**5-2 STREAM FUNCTION FOR TWO-DIMENSIONAL INCOMPRESSIBLE FLOW

It is convenient to have a means of describing mathematically any particular pattern of flow. An adequate description should portray the notion of the shape of the streamlines (including the boundaries) and the scale of the velocity at representative points in the flow. A mathematical device that serves this purpose is the stream function, ψ. The stream function is formulated as a relation between the streamlines and the statement of conservation of mass.

For a two-dimensional incompressible flow in the xy plane, conservation of mass, Eq. 5.1a, can be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{5.3}$$

If a continuous function, $\psi(x, y, t)$, called the stream function, is defined such that

$$u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad v \equiv -\frac{\partial \psi}{\partial x} \tag{5.4}$$

then the continuity equation, Eq. 5.3, is satisfied exactly, since

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial^2 \psi}{\partial y\, \partial x} = 0$$

Recall that streamlines are lines drawn in the flow field, such that, at a given instant of time, they are tangent to the direction of flow at every point in the flow field. Thus, if $d\vec{r}$ is an element of length along a streamline, the equation of the streamline is given by

$$\begin{aligned} \vec{V} \times d\vec{r} = 0 &= (\hat{i} u + \hat{j} v) \times (\hat{i}\, dx + \hat{j}\, dy) \\ &= \hat{k}(u\, dy - v\, dx) \end{aligned}$$

The equation of a streamline in a two-dimensional flow is

$$u\, dy - v\, dx = 0$$

Substituting for the velocity components u and v in terms of the stream function, ψ, from Eq. 5.4, then along a streamline

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0 \tag{5.5}$$

Since $\psi = \psi(x, y, t)$, then at an instant, t_0, $\psi = \psi(x, y, t_0)$; at this instant, a change in ψ may be evaluated as though $\psi = \psi(x, y)$. Thus, at any instant,

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \tag{5.6}$$

Comparing Eqs. 5.5 and 5.6, we see that along an instantaneous streamline, $d\psi = 0$; ψ is a constant along a streamline. Since the differential of ψ is exact, the integral of $d\psi$

** This section may be omitted without loss of continuity in the text material.

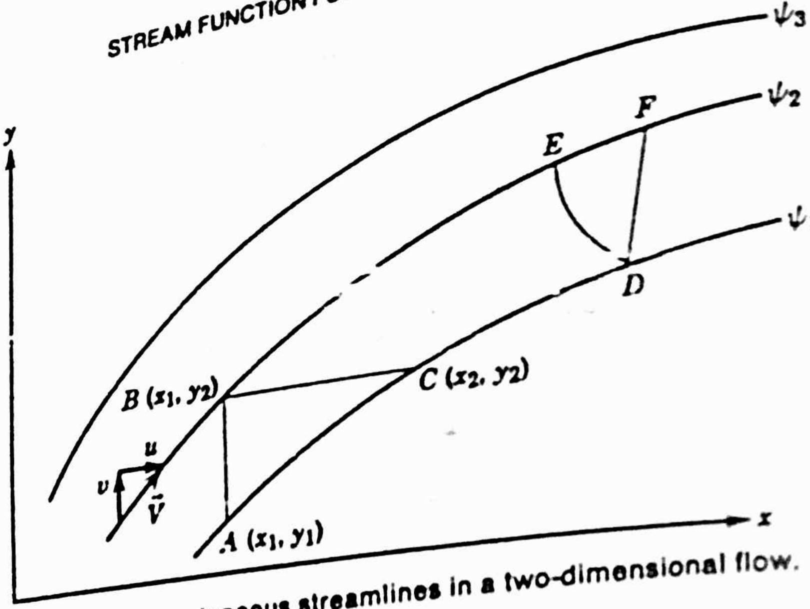


Fig. 5.3 Instantaneous streamlines in a two-dimensional flow.

between any two points in a flow field, $\psi_2 - \psi_1$, depends only on the end points of integration.

From the definition of a streamline, we recognize that there can be no flow across a streamline. Thus, if the streamlines in a two-dimensional, incompressible flow field, at a given instant are as shown in Fig. 5.3, the rate of flow between streamlines ψ_1 and ψ_2 across the lines AB, BC, DE, and DF must be equal.

