5-3 MOTION OF A FLUID ELEMENT (KINEMATICS)

Before formulating the effects of forces on fluid motion (dynamics), let us consider first the motion (kinematics) of a fluid element in a flow field. For convenience, we follow an infinitesimal element of fixed identity (mass), as shown in Fig. 5.4.

As the infinitesimal element of mass, *dm*, moves in a flow field, several things may happen to it. Perhaps the most obvious of these is that the element translates; it undergoes a linear displacement from a location x, y, z to a different location x_1, y_1, z_1. The element may also rotate; the orientation of the element as shown in Fig. 5.4, where the sides of the element are parallel to the coordinate axes x, y, z, may change as a result of pure rotation about any one (or all three) of the coordinate axes. In addition the

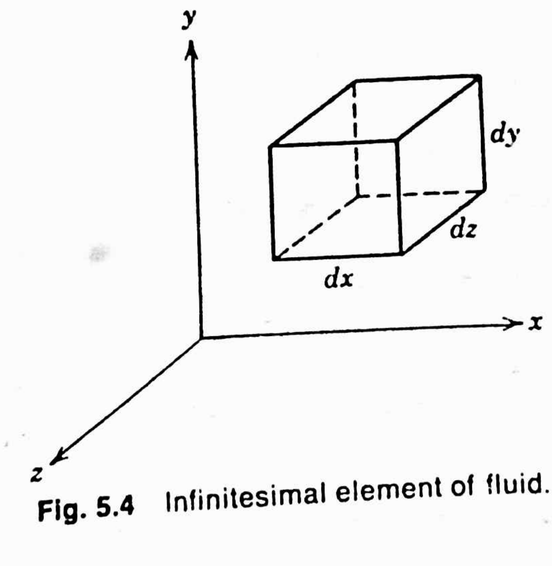


Fig. 5.4 Infinitesimal element of fluid.

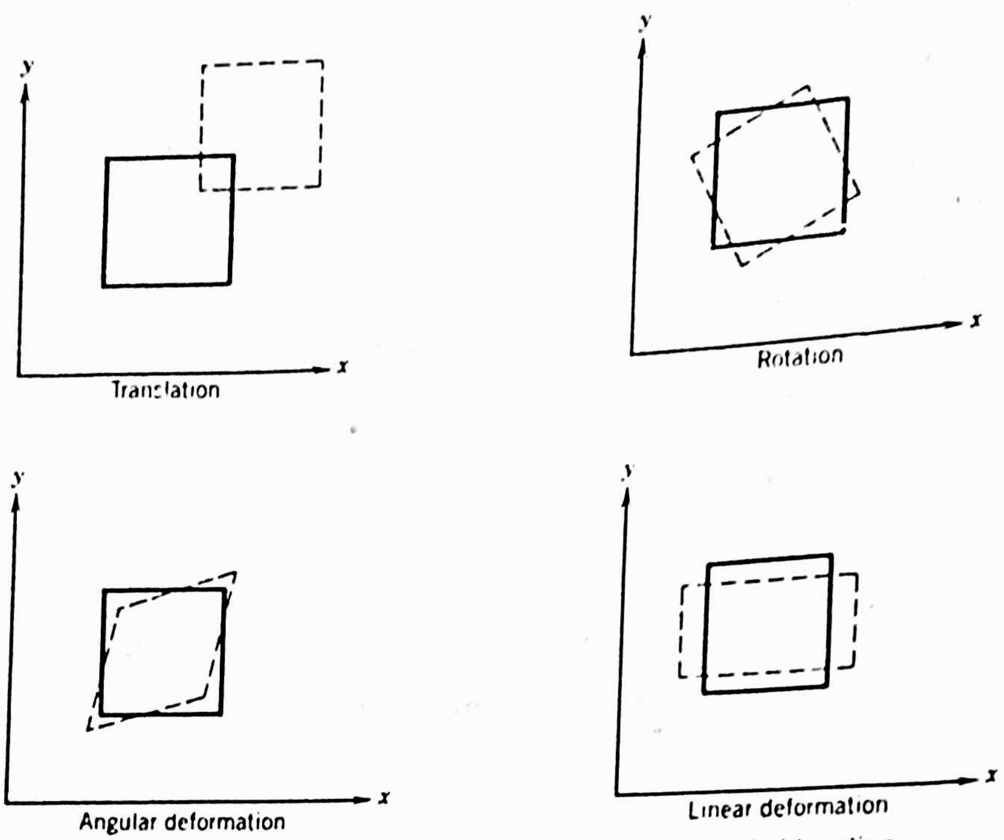


Fig. 5.5 Pictorial representation of the components of fluid motion.

element may deform. The deformation may be subdivided into two parts—linear and angular deformation. Linear deformation involves a change in shape without a change in orientation of the element: a deformation in which planes of the element that were originally perpendicular (e.g. the top and side of the element) remain perpendicular. Angular deformation involves a distortion of the element in which planes that were originally perpendicular are no longer perpendicular. In general, a fluid element may undergo a combination of translation, rotation, and linear and angular deformation during the course of its motion.

