial}{\partial\theta}\ln L(\cdot)\right)^2 = \left(\frac{n}{\theta} - \sum x_i\right)^2$$
$$E\left[\frac{\partial l}{\partial\theta}\right]^2 = E\left[\sum x_i - n/\theta\right]^2 \Rightarrow E\left[n\bar{x} - n/\theta\right]^2$$
$$= n^2 E\left[\bar{x} - \cancel{n}/\theta\right]^2 \qquad \because E(x) = E(\bar{x}) = 1/\theta$$
$$= n^2 V(\bar{x}) \qquad V(\bar{x}) = \frac{1}{n^2}\sum_i^n V(x) =$$
$$= n^{\cancel{2}}\cdot\frac{1}{\cancel{n}\theta^2} = \frac{n}{\theta^2} = I_x(\theta) \qquad = 1/n\cdot\frac{1}{\theta^2}$$

<u>Note</u> Fisher Information can be applied only when regularity condition (i.e
$$E\left(\frac{\partial}{\partial\theta}\ln L(\cdot)\right)^2 = -E\left[\frac{\partial^2\ln L(\cdot)}{\partial\theta^2}\right]$$
) holds.

e.g in case of uniform, $U[-\theta,\theta]$
$$f(x) = \frac{1}{2\theta}$$
$$L(x) = \frac{1}{(2\theta)^n} \Rightarrow l = \ln L(\cdot) = -n\ln(2\theta)$$
$$\frac{\partial l}{\partial\theta} = \frac{\partial}{\partial\theta}\left[-n\ln 2 - n\ln\theta\right] = -\frac{n}{\theta} \quad = E\left(\frac{\partial}{\partial\theta}\ln L\right)^2$$
$$\left(\frac{\partial l}{\partial\theta}\right)^2 = \frac{n}{\theta^2} \Rightarrow E\left[\frac{\partial l}{\partial\theta}\right]^2 = n/\theta^2$$
$$\frac{\partial^2 l}{\partial\theta^2} = \frac{n}{\theta^2} \Rightarrow -E\left[\frac{\partial^2 l}{\partial\theta^2}\right] = -n/\theta^2$$
hence
$$E\left[\frac{\partial l}{\partial\theta}\right]^2 \neq -E\left[\frac{\partial^2 l}{\partial\theta^2}\right]$$
$$\frac{n}{\theta^2} \neq \frac{-n}{\theta^2}$$

Imp

Remark ::

Suppose the p.m.f or p.d.f $f(x;\theta)$ is such that $\frac{\partial^2}{\partial\theta^2} f_X(x;\theta)$ is finite or $x \in X$

and $E_\theta\left\{\frac{\partial^2}{\partial\theta^2} f(x;\theta)\right\}$ is finite for all $\theta \in \Theta$,

then, the Fisher Information can alternatively be found as follows

$$I_X(\theta) = -E_\theta\left[\frac{\partial^2}{\partial\theta^2}\log_e f(x;\theta)\right]$$

$E\left(\frac{\partial}{\partial\theta}\ln L(\cdot)\right)^2$

Proof ::

As we know that

$$1 = \int_{-\infty}^{\infty} f(x;\theta)\,dx$$

Differentiate w.r.t θ,

$$0 = \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} f(x;\theta)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} f(x;\theta)\,\frac{f(x;\theta)}{f(x;\theta)}\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta}\left(\ln f(x;\theta)\right)\cdot f(x;\theta)\,dx.$$

differentiate w.r.t θ, we get

$$0 = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\ln f(x;\theta)\cdot\frac{\partial}{\partial\theta}f(x;\theta) + f(x;\theta)\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]dx$$

$$0 = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\ln f(x;\theta)\frac{\partial}{\partial\theta}f(x;\theta)\,dx + \int_{-\infty}^{\infty}\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\,f(x;\theta)\,dx$$

$$0 = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\ln f(x;\theta)\frac{\partial}{\partial\theta}f(x;\theta)\frac{f(x;\theta)}{f(x;\theta)}\,dx + E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]$$

$$0 = \int_{-\infty}^{\infty}\frac{\partial}{\partial\theta}\ln f(x;\theta)\frac{\partial}{\partial\theta}\ln f(x;\theta)\,f(x;\theta)\,dx + E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]$$

$$0 = \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial\theta}\ln f(x;\theta)\right)^2 f(x;\theta)\,dx + E\left[\frac{\partial^2}{\partial\theta^2}\ln f(x;\theta)\right]$$

$E\left[(-)^2\right]$