Possible Values	Values of S	Values of T	$f(x_1, x_2, x_3 \mid S)$	$f(x_1, x_2, x_3 \mid T)$
0,0,0	0	0	1	$(1-p)/(1+p)$
0,0,1	1	1	1/3	$(1-p)/(1+2p)$
0,1,0	1	0	1/3	$p/(1+p)$
1,0,0	1	0	1/3	$p/(1+p)$
0,1,1	2	1	1/3	$p/(1+2p)$
1,0,1	2	1	1/3	$p/(1+2p)$
1,1,0	2	1	1/3	$p/(1+2p)$
1,1,1	3	2	1	1

Independent of θ

depend on p's.

Same

should be

m.f of θ

B

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~~$f(0,0,1)/S=$~~

$$f(0,0,1/S=1) = \frac{f(0,0,1)}{P(S=1)}.$$

$$= \frac{P(X_1=0)\,P(X_2=0)\,P(X_3=1)}{P(X_1=0, X_2=0, X_3=1) + P(X_1=0, X_2=1, X_3=0) + P(X_1=1, X_2=0, X_3=0)}$$

$$= \frac{p^0(1-p)^{1-0}\,p^0(1-p)^{1-0}\,p^1(1-p)^{1-1}}{p^1(1-p)^{3-1} + p^1(1-p)^2 + p^1(1-p)^2}$$

$$= \frac{p(1-p)^2}{3p(1-p)^2}$$

$$= \frac{1}{3}$$

Gossip!

Day/Date $2^4 = 16$

Y_1				
	SSSS	SFSS	FSSS	FFSS
	SSSF	SFSF	FSSF	FFSF
	SSFS	SFFS	FSFS	FFFS
	SSFF	SFFF	FSFF	FFFF

(i) $Y_1 = \{\text{Outcome of first trial}\}$

(ii) $Y_2 = \{\text{no of success}\}$

(iii) $Y_3 = \{Y_1, Y_2\}$

$Y_2 =$

4		3	2
SSSS	SFSS	FSSS	FFSS
SSSF	SFSF	FSSF	FFSF 1
SSFS	SFFS	FSFS	FFFS
SSFF	SFFF	FSFF	FFFF 0

This is minimal sufficient

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Remark

If $f(x;\theta_1,\dots,\theta_k) = a(\theta_1,\dots,\theta_k)\, b(x) \exp\left[\sum_{j=1}^{k} c_j(\theta_1,\dots,\theta_k)\, d_j(x)\right]$

then

$\prod_{i=1}^{n} f(x_i;\theta_1,\dots,\theta_k) = a^n(\theta_1,\dots,\theta_k)\left[\prod_{i=1}^{n} b(x_i)\right] \exp\left[\sum c_j(\theta_1,\dots,\theta_k) \sum d_j(x_i)\right]$

So by factorization criteria

$\sum_{i=1}^{n} d_1(x_i), \dots \sum_{i=1}^{n} d_k(x_i)$ is a set of jointly sufficient statistics. $\sum d_1(x_i), \dots \sum_i d_k(x_i)$ in fact minimal sufficient statistics.

$\Rightarrow$ Uniformally minimum variance U/B estimator (UMVUE)

First discuss fisher information

What is Fisher Information?

We marked earlier that we would work with a sufficient or minimal sufficient statistics "T" because it would reduce the data and preserve all the information about θ contained in the original data. But how

Gossip!

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⇒ 2.00 page for change of variable.

⇒ Exponential family of densities:-

A one parameter family (θ is uni-dim) of densities $f(\cdot;\theta)$ that can be expressed as

$$f(x;\theta) = a(\theta)b(x)\exp[c(\theta)d(x)]$$

for $-\infty < x < \infty$ and $\theta \in \Theta$ and for a suitable choice of functions $a(\cdot)$, $b(\cdot)$, $c(\cdot)$ and $d(\cdot)$ is defined to belong to the exponential family or exponential class.

Example If $f(x;\theta) = \theta e^{-\theta x}$, show that this belongs to exponential family.

