2

8 (ii)

Show that $\hat{\theta}$ is consistent but bias($\hat{\theta}$).
i.e bias($\hat{\theta}$) $\nrightarrow 0$ as $n \to \infty$.

Sol As bias is not zero then MSE can't be zero

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$E(\hat{\theta}) = \sum_{\hat{\theta} = \theta, \theta+n} \hat{\theta} P(\hat{\theta})$$

$$= \theta P(\hat{\theta} = \theta) + (\theta + n) P(\hat{\theta} = \theta + n)$$

$$= \theta \left(\frac{n-1}{n}\right) + (\theta + n)\frac{1}{n}$$

$$= \frac{n\theta}{n} - \frac{\theta}{n} + \frac{\theta}{n} + 1$$

$$E(\hat{\theta}) = \theta + 1$$

so $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta = \cancel{\theta} + 1 - \cancel{\theta} = 1$

which is constant numbers and constant nover approches to zero as $n \to \infty$, so

$$bias(\hat{\theta}) \neq 0$$

$$E(\hat{\theta}^2) = \theta^2 \left(\frac{n-1}{n}\right) + (\theta + n)^2 \frac{1}{n}$$

$$= \theta^2 - \frac{\theta^2}{n} + \frac{\theta^2}{n} + n + 2\theta$$

$$= \theta^2 + 2n + n$$

$$V(\hat{\theta}) = E(\hat{\theta}^2) - (E(\hat{\theta}))^2$$

$$= \theta^2 + 2\theta + n - (\theta + 1)^2$$

$$= \cancel{\theta^2} + \cancel{2\theta} + n - \cancel{\theta^2} - 1 - \cancel{2\theta} = n-$$

$$E(\hat{\theta}_n^2) = \int_0^\theta y^2 \cdot \frac{n y^{n-1}}{\theta^n}\, dy$$

$$= \frac{n}{\theta^n} \int_0^\theta y^{n+1}\, dy = \frac{n}{\theta^n} \left(\frac{y^{n+2}}{n+2} \Big|_0^\theta \right)$$

$$= \frac{n}{\theta^n} \frac{\theta^{n+2}}{n+2} = \frac{n}{n+2} \theta^{\cancel{n}+2-\cancel{n}}$$

$$E(\hat{\theta}_n^2) = \frac{n}{n+2}\theta^2$$

Now from (2)

$$\mathrm{var}(\hat{\theta}_n) = E(\hat{\theta}_n^2) - (E(\hat{\theta}_n))^2$$

$$= \frac{n}{n+2}\theta^2 - \left(\frac{n\theta}{n+1}\right)^2$$

$$= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$$

$$\mathrm{var}(\hat{\theta}_n) = \frac{n\theta^2(n+1)^2 - n^2\theta^2(n+2)}{(n+2)(n+1)^2}$$

$$= \frac{n\theta^2(n^2+1+2n) - n^3\theta^2 - 2n^2\theta^2}{(n+2)(n+1)^2}$$

$$= \frac{\cancel{n^3\theta^2} + n\theta^2 + \cancel{2n^2\theta^2} - \cancel{\theta^2 n^3} - \cancel{2n^2\theta^2}}{(n+2)(n+1)^2}$$

$$\mathrm{var}(\hat{\theta}_n) = \frac{n\theta^2}{(n+2)(n+1)^2}$$

Lecture 6 : 22-6-2020

Problem:- Let $X_1, \dots, X_n$ be a r.s from a $uni(0,\theta)$ where θ is unknown. Define the estimator.

$$\hat{\theta}_n = \max\{X_1, X_2, \dots, X_n\}$$

(a) Find bias of $\hat{\theta}_n$ $B(\hat{\theta}_n)$

(b) Find the MSE of $\hat{\theta}_n$ $MSE(\hat{\theta}_n)$

(c) Is $\hat{\theta}_n$ a consistent estimator of θ?

