Lecture 5. 21-10-2020

Exam Let the simple linear regression $Y_i = \hat{\alpha} + \hat{\beta} X_i + e_i$ where each Y_i is normally distributed and X_i's are fixed. Then show that $\hat{\alpha}$ and $\hat{\beta}$ are U/B estimators of the parameters α and β.

Sol Let the sample reg coefficient may be expressed as

$$\hat{\beta} = \frac{n\Sigma XY - \Sigma X \Sigma Y}{n\Sigma X^2 - (\Sigma X)^2}$$

$$\hat{\beta} = \frac{n\Sigma X(\alpha + \beta X + \varepsilon) - \Sigma X(\alpha + \beta X + \varepsilon)}{n\Sigma X^2 - (\Sigma X)^2}$$

$$\hat{\beta} = \frac{\cancel{\alpha n \Sigma X} + n\beta \Sigma X^2 + n\Sigma X\varepsilon - \cancel{n\alpha \Sigma X} - \beta(\Sigma X)^2 - \Sigma X \Sigma \varepsilon}{n\Sigma X^2 - (\Sigma X)^2}$$

$$\hat{\beta} = \frac{\beta\left[n\Sigma X^2 - (\Sigma X)^2\right] + n\Sigma X\varepsilon - n\Sigma \bar{X}\varepsilon}{n\Sigma X^2 - (\Sigma X)^2}$$

$\because \Sigma X = n\bar{X}$

$$\hat{\beta} = \frac{\beta\left[\cancel{n\Sigma X^2 - (\Sigma X)^2}\right]}{\cancel{n\Sigma X^2 - (\Sigma X)^2}} + \frac{n\Sigma X\varepsilon - n\Sigma \bar{X}\varepsilon}{n\Sigma X^2 - (\Sigma X)^2}$$

$$= \beta + \frac{n\Sigma (X - \bar{X})\varepsilon}{\Sigma (X - \bar{X})^2}$$

$\because n\Sigma X^2 - (\Sigma X)^2 = \Sigma (X - \bar{X})^2$

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By taking expectations

$$E(\hat{\beta}) = \beta + \frac{n\Sigma(X-\bar{X})E(\varepsilon)}{\Sigma(X-\bar{X})^2} \qquad \& \quad E(\varepsilon)=0$$

$$E(\hat{\beta}) = \beta + 0 = \beta$$

hence proved that $E(\hat{\beta}) = \beta$, i.e. $\hat{\beta}$ is an U/B estimator of β.

Now for α

$$E(\hat{\alpha}) = E(\bar{Y} - \hat{\beta}\bar{X}) \qquad \alpha = \bar{Y} - \hat{\beta}\bar{X}$$

$$E(\hat{\alpha}) = E(\bar{Y}) - E(\hat{\beta})\bar{X}$$

$$E(\hat{\alpha}) = E\left(\frac{\Sigma Y_i}{n}\right) - \beta\bar{X} \qquad \because E(\hat{\beta}) = \beta$$

$$E(\hat{\alpha}) = \frac{\Sigma(\alpha + \beta X)}{n} - \beta\bar{X}$$

$$E(\hat{\alpha}) = E\frac{(n\alpha + \beta\Sigma X)}{n} - \beta\bar{X}$$

$$E(\hat{\alpha}) = \frac{\cancel{n}\alpha}{\cancel{n}} + \beta\frac{\Sigma X}{n} - \beta\bar{X}$$

$$E(\hat{\alpha}) = \alpha + \cancel{\beta\bar{X}} - \cancel{\beta\bar{X}} = \alpha$$

hence proved that $\hat{\alpha}$ is an U/B estimator of α.

Example For a random sample $X_1, \dots, X_n$ from a gamma dist^n with p.d.f $f(x;\theta) = \frac{1}{\theta^4 3!} x^3 e^{-x/\theta}$, $x > 0$

Investigate $\hat{\theta}$ is equal to $\bar{X}/4$ i.e. $\hat{\theta} = \frac{\bar{X}}{4}$ is an U/B estimator of θ.

Sol

$$\because f(x) = \frac{1}{\Gamma\alpha \beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$\alpha = 4$

$\beta = \theta$

$$E_\theta(X) = \alpha\beta, \quad V(X) = \alpha\beta^2$$

$$E(\hat{\theta}) = \frac{1}{4} E(\bar{X}) = \frac{1}{4} E(X)$$

$$= \cancel{\frac{1}{4}(\alpha\beta)} \quad \text{i.e} = \frac{1}{4}(4\theta)$$

$$E(\hat{\theta}) = \frac{1}{\cancel{4}}(\cancel{4}\theta) = \theta$$

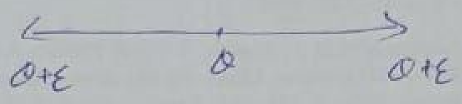
So $\hat{\theta}$ is an U/B estimator of θ.

$\theta+\varepsilon \quad \theta \quad \theta+\varepsilon$

2 Consistency:- Although some bias may be acceptable in an estimator, we would like the bias to tends to zero as the sample size tends to infinity. In addition we would like the variance to tends to zero as $n \to \infty$. These requirements are related to the idea of consistency.

Two types of consistency:-

(1) Weak consistent estimator:- Probability that $\hat{\theta}$ is an estimator which take any value within this interval



ε is very small

(2) Strongly consistent estimator:- If $\hat{\theta}$ approaches a single value, this is called strongly consistent estimator.

If $\varepsilon = 0$, then weakly consistent estimator becomes strongly consistent

Mean Square Error: Mean square error is the mean of squared difference of an estimator from the true value.

$$MSE = E(\hat{\theta}-\theta)^2 = E(\hat{\theta}-E(\hat{\theta})+E(\hat{\theta})-\theta)^2$$

$$= E(\hat{\theta}-E(\hat{\theta}))^2 + E(E(\hat{\theta})-\theta)^2$$

$$+ 2E(\hat{\theta}-E(\hat{\theta}))(E(\hat{\theta})-\theta)$$

$$= Var(\hat{\theta}) + (Bias)^2 + 2(E(\hat{\theta}-E(\hat{\theta})(E(\hat{\theta})-\theta))$$

as $E(\hat{\theta}) = \theta$

$$= Var(\hat{\theta}) + (Bias)^2 + \underbrace{2(E(\hat{\theta})-\theta)(E(\hat{\theta} - E(\hat{\theta}))}_{0}$$

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + (Bias)^2$$

MSE approaches to zero if $Var(\hat{\theta})$ and $bias(\hat{\theta})$ approaches to zero.

→ <u>Mean square error consistency</u> :-

Let $T_1, \ldots, T_n$ be sequence of estimators of $\tau(\theta)$ where $T_n = t_n(X_1, \ldots, X_n)$ is biased on a sample of size n. This sequence of estimators is defined to be a MSE consistent sequence of estimators of $\tau(\theta)$ iff

$$\lim_{n \to \infty} E_\theta (T_n - \tau(\theta))^2 = 0$$

Remark: MSE consistency implies that both the bias and the variance of T_n approach zero, since

$$E_\theta (T_n - \tau(\theta))^2 = Var(T_n) + \cancel{(\tau(\theta) - E(} (E(T_n) - \tau(\theta))^2$$