These four components of fluid motion are illustrated in Fig. 5.5 for motion in the xy plane. For a general three-dimensional flow, similar motions of the particle would be depicted in the yz and xz planes. For pure translation or rotation, the fluid element retains its shape; there is no deformation. Thus shear stresses do not arise as a result of pure translation or rotation (recall from Chapter 2 that in a Newtonian fluid the shear stress is directly proportional to the rate of angular deformation).

5-3.1 Acceleration of a Fluid Particle in a Velocity Field

Let us remember first that we are dealing with an element of fixed mass, dm. As discussed in Section 1-5.3, one may obtain the equation of motion for a particle by applying Newton's second law to that particle. The disadvantage of this approach is that a separate equation is required for each particle. Thus the bookkeeping for many particles becomes a problem.

A more general description of acceleration can be obtained by considering a particle moving in a velocity field. The basic hypothesis of continuum fluid mechanics has led us to a field description of fluid flow in which the properties of a flow field are defined by continuous functions of the space coordinates and time. In particular, the velocity field is given by $\vec{V} = \vec{V}(x, y, z, t)$. The field description is very powerful since information for the entire flow is given by one equation.

The problem, then, is to retain the field description for fluid properties and obtain an expression for the acceleration of a fluid particle as it translates in a flow field. Stated simply, the problem is:

Given the velocity field, $\vec{V} = \vec{V}(x, y, z, t)$, find the acceleration of a fluid particle, $\vec{a}_p$.

Consider a particle moving in a velocity field. At time, t, the particle is at the position x, y, z and has a velocity corresponding to the velocity at that point in space at t,

$$\vec{V}_p\big]_t = \vec{V}(x, y, z, t)$$

At $t + dt$, the particle has moved to a new position, with coordinates $x + dx$, $y + dy$, $z + dz$, and has a velocity given by

$$\vec{V}_p\big]_{t+dt} = \vec{V}(x + dx, y + dy, z + dz, t + dt)$$

This is shown pictorially in Fig. 5.6.

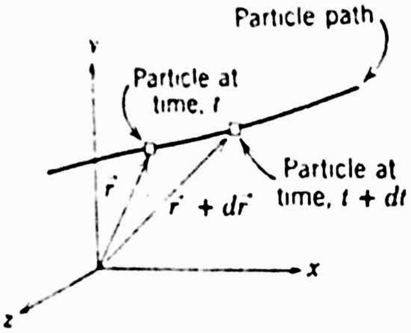


Fig. 5.6 Motion of a particle in a flow field.

The particle velocity at t (position $\vec{r}$) is given by $\vec{V}_p = \vec{V}(x, y, z, t)$. Then $d\vec{V}_p$, the change in the velocity of the particle, in moving from location $\vec{r}$ to $\vec{r} + d\vec{r}$, is given by

$$d\vec{V}_p = \frac{\partial \vec{V}}{\partial x} dx_p + \frac{\partial \vec{V}}{\partial y} dy_p + \frac{\partial \vec{V}}{\partial z} dz_p + \frac{\partial \vec{V}}{\partial t} dt$$

The total acceleration of the particle is given by

$$\vec{a}_p = \frac{d\vec{V}_p}{dt} = \frac{\partial \vec{V}}{\partial x}\frac{dx_p}{dt} + \frac{\partial \vec{V}}{\partial y}\frac{dy_p}{dt} + \frac{\partial \vec{V}}{\partial z}\frac{dz_p}{dt} + \frac{\partial \vec{V}}{\partial t}$$

Since

$$\frac{dx_p}{dt} = u, \quad \frac{dy_p}{dt} = v, \quad \text{and} \quad \frac{dz_p}{dt} = w$$

then

$$\vec{a}_p = \frac{d\vec{V}_p}{dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

To remind us that calculation of the acceleration of a fluid particle in a velocity field requires a special derivative, it is given the symbol $D\vec{V}/Dt$. Thus

$$\frac{D\vec{V}}{Dt} \equiv \vec{a}_p = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \tag{5.9}$$

The derivative, $D\vec{V}/Dt$, defined by Eq. 5.9, is commonly called the *substantial derivative* to remind us that it is computed for a particle of "substance." It often is called the material or particle derivative.

From Eq. 5.9 we recognize that a fluid particle moving in a flow field may undergo an acceleration for either of two reasons. It may be accelerated because it is convected into a region of higher (or lower) velocity. For example, in the steady flow through a nozzle, in which, by definition, the velocity field is not a function of time, a fluid particle will accelerate as it moves through the nozzle. The particle is convected into a region of higher velocity. If a flow field is unsteady, a fluid particle will undergo an acceleration, a "local" acceleration, because the velocity field is a function of time.

The physical significance of the terms in Eq. 5.9 is

$$\vec{a}_p = \underbrace{\frac{D\vec{V}}{Dt}}_{\substack{\text{total}\\\text{acceleration}\\\text{of a particle}}} = \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\substack{\text{convective}\\\text{acceleration}}} + \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\substack{\text{local}\\\text{acceleration}}}$$

For a two-dimensional flow, say $\vec{V} = \vec{V}(x, y, t)$, Eq. 5.9 reduces to

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + \frac{\partial \vec{V}}{\partial t}$$

For a one-dimensional flow, say $\vec{V} = \vec{V}(x, t)$, Eq. 5.9 becomes

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + \frac{\partial \vec{V}}{\partial t}$$

Finally, for a steady flow in three dimensions, Eq. 5.9 becomes

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}$$

which is not necessarily zero. Thus a fluid particle can undergo a convective acceleration due to its motion, even in a steady velocity field.