Sol If $X \sim Uni(0,\theta)$, then the PDF and CDF of X are given by

$$f_X(x) = \begin{cases} \frac{1}{\theta} & 0 \le x \le \theta \\ 0 & o.w \end{cases}$$

and

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{\theta} & 0 \le x \le \theta \end{cases}$$

the pdf of $\hat{\theta}_n$ is given by

$$f_{\hat{\theta}_n}(y) = n f_X(x) \{F_X(x)\}^{n-1}$$

$$= n \frac{1}{\theta}\left(\frac{y}{\theta}\right)^{n-1} = n\left(\frac{1}{\theta}\right) \frac{y^{n-1}}{\theta^{n-1}}$$

$$= \begin{cases} \frac{n y^{n-1}}{\theta^n} & 0 \le y \le \theta \end{cases}$$

by direct method

→ By definition

$$= P[|\hat{\theta} - \theta| > \varepsilon] = P(|\tfrac{\bar{x}}{4} - \theta| > \varepsilon)$$

$$= P\left[\frac{|\bar{x}/4 - \theta|}{\sqrt{V(\bar{x}/4)}} > \frac{\varepsilon}{\sqrt{V(\bar{x}/4)}}\right]$$

$$= P\left(\frac{|\bar{x}/4 - \theta|}{\theta^2/4n} > \frac{\varepsilon}{\theta^2/4n}\right)$$

$$= P\left\{|Z| > \frac{4n\varepsilon}{\theta^2}\right\}$$

$$= 1 - P\left\{|Z| \le \frac{4n\varepsilon}{\theta^2}\right\}$$

$$\lim_{n \to \infty} P[|\hat{\theta} - \theta| > \varepsilon] = 1 - P(|Z| \le \infty))$$

$$= 1 - P(-\infty \le Z \le \infty)$$

$$= 1 - \int_{-\infty}^{\infty} f(z)\,dz = \cancel{1} - \cancel{1} = 0$$

hence proved $P[|\hat{\theta} - \theta| > \varepsilon] = 0$.

consistent.

Example : Suppose $\hat{\theta}$ is an estimator of θ with prob function

$$P(\hat{\theta} = \theta) = \frac{n-1}{n}$$

$$P(\hat{\theta} = \theta + n) = 1/n$$

and no other value of θ are possibl

A consistent estimate has insignificant errors as sample size increases. More specifically the more data to collect, a consistent estimator will be close to the real pop^n parameter you are trying to measure.

The sample mean and sample variance are two well known consistent estimators.

Let $\hat{\theta}$ be an estimator of θ based on a sample of size n. Then $\hat{\theta}$ is a consistent estimator of θ, if $var(\hat{\theta}) \to 0$ as $n \to \infty$

To illustrate, let us consider $\bar{X}$, the sample mean based on random of size n. We know that

$$E(\bar{X}) = \mu \quad , \quad var(\bar{X}) = \frac{\sigma^2}{n}$$

Now as $n \to \infty$ $\frac{\sigma^2}{n} \to 0$

$$\text{i.e} \quad \lim_{n\to\infty} var(\bar{X}) = \lim_{n\to\infty} \frac{\sigma^2}{n} = 0$$

Hence $\bar{X}$ is an U/B estimator of μ.

Now from (1),

$$MSE(\hat{\theta}_n) = var(\hat{\theta}_n) + (B(\hat{\theta}_n))^2.$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2} + \left(\frac{-\theta}{n+1}\right)^2$$

$$= \frac{n\theta^2}{(n+2)(n+1)^2} + \frac{\theta^2}{(n+1)^2}$$

$$= \frac{n\theta^2 + \theta^2(n+2)}{(n+2)(n+1)^2}.$$

$$= \frac{n\theta^2 + \theta^2 n + 2\theta^2}{(n+2)(n+1)^2}$$

$$= \frac{2n\theta^2 + 2\theta^2}{(n+2)(n+1)^2}$$

$$MSE(\hat{\theta}_n) = \frac{2\theta^2\cancel{(n+1)}}{(n+2)(n+1)^{\cancel{2}}} = \frac{2\theta^2}{(n+2)(n+1)}$$

$$MSE(\hat{\theta}_n) = \frac{2\theta^2}{(n+2)(n+1)}$$

(c) Is $\hat{\theta}_n$ is consistent estimator of θ.