Sol

$$f(x;\theta) = \theta e^{-\theta x}.$$

$$a(\theta) = \theta$$
$$b(x) = 1$$
$$c(\theta) = -\theta$$
$$d(x) = x.$$

$$f(x;\theta) = \theta \, 1 . \exp[-\theta(x)]$$

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$\Rightarrow$ K-parameter exponential family:—

A family of densities $f(x;\theta_1,\theta_2,\dots,\theta_k)$ that can be expressed as

$$f(x;\theta_1,\dots,\theta_k) = a(\theta_1,\dots,\theta_k)\, b(x) \exp\left(\sum c_j(\theta_1,\dots,\theta_k) d_j(x)\right)$$

Example: If $f(x;\theta_1,\theta_2) = \phi_{\mu,\sigma^2}(x)$ where $(\theta_1,\theta_2)=(\mu,\sigma^2)$

then $f(x;\theta_1,\theta_2)$ belongs to the exponential family

Sol
$$f(x;\theta_1,\theta_2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}\right) \exp\left(\frac{-1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x\right)$$

$(a(\hat{\theta}_1))\ (a(\hat{\theta}_2))\quad b(x)=1$ &

$$c_1(\mu,\sigma) = \frac{-1}{2\sigma^2}, \quad c_2(\mu,\sigma) = \frac{\mu}{\sigma^2}.$$

$$d_1(x) = x, \quad d_2(x)\ x^2.$$

so $\phi_{\mu,\sigma}(x)$ belongs to the exponential family.

Example: If $f(x;\theta_1,\theta_2) = \frac{1}{B(\theta_1,\theta_2)} x^{\theta_1-1}(1-x)^{\theta_2-1}$ $(0<x<1)$

then $f(x;\theta_1,\theta_2)$ belongs to exponential family.

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example

$$f(x;\theta_1,\theta_2) = \frac{1}{B(\theta_1,\theta_2)} x^{\theta_1-1}(1-x)^{\theta_2-1}$$

$$f(x;\theta_1,\theta_2) = \frac{1}{\underset{a(\theta_1,\theta_2)}{B(\theta_1,\theta_2)}} \cdot \underset{b(x)}{(1)} \cdot \exp\left\{(\theta_1-1)\log x + (\theta_2-1)\log(1-x)\right\}$$

here to get minimal sufficient we take

$$\prod_{i=1}^{n} f(x_i;\theta_1,\theta_2) = \frac{1}{B^n(\theta_1,\theta_2)} \exp\left\{(\theta_1-1)\Sigma\log x + (\theta_2-1)\Sigma\log(1-x)\right\}$$

$d_1(x) = \Sigma \log x_i$

$d_2(x) = \Sigma \log(1-x_i)$ are jointly minimal sufficient statistics.

$$\rightarrow \quad f(x;\alpha,\beta) = \frac{1}{\Gamma\alpha\,\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad \begin{matrix} x>0 \\ \alpha,\beta>0 \end{matrix}$$

$$f(x;\alpha,\beta) = \frac{1}{\Gamma\alpha\,\beta^\alpha}(1)\exp\left(-x\cdot\frac{1}{\beta} + (\alpha-1)\log x\right)$$

$$= \frac{1}{\Gamma\alpha\,\beta^\alpha}\cdot\frac{1}{x}\exp\left(-\frac{1}{\beta}x\cdot\alpha\log x\right).$$

$a_1(\alpha) = \frac{1}{\Gamma\alpha}$

$a_2(\beta) = \beta^\alpha$

$b(x) = 1/x$

$d_1(x) = \cancel{-\frac{1}{\beta}}\ x.$

$d_2(x) = \cancel{\xi}\alpha\ \cancel{\frac{1}{x}}\ \log x.$

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Theorem: Let $X_1, X_2, \ldots, X_n$ be iid with a common pdf or pmf $f(x;\theta)$ we denote $I_{X_1}(\theta)$, the information contained in one observation $I_{\underline{X}}(\theta)$ contained in $\underline{X} = (X_1, \ldots, X_n)$ is.

$$I_{X_1}(\theta) = I_{\underline{x}}(\theta)$$

$\Rightarrow$ means Fisher information contained in variable X_1 for parameter θ.