Equation 5.9 is a vector equation. As with all vector equations, it may be written in scalar component equations. Relative to an xyz coordinate system, the scalar components of Eq. 5.9 are written:

$$a_{x_p} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \tag{5.10a}$$

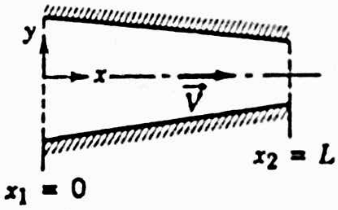
$$a_{y_p} = \frac{Dv}{Dt} = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t} \tag{5.10b}$$

$$a_{z_p} = \frac{Dw}{Dt} = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t} \tag{5.10c}$$

We have obtained an expression for the acceleration of a particle anywhere in the flow field; this is the Eulerian method of description. To determine the acceleration of a particle at a particular point in the flow field, one substitutes the coordinates of the point into the field expression for acceleration. In the Lagrangian method of description, the motion (position, velocity, and acceleration) of the particle is described as a function of time. The Eulerian and Lagrangian methods of description are illustrated in Example Problem 5.5.

Example 5.5 Particle Acc. In Eulerian & Lag. Descriptions.

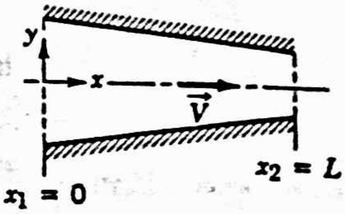
Consider one-dimensional, steady, incompressible flow through the plane converging channel shown. The velocity field is given by $\vec{V} = V_1[1 + (x/L)]\hat{i}$. Find the x component of acceleration for a particle moving in the flow field. If we use the method of description of particle mechanics, the position of the particle, located at $x = 0$ at time $t = 0$, will be a function of time, $x_p = f(t)$. Obtain the expression for $f(t)$ and then, by taking the second derivative of the function with respect to time, obtain an expression for the x component of the particle acceleration.



EXAMPLE PROBLEM 5.5

GIVEN: Steady, one-dimensional, incompressible flow through the converging channel shown.

$$\vec{V} = V_1\left(1 + \frac{x}{L}\right)\hat{i}$$



FIND: (a) The x component of the acceleration of a particle moving in the flow field.
(b) For the particle located at $x = 0$ at $t = 0$, obtain an expression for its
 (1) position, x_p, as a function of time.
 (2) x component of acceleration, a_{x_p}, as a function of time.

SOLUTION:

The acceleration of a particle moving in a velocity field is given by

$$\frac{D\vec{V}}{Dt} = u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t}$$

The x component of the acceleration is given by

$$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t}$$

For the flow field given,

$$v = w = 0 \qquad u = V_1\left(1 + \frac{x}{L}\right)$$

Therefore, $$\frac{Du}{Dt} = u\frac{\partial u}{\partial x} = V_1\left(1 + \frac{x}{L}\right)\frac{V_1}{L} = \frac{V_1^2}{L}\left(1 + \frac{x}{L}\right)$$ $\frac{Du}{Dt}$

{To determine the acceleration of a particle at any point in the flow field, we merely substitute the present location of the particle into the above result.}

In the second part of this problem we are interested in following a particular particle, namely, the one located at $x = 0$ at $t = 0$, as it flows through the channel.

The x coordinate that locates this particle will be a function of time, $x_p = f(t)$. Furthermore, $u_p = df/dt$ will be a function of time. The particle will have the velocity corresponding to its location in the velocity field. At $t = 0$, the particle is at $x = 0$, and its velocity $u_p = V_1$. At some later time, t, the particle will reach the exit, $x = L$; at that time it will have a velocity $u_p = 2V_1$. To find the expression for $x_p = f(t)$, we write

$$u_p = \frac{dx_p}{dt} = \frac{df}{dt} = V_1\left(1 + \frac{x}{L}\right) = V_1\left(1 + \frac{f}{L}\right)$$

Separating variables gives

$$\frac{df}{(1 + f/L)} = V_1\, dt$$

Since at $t = 0$, the particle in question was located at $x = 0$, and at t, this particle is located at $x_p = f$, then

$$\int_0^f \frac{df}{(1 + f/L)} = \int_0^t V_1\, dt$$

$$L \ln\left(1 + \frac{f}{L}\right) = V_1 t$$

$$\ln\left(1 + \frac{f}{L}\right) = \frac{V_1 t}{L}$$

$$1 + \frac{f}{L} = e^{V_1 t/L}$$

and

$$f = L[e^{V_1 t/L} - 1]$$

Then the position of the particle, located at $x = 0$ at $t = 0$, as a function of time is given by

$$x_p = f(t) = L[e^{V_1 t/L} - 1]$$ x_p

The x component of acceleration of this particle is given by

$$a_{x_p} = \frac{d^2 x_p}{dt^2} = \frac{d^2 f}{dt^2} = \frac{V_1^2}{L} e^{V_1 t/L}$$ a_{x_p}

We now have two different ways of expressing the acceleration of the particle that was located at $x = 0$ at $t = 0$. Note that although the flow field is steady, when we follow a particular particle, its position and acceleration (and velocity for that matter) are functions of time.