$$\lim_{n\to\infty} MSE(\hat{\theta}_n) = \lim_{n\to\infty} \frac{2\theta^2}{(n+2)(n+1)} = 0$$

$\hat{\theta}_n$ is a consistent estimator of θ.

Also for proportion, $\hat{p} = \frac{X}{n}$

$$E(\hat{p}) = E\left(\frac{X}{n}\right) = \frac{E(X)}{n} = \frac{np}{n} = p$$

So $\hat{p}$ is U/B estimator of P

Now $var(p) = var\left(\frac{X}{n}\right) = \frac{1}{n^2} var(X)$

$var(p) = \frac{1}{n^2}(npq) = \frac{pq}{n}$

$$\lim_{n\to\infty} var(p) = \lim_{n\to\infty} \frac{pq}{n} = 0$$

hence $\hat{p}$ is a consistent estimator of P.

Example (Lecture 5, page 3)

Investigate $\hat{\theta}$ is equal to $\bar{X}/4$ i.e. $\hat{\theta} = \bar{X}/4$ is an U/B estimator and consistent.

$E(\hat{\theta}) = \theta$ where $\hat{\theta} = \bar{X}/4$

Sol

$$var(\hat{\theta}) = \frac{1}{16} var(\bar{X})$$ $V(\bar{x}) = \frac{\sigma^2}{n}$

$$= \frac{1}{16}\frac{\sigma^2}{n} = \frac{1}{4 \cdot 16}\frac{4\theta^2}{n}$$ $\sigma^2 = 4\theta^2$

$$Var(\hat{\theta}) = \frac{\theta^2}{4n}$$ $\rightarrow$ Now $\lim_{n\to\infty} var(\hat{\theta}) = \lim_{n\to\infty} \frac{\theta^2}{4n} = 0$ hence proved

To find the bias we have

$$B(\hat{\theta}_n) = E[\hat{\theta}_n] - \theta$$

Now find $E[\hat{\theta}_n]$

$$E(\hat{\theta}_n) = \int_0^{\theta} y \cdot \frac{n y^{n-1}}{\theta^n}\, dy$$

$$= \frac{n}{\theta^n} \int_0^{\theta} y^n\, dy = \frac{n}{\theta^n} \left(\frac{y^{n+1}}{n+1} \Big|_0^{\theta} \right)$$

$$E(\hat{\theta}_n) = \frac{n}{\theta^n} \left(\frac{\theta^{n+1}}{n+1} \right) = \frac{n}{n+1} \theta^{\cancel{n} - \cancel{n} + 1} = \frac{n}{n+1} \theta$$

Now find bias

$$B(\hat{\theta}_n) = \frac{n}{n+1}\theta - \theta = \frac{\cancel{n\theta} - \cancel{n\theta} - \theta}{n+1} = \frac{-\theta}{n+1}$$

(b) To find $MSE(\hat{\theta}_n)$, we can write

$$MSE(\hat{\theta}_n) = Var(\hat{\theta}_n) + (B(\hat{\theta}_n))^2$$

$$MSE(\hat{\theta}_n) = Var(\hat{\theta}_n) + \frac{\theta^2}{(n+1)^2} \quad \text{——— } (1)$$

Thus

$$Var(\hat{\theta}_n) = E[\hat{\theta}_n^2] - (E(\hat{\theta}_n))^2 \quad \text{——— } (2)$$

$E(\hat{\theta}_n^2)$ will be solve in similar way as $E(\hat{\theta}_n)$