Proof: $L(\theta;\underline{x}) = f(x_1;\theta) \cdots f(x_n;\theta)$

$$L(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta)$$

$$\log_e L(\underline{x};\theta) = \sum_{i=1}^{n} \log_e f_x(x_i;\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{\partial}{\partial\theta} \log L(\underline{x};\theta) = \sum_{i=1}^{n} \frac{\partial}{\partial\theta} \log f(x_i;\theta)$$

taking square on both sides

$$\left(\frac{\partial}{\partial\theta} \log_e L(\theta;\underline{x})\right)^2 = \left(\sum \frac{\partial}{\partial\theta} \log_e f(x_i;\theta)\right)^2$$

$$\because \left(\sum a_i\right)^2 = \sum a_i^2 + \sum_{i\neq j}\sum a_i a_j$$

$$\left(\frac{\partial}{\partial\theta} \log L(\theta;\underline{x})\right)^2 = \sum \left(\frac{\partial}{\partial\theta} \log f(x;\theta)\right)^2 +$$

$$\sum\sum \frac{\partial}{\partial\theta} \log f(x_i;\theta) \frac{\partial}{\partial\theta} \log f(x_j;\theta)$$

GOSSIP!

much information does one have in original data that one tries to preserve it?

Suppose that X is an observable real valued r.v with its p.m.f or p.d.f $f(x;\theta)$ where the unknown parameter $\theta \in \Theta$ and open interval of $\mathbb{R}$ (real no.s) then $\mathcal{X}$ space is assumed not to depend on θ.

⇒ Assumptions:

(i) The partial derivative $\frac{\partial}{\partial\theta} f(x;\theta)$ $\forall x \in \mathcal{X}$ is finite for all $x \in \mathcal{X}$ and $\theta \in \Theta$

(ii) We can interchange the derivatives (w.r.t. θ) and (w.r.t. x)

<u>Definition:</u> The Fisher information or simply information about θ contained in X is

$$I_X(\theta) = E_\theta\left[\left\{\frac{\partial}{\partial\theta}\log f(x;\theta)\right\}^2\right]$$

X has this information about θ.

Example: If $f(x;\theta) = f(x;\lambda)$ is the poisson density, then

$$f(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \qquad x>0$$

$$= e^{-\lambda}\cdot\frac{1}{x!}\exp[x\log\lambda]$$

here $a(\theta) = e^{-\lambda}$

$b(x) = \frac{1}{x!}$

$c(\theta) = \log\lambda$

$d(x) = x$.

So $f(x;\lambda)$ belongs to a exponential family.

Example:

Remark: If $f(x;\theta) = a(\theta)\,b(x)\exp[c(\theta)d(x)]$

if $\prod_{i=1}^{n} f(x_i;\theta) = a^{n}(\theta)\left[\prod_{i=1}^{n} b(x_i)\right]\exp\left[c(\theta)\sum d(x_i)\right]$

and hence by factorization criterion $\sum d(x_i)$ is a sufficient statistics.

The one parameter family can be generalized to the k-parameter exponential family.

Day/Date

last term vanish due Σ iid's

$$\left(\frac{\partial}{\partial\theta}\log L(\underline{x};\theta)\right)^2 = \sum_{i=1}^{n}\left(\frac{\partial}{\partial\theta}\ln f(x_i;\theta)\right)^2.$$

applying expectations on both sides

$$E\left[\frac{\partial}{\partial\theta}\log L(\underline{x};\theta)\right]^2 = n\,E\left[\frac{\partial}{\partial\theta}\ln f(x_i;\theta)\right]^2.$$

$$I_{\underline{x}}(\theta) = n\cdot I_{x_i}(\theta).$$

proved.

Now suppose $\underline{X}$ is the date and $T = T(\underline{X})$ is a statistic. Then $I_{\underline{X}}(\theta) = I_T(\theta)$ for all $\theta \in \underline{\Theta}$. The equality holds for all $\forall$ θ iff T is sufficient for θ.

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$$\ln f(x;\mu,\sigma^2) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x-\mu)^2$$

$$\frac{\partial}{\partial\mu}\ln f(x;\mu,\sigma^2) = 0 - \frac{1}{2\sigma^2} 2(x-\mu)(-1).$$

$$\frac{\partial}{\partial\mu}\ln f(x;\mu,\sigma^2) = \left(\frac{x-\mu}{\sigma^2}\right)$$

$$E\left[\frac{\partial}{\partial\mu}\ln f(x;\mu,\sigma^2)\right]^2 = E\left(\frac{x-\mu}{\sigma^2}\right)^2$$

$$= \frac{1}{\sigma^4} E(x-\mu)^2$$

$$= \frac{1}{\sigma^4} \operatorname{var}(X)$$

$$E\left[\left\{\frac{\partial}{\partial\mu}\ln f(x;\mu,\sigma^2)\right\}^2\right] = \frac{\sigma^2}{\sigma^4} = \frac{1}{\sigma^2}.$$

$$I_X(\mu) = \frac{1}{\sigma^2}$$

X has σ^{-2} information for parameter μ.