We check to see that both expressions for the acceleration give identical results.

$$a_{x_p} = \frac{V_1^2}{L} e^{V_1 t/L}$$

$$a_{x_p} = \frac{Du}{Dt} = \frac{V_1^2}{L}\left(1 + \frac{x}{L}\right)$$

(a) At $t = 0$, $x_p = 0$

$$a_{x_p} = \frac{V_1^2}{L} e^0 = \frac{V_1^2}{L} \qquad \text{(a)}$$

At $t = 0$, the particle is at $x = 0$

$$\frac{Du}{Dt} = \frac{V_1^2}{L}(1 + 0) = \frac{V_1^2}{L} \qquad \text{(a)}$$

Check.

(b) When $x_p = \frac{L}{2}$, $t = t_1$,

$$x_p = \frac{L}{2} = L[e^{V_1 t_1/L} - 1]$$

Therefore, $e^{V_1 t_1/L} = 1.5$, and

$$a_{x_p} = \frac{V_1^2}{L} e^{V_1 t_1/L}$$

$$a_{x_p} = \frac{V_1^2}{L}(1.5) = \frac{1.5V_1^2}{L} \qquad \text{(b)}$$

At $x = 0.5L$

$$\frac{Du}{Dt} = \frac{V_1^2}{L}(1 + 0.5)$$

$$\frac{Du}{Dt} = \frac{1.5V_1^2}{L} \qquad \text{(b)}$$

Check.

(c) When $x_p = L$, $t = t_2$,

$$x_p = L = L[e^{V_1 t_2/L} - 1]$$

Therefore, $e^{V_1 t_2/L} = 2$, and

$$a_{x_p} = \frac{V_1^2}{L} e^{V_1 t_2/L}$$

$$a_{x_p} = \frac{V_1^2}{L}(2) = \frac{2V_1^2}{L} \qquad \text{(c)}$$

At $x = L$

$$\frac{Du}{Dt} = \frac{V_1^2}{L}(1 + 1)$$

$$\frac{Du}{Dt} = \frac{2V_1^2}{L} \qquad \text{(c)}$$

Check.

This problem illustrates the Eulerian and Lagrangian methods of describing the motion of a particle.

5-3.2 Fluid Rotation

The rotation, $\vec{\omega}$, of a fluid particle is defined as the average angular velocity of any two mutually perpendicular line elements of the particle. Rotation is a vector quantity. A particle moving in a general three-dimensional flow field may rotate about all three coordinate axes. Thus, in general,

$$\vec{\omega} = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z$$

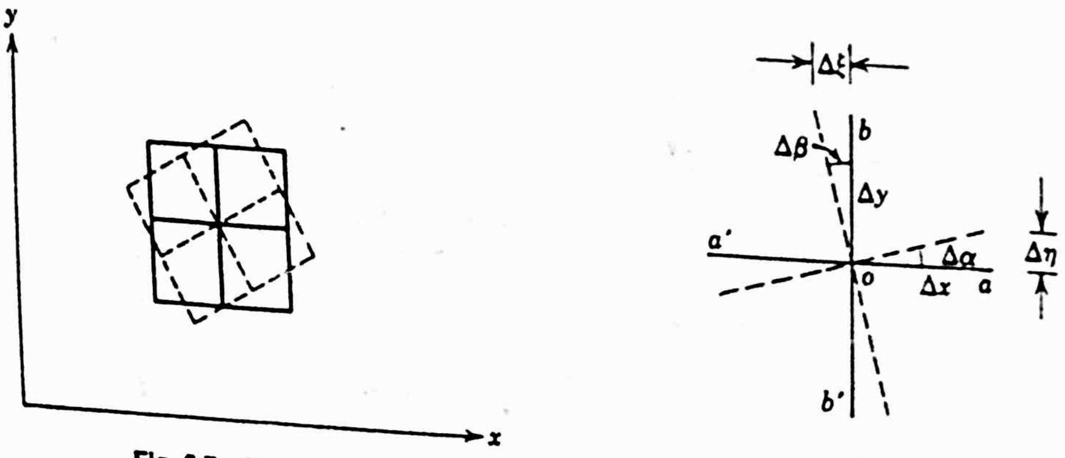


Fig. 5.7 Rotation of a fluid element in a two-dimensional flow field.

where ω_x is the rotation about the x axis, ω_y is the rotation about the y axis, and ω_z is the rotation about the z axis. The positive sense of rotation is given by the right-hand rule.

To obtain a mathematical expression for fluid rotation, consider motion of a fluid element in the xy plane. The components of the velocity at every point in the flow field are given by $u(x, y)$ and $v(x, y)$. The rotation of a fluid element in such a flow field is illustrated in Fig. 5.7. The two mutually perpendicular lines, oa and ob, will rotate to the positions shown during the interval, Δt, only if the velocities at points a and b are different from the velocity at o.