The volume flow rate, Q, between streamlines ψ_1 and ψ_2 can be evaluated by considering the flow across AB or across BC. For a unit depth, the flow rate across AB is

$$Q = \int_{y_1}^{y_2} u \, dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} \, dy$$

Along AB, x = constant, and $d\psi = \partial\psi/\partial y \, dy$. Therefore,

$$Q = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} \, dy = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

For a unit depth, the flow rate across BC is

$$Q = \int_{x_1}^{x_2} v \, dx = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} \, dx$$

Along BC, y = constant, and $d\psi = \partial\psi/\partial x \, dx$. Therefore,

$$Q = -\int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} \, dx = -\int_{\psi_2}^{\psi_1} d\psi = \psi_2 - \psi_1$$

Thus the volume rate of flow (per unit depth) between any two streamlines can be written as the difference between the constant values of ψ defining the two streamlines.[2]

[2] For two-dimensional steady compressible flow in the xy plane, the stream function, ψ, is defined such that

$$\rho u \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad \rho v \equiv -\frac{\partial \psi}{\partial x}$$

The difference between the constant values of ψ defining two streamlines is the mass rate of flow (per unit depth) between the two streamlines.

For a two-dimensional, incompressible flow in the $r\theta$ plane, the conservation of mass, Eq. 5.2, can be written as

$$\frac{\partial r V_r}{\partial r} + \frac{\partial V_\theta}{\partial \theta} = 0 \tag{5.7}$$

The stream function, $\psi(r, \theta, t)$, then is defined such that

$$V_r \equiv \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad V_\theta \equiv -\frac{\partial \psi}{\partial r} \tag{5.8}$$

With ψ defined according to Eq. 5.8, the continuity equation, Eq. 5.7, is satisfied exactly.

Example 5.4

Given the velocity field for the steady, incompressible flow of Example 5.1, $\vec{V} = Ax\hat{i} - Ay\hat{j}$, determine the stream function that will yield this velocity field. Plot and interpret the streamline pattern in the first quadrant of the xy plane.

EXAMPLE PROBLEM 5.4

GIVEN: Velocity field, $\vec{V} = Ax\hat{i} - Ay\hat{j}$.

FIND: (a) Stream function, ψ.
(b) Plot in first quadrant and interpret.

SOLUTION:

The flow is incompressible, so the stream function satisfies Eq. 5.4.

From Eq. 5.4, $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$. From the given velocity field,

$$u = Ax = \frac{\partial \psi}{\partial y}$$

Integrating with respect to y gives

$$\psi = \int \frac{\partial \psi}{\partial y}\, dy + f(x) = Axy + f(x) \tag{1}$$

where $f(x)$ is arbitrary. The function $f(x)$ may be evaluated using the equation for v. Thus, from Eq. 1,

$$v = -\frac{\partial \psi}{\partial x} = -Ay - \frac{df}{dx} \tag{2}$$

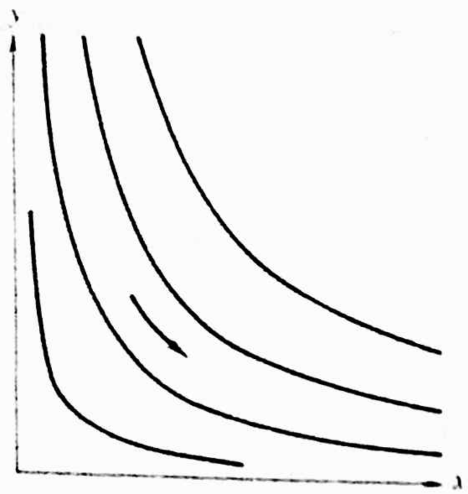
From the given velocity field,

$$v = -Ay$$

Comparing this with Eq. 2 shows that $\frac{df}{dx} = 0$, or $f(x) =$ constant. Therefore, Eq. 1 becomes

$$\psi = Axy + c$$

Lines $\psi =$ constant represent streamlines in the flow field. The constant may be chosen as any convenient value for plotting purposes. A few streamlines are plotted in the following sketch:



Flow velocity components are to the right and down, since $u > 0$ and $v < 0$. Regions of high-speed flow occur where the streamlines are close together. Lower speed flow occurs near the origin where streamline spacing is wider. Flow qualitatively looks like flow in a "corner," formed by a pair of walls.