Assignment

Do for μ known but σ^2 unknown

Aso find for gamma, beta, exponential and uniform distributions.

Day/Date

Example:- Suppose X is Poisson (λ), $\lambda > 0$

now

$$f_X(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, \dots, \infty$$

$$\ln f(x;\lambda) = -\lambda + x\ln\lambda - \ln x!$$

$$\frac{\partial}{\partial\lambda}\ln f(x;\lambda) = -1 + \frac{x}{\lambda}$$

$$E\left[\left(\frac{\partial}{\partial\lambda}\ln f(x;\lambda)\right)^2\right] = E\left(-1+\frac{x}{\lambda}\right)^2$$

$$= E\left(\frac{-\lambda+x}{\lambda}\right)^2$$

$$= \frac{1}{\lambda^2}E(-\lambda+x)^2 = \frac{1}{\lambda^2}E(x-\lambda)^2$$

$$= \cancel{\frac{1}{\lambda^2}E(x^2+\lambda^2-2x\lambda)}$$

$$= \frac{1}{\lambda^2}\,\mathrm{var}(x)$$

$$\mathrm{var}(x) = \lambda$$

$$E\left[\left\{\frac{\partial}{\partial\lambda}\ln f(x;\lambda)\right\}^2\right] = \frac{1}{\lambda^2}\lambda = \frac{1}{\lambda}$$

X has λ^{-1} information for parameter λ.

Example: Suppose X is $N(\mu, \sigma^2)$, where μ is unknown and σ^2 is known

$$f_X(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

$$\ln f(x;\mu,\sigma^2) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) - \frac{1}{2\sigma^2}(x-\mu)^2$$

For verification of dist of $T=\sum_{i=1}^{n} x_i$, we can use m.g.f Technique.
Note that $X_i \sim$ Gamma(α,β) then

$E(X)=\alpha/\beta$, $V(X)=\frac{\alpha}{\beta^2}$ for Gamma I

$E(X)=\alpha\beta$, $V(X)=\alpha\beta^2$ for Gamma II i.e

$$f(x)=\frac{1}{\Gamma\alpha\,\beta^{\alpha}}x^{\alpha-1}e^{-x/\beta}, \quad x>0$$

Now

$$E(e^{tT})=E(e^{t\sum_{i=1}^{n}x_i})=E(e^{tx_1+tx_2+\dots+tx_n})$$

$$=E(e^{tx_1})E(e^{tx_2})\dots E(e^{tx_n}) \quad \text{due to iid's}$$

$$=(1-\beta t)^{-\alpha}(1-\beta t)^{-\alpha}\dots(1-\beta t)^{-\alpha}$$

$$=((1-\beta t)^{-\alpha})^{n}$$

$$=(1-\beta t)^{-n\alpha}$$

i.e $T=\sum_{i=1}^{n}x_i \sim$ Gamma$(n\alpha,\beta)$.

M-I: Fisher Information Through Sufficient Statistics:

$$g(t,n\alpha,\beta)=\frac{1}{\Gamma n\alpha}\beta^{n\alpha}t^{n\alpha-1}e^{-t\beta} \quad \text{i.e } E(t)=n\alpha/\beta, \; V(t)=n\alpha/\beta^2$$

$$\log_e g(t,n\alpha,\beta)=n\alpha\log_e\beta+(n\alpha-1)\log_e t-t\beta\log_e e-\log_e\Gamma n\alpha$$

$$\frac{\partial}{\partial\beta}\log_e g(t;n\alpha,\beta)=\frac{n\alpha}{\beta}-t$$

$$E\left(\frac{\partial}{\partial\beta}\log_e g(t;n\alpha,\beta)\right)^2=E\left(\frac{n\alpha}{\beta}-t\right)^2$$

$$=E\left[t-\frac{n\alpha}{\beta}\right]^2=E[t-E(t)]^2=V(t)$$

$$I_T(\theta)=\frac{n\alpha}{\beta^2}$$

M-II: Fisher Information Through likelihood fn :-

$$f(x)=\frac{1}{\Gamma\alpha}\beta^{\alpha}x^{\alpha-1}e^{-x\beta}, \quad x>0$$