Consider first the rotation of line oa, of length, Δx. Rotation of this line is due to variations of the y component of velocity. If the y component of velocity at the point o is taken as v_o, then the y component of velocity at point a can be written, using a Taylor series expansion, as

$$v = v_o + \frac{\partial v}{\partial x}\Delta x$$

The angular velocity of line oa is given by

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta/\Delta x}{\Delta t}$$

Since

$$\Delta\eta = \frac{\partial v}{\partial x}\Delta x\,\Delta t$$

$$\omega_{oa} = \lim_{\Delta t \to 0} \frac{(\partial v/\partial x)\Delta x\,\Delta t/\Delta x}{\Delta t} = \frac{\partial v}{\partial x}\,.$$

Rotation of line ob, of length Δy, results from variations in the x component of velocity. If the x component of velocity at point o is taken as u_o, then the x component of velocity at point b can be written, using a Taylor series expansion, as

$$u = u_o + \frac{\partial u}{\partial y}\Delta y$$

the angular velocity of line *ob* is given by

$$\omega_{ob} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi/\Delta y}{\Delta t}$$

Since

$$\Delta\xi = -\frac{\partial u}{\partial y}\Delta y\,\Delta t$$

$$\omega_{ob} = \lim_{\Delta t \to 0} \frac{-(\partial u/\partial y)\Delta y\,\Delta t/\Delta y}{\Delta t} = -\frac{\partial u}{\partial y}$$

(The negative sign is introduced to give a positive value of ω_{ob}. According to our sign convention, counterclockwise rotation is positive.)

The rotation of the fluid element about the z axis is the average angular velocity of the two mutually perpendicular line elements, *oa* and *ob*, in the xy plane.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

By considering the rotation of two mutually perpendicular lines in the yz and xz planes, one can show that

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

and

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

Then

$$\vec{\omega} = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z = \frac{1}{2}\left[\hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \quad (5.11)$$

We recognize the term in the square brackets as

$$\text{curl}\,\vec{V} = \nabla \times \vec{V}$$

Then, in vector notation, we can write

$$\vec{\omega} = \tfrac{1}{2}\nabla \times \vec{V} \quad (5.12)$$

Under what conditions might we expect to have an irrotational flow? A fluid particle moving, without rotation, in a flow field cannot develop a rotation under the action of a body force or normal surface (pressure) forces. Development of rotation in a fluid particle, initially without rotation, requires the action of a shear stress on the surface of the particle. Since shear stress is proportional to the rate of angular deformation, then a particle that is initially without rotation will not develop a rotation without a simultaneous angular deformation. The shear stress is related to the rate of angular deformation through the viscosity. The presence of viscous forces means the flow is rotational.[3]

The condition of irrotationality may be a valid assumption for those regions of a flow in which viscous forces are negligible.[4] (For example, such a region exists outside the boundary layer in the flow over a solid surface.) The factor of ½ can be eliminated in Eq. 5.12 by defining a quantity called the *vorticity*, $\vec{\zeta}$, to be twice the rotation.

$$\vec{\zeta} \equiv 2\vec{\omega} = \nabla \times \vec{V} \tag{5.13}$$

In cylindrical coordinates,

$$\vec{V} = \hat{i}_r V_r + \hat{i}_\theta V_\theta + \hat{i}_z V_z$$

and

$$\nabla = \hat{i}_r \frac{\partial}{\partial r} + \hat{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{i}_z \frac{\partial}{\partial z}$$

The vorticity, in cylindrical coordinates, is then[5]

$$\nabla \times \vec{V} = \hat{i}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \hat{i}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \hat{i}_z\left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) \tag{5.14}$$

The vorticity is a measure of the rotation of a fluid element as it moves in the flow field. The *circulation*, Γ, is defined as the line integral of the tangential velocity component about a closed curve fixed in the flow.

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} \tag{5.15}$$

where $d\vec{s}$ is an elemental vector, of length ds, tangent to the curve; a positive sense corresponds to a counterclockwise path of integration around the curve. A relationship between circulation and vorticity can be obtained by considering the fluid element of Fig. 5.7. The element has been redrawn in Fig. 5.8; the velocity variations shown are consistent with those used in determining the fluid rotation.

For the closed curve *oacb*

$$d\Gamma = u\Delta x + \left(v + \frac{\partial v}{\partial x}\Delta x\right)\Delta y - \left(u + \frac{\partial u}{\partial y}\Delta y\right)\Delta x - v\,\Delta y$$

$$d\Gamma = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Delta x\,\Delta y$$

$$d\Gamma = 2\omega_z\,\Delta x\,\Delta y$$

[3] A rigorous proof using the complete equations of motion for a fluid particle is given in W. H. Li and S. H. Lam, *Principles of Fluid Mechanics* (Reading, Mass.: Addison-Wesley, 1964), pp. 142–145.

[4] Examples of rotational and irrotational motion are shown in the film loops: S-FM014A, *Visualization of Vorticity with Vorticity Meter—Part I*; S-FM014B, *Visualization of Vorticity with Vorticity Meter—Part II*.

[5] In carrying out the curl operation, recall that $\hat{i}_r$ and $\hat{i}_\theta$ are functions of θ (see footnote[1] on p. 209).

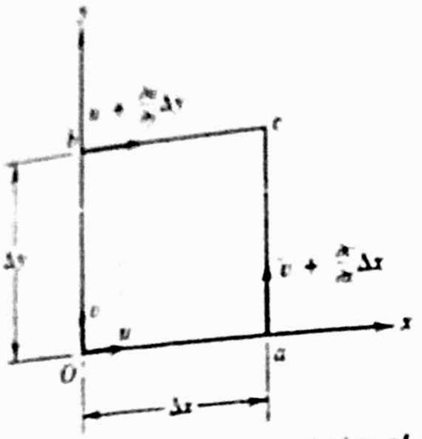


Fig. 5.8 Velocity components on the boundaries of a fluid element.

Then

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_A 2\omega_z \, dA = \int_A (\nabla \times \vec{V})_z \, dA \tag{5.16}$$

Equation 5.16 is a statement of Stokes theorem in two dimensions. Thus the circulation around a closed contour is the sum of the vorticity enclosed within it.

Example 5.6 Free & Forced Vortex Flows.

Consider flow fields with purely tangential motion (circular streamlines): $V_r = 0$ and $V_\theta = f(r)$. Evaluate the rotation, vorticity, and circulation for solid-body rotation, a *forced vortex*. Show that it is possible to choose $f(r)$ so that flow is irrotational, to produce a *free vortex*.

EXAMPLE PROBLEM 5.6

GIVEN: Flow field with tangential motion, $V_r = 0$ and $V_\theta = f(r)$

FIND: (a) Rotation, vorticity, and circulation for solid-body motion (a *forced vortex*).
(b) Evaluate $f(r)$ for irrotational motion (a *free vortex*).

SOLUTION:

Basic equation: $\vec{\zeta} = 2\vec{\omega} = \nabla \times \vec{V}$ (5.13)

For motion in the $r\theta$ plane, the only components of rotation and vorticity are in the z direction,

$$\zeta_z = 2\omega_z = \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}$$

Because $V_r = 0$ everywhere in this field, this reduces to $\zeta_z = 2\omega_z = \frac{1}{r}\frac{\partial r V_\theta}{\partial r}$

(a) For solid-body rotation, $V_\theta = \omega r$

Then $\omega_z = \frac{1}{2}\frac{1}{r}\frac{\partial r V_\theta}{\partial r} = \frac{1}{2}\frac{1}{r}\frac{\partial}{\partial r}\omega r^2 = \frac{1}{2r}(2\omega r) = \omega$ and $\zeta_z = 2\omega$

The circulation is $\Gamma = \oint_C \vec{V} \cdot d\vec{s} = \int_A 2\omega_z \, dA$ (5.16)

Since $\omega_z = \omega =$ constant, the circulation about any closed contour is given by $\Gamma = 2\omega A$, where A is the area enclosed by the contour.

Thus for solid-body motion (a forced vortex), the rotation and vorticity are constants; the circulation depends on the area enclosed by a contour.

(b) For irrotational flow, $\frac{1}{r}\frac{\partial}{\partial r} rV_\theta = 0$. Integrating, we find

$$rV_\theta = \text{constant} \quad \text{or} \quad V_\theta = f(r) = \frac{C}{r}$$

For this flow, the origin is a singular point where $V_\theta \to \infty$. The circulation for any contour enclosing the origin is

$$\Gamma = \oint_C \vec{V} \cdot d\vec{s}$$

$$\Gamma = \int_0^{2\pi} \frac{C}{r} r \, d\theta = 2\pi C$$

The circulation around any contour not enclosing the singular point at the origin is zero.

5-3.3 Fluid Deformation

Angular deformation of a fluid element involves changes in the angle between two mutually perpendicular lines in the fluid. Referring to Fig. 5.9, we see that the rate of angular deformation of the fluid element is the rate of decrease of the angle between lines oa and ob. The rate of angular deformation is given by

$$-\frac{d\gamma}{dt} = \frac{d\alpha}{dt} + \frac{d\beta}{dt}$$

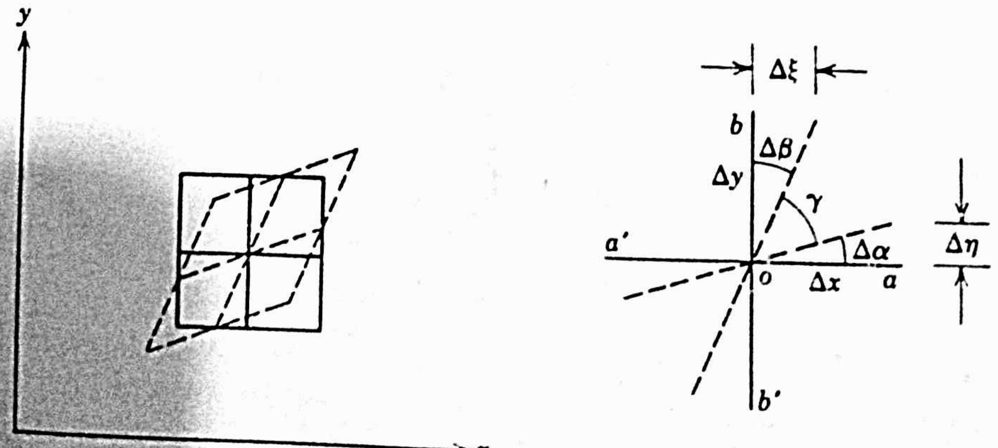


Fig. 5.9 Angular deformation of a fluid element in a two-dimensional flow field.

Now,

$$\frac{d\alpha}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\alpha}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\eta/\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial v/\partial x)\Delta x\, \Delta t/\Delta x}{\Delta t} = \frac{\partial v}{\partial x}$$

and

$$\frac{d\beta}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\beta}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\xi/\Delta y}{\Delta t} = \lim_{\Delta t \to 0} \frac{(\partial u/\partial y)\Delta y\, \Delta t/\Delta y}{\Delta t} = \frac{\partial u}{\partial y}$$

Consequently, the rate of angular deformation in the xy plane is

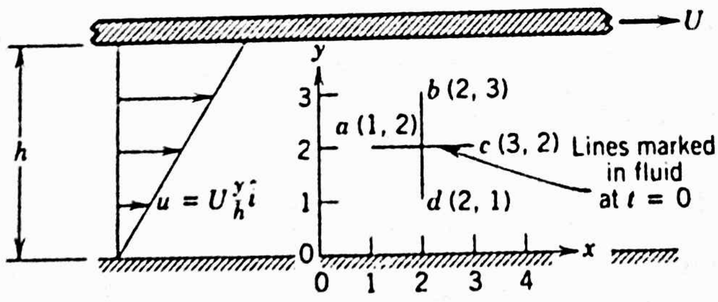
$$\frac{d\alpha}{dt} + \frac{d\beta}{dt} = -\frac{d\gamma}{dt} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{5.17}$$

The shear stress is related to the rate of angular deformation through the fluid viscosity. In a viscous flow (where velocity gradients are present) it is highly unlikely that $\partial v/\partial x$ will be equal and opposite to $\partial u/\partial y$ throughout the flow field (e.g. consider the boundary-layer flow of Fig. 2.11 and the flow over a cylinder, shown in Fig. 2.12). The presence of viscous forces means the flow is rotational.

Calculation of angular deformation is illustrated for a simple flow field in Example Problem 5.7.

Example 5.7 Rotation in viscometric flow.

A viscometric flow in the narrow gap between large parallel plates is shown. The velocity field in the narrow gap is given by $\vec{V} = U(y/h)\hat{i}$, where $U = 4$ mm/sec and $h = 4$ mm. At $t = 0$ two lines, ac and bd, are marked in the fluid as shown. Evaluate the positions of the marked points at $t = 1.5$ sec and sketch for comparison. Calculate the rate of angular deformation and the rate of rotation of a fluid particle in this velocity field. Comment on your results.



EXAMPLE PROBLEM 5.7

GIVEN: Velocity field, $\vec{V} = U\dfrac{y}{h}\hat{i}$; $U = 4$ mm/sec, and $h = 4$ mm. Fluid particles marked at $t = 0$ to form cross as shown.

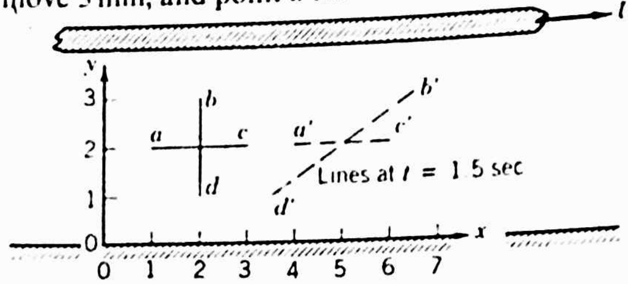
FIND:
(a) Positions of points a', b', c', and d' at $t = 1.5$ sec; plot.
(b) Rate of angular deformation.
(c) Rate of rotation of a fluid particle.
(d) Comment on the significance of these results.

SOLUTION:

For the given flow field, $v = 0$, so there is no vertical motion. The velocity of each point stays constant, so $\Delta x = u\,\Delta t$ for each point. At point b, $u = 3$ mm/sec, so

$$\Delta x_b = \frac{3\text{ mm}}{\text{sec}} \times 1.5\text{ sec} = 4.5\text{ mm}$$

Points a and c each move 3 mm, and point d moves 1.5 mm. The plot at $t = 1.5$ sec is



The rate of angular deformation is

$$-\dot{\gamma} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = U\frac{1}{h} + 0 = \frac{U}{h} = \frac{4\text{ mm}}{\text{sec}} \times \frac{1}{4\text{ mm}} = 1\text{ sec}^{-1}$$

The rate of rotation is

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left(0 - \frac{U}{h}\right) = -\frac{1}{2} \times \frac{4\text{ mm}}{\text{sec}} \times \frac{1}{4\text{ mm}} = -0.5\text{ sec}^{-1}$$

This flow is viscous, so we expect to have both angular deformation and rotation: shape and orientation of a fluid particle both change.

During linear deformation, the shape of the fluid element, described by the angles at its vertices, remains unchanged, since all right angles continue to be right angles (see Fig. 5.5). The element will change length in the x direction only if $\partial u/\partial x$ is other than zero. Similarly, a change in the y dimension requires a nonzero value of $\partial v/\partial y$ and a change in the z dimension requires a nonzero value of $\partial w/\partial z$. These quantities represent the components of longitudinal rates of strain in the x, y, and z directions, respectively. Changes in length of the sides may produce changes in volume of the element. The rate of local instantaneous *volume dilation* is given by

$$\text{Volume dilation rate} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{V} \tag{5.18}$$

For incompressible flow, the rate of volume dilation is zero.

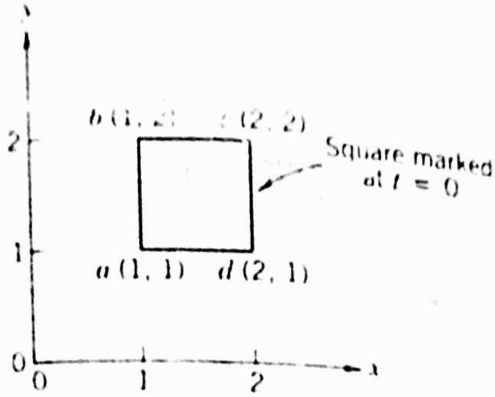
Example 5.8 Deformation rates for flow in corner.

The velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$, represents flow in a "corner," as shown in Example Problem 5.4. Consider the case where $A = 0.3\text{ sec}^{-1}$ and the coordinates are measured in meters. A square is marked in the fluid as shown at $t = 0$. Evaluate the new positions of the four corner points when point a has moved to $x = \frac{3}{2}$ m after τ seconds. Evaluate the rates of linear deformation in the x and y directions. Compare area $a'b'c'd'$ at $t = \tau$ with area $abcd$ at $t = 0$. Comment on this result.

EXAMPLE PROBLEM 5.8

GIVEN: $\vec{V} = Ax\hat{i} - Ay\hat{j}$; $A = 0.3\ \text{sec}^{-1}$, x and y in meters.

FIND: (a) Position of square at $t = \tau$ when a is at a' at $x = \frac{3}{2}$.
(b) Rates of linear deformation.
(c) Compare area $a'b'c'd'$ with $abcd$.
(d) Comment on the results.



SOLUTION:

First we must find τ, so we must follow a fluid particle using Lagrangian description. Thus

$$u = \frac{dx_p}{dt} = Ax_p; \qquad \frac{dx}{x} = A\,dt; \qquad \int_{x_0}^{x} \frac{dx}{x} = \int_0^{\tau} A\,dt; \qquad \ln\frac{x}{x_0} = A\tau$$

$$\tau = \frac{\ln x/x_0}{A} = \frac{\ln(\frac{3}{2})}{0.3\ \text{sec}^{-1}} = 1.35\ \text{sec}$$

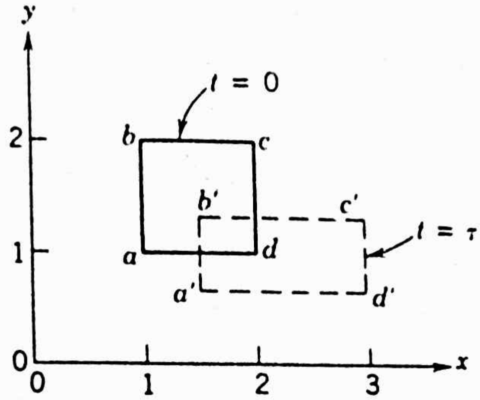
In the y direction

$$v = \frac{dy_p}{dt} = -Ay_p; \qquad \frac{dy}{y} = -A\,dt; \qquad \frac{y}{y_0} = e^{-A\tau}$$

The point coordinates at τ are:

Point	$t = 0$	$t = \tau$
a	(1, 1)	$(\frac{3}{2}, \frac{2}{3})$
b	(1, 2)	$(\frac{3}{2}, \frac{4}{3})$
c	(2, 2)	$(3, \frac{4}{3})$
d	(2, 1)	$(3, \frac{2}{3})$

The plot is:



The rates of linear deformation are:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} Ax = A = 0.3\ \text{sec}^{-1} \qquad \text{in the } x \text{ direction}$$

$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(-Ay) = -A = -0.3\ \text{sec}^{-1} \qquad \text{in the } y \text{ direction}$$

The rate of volume dilation is

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = A - A = 0$$

Area $abcd = 1\ \text{m}^2$ and area $a'b'c'd' = (3 - \frac{3}{2})(\frac{4}{3} - \frac{2}{3}) = 1\ \text{m}^2$

Note that parallel planes remain parallel; there is linear deformation but no angular deformation. The rates of linear deformation are equal and opposite, so the area of the marked region is conserved.

We have shown in this section that the velocity field contains all information needed to determine translation, rotation, deformation, and acceleration of a particle in a flow.