Chapter 5

INTRODUCTION TO DIFFERENTIAL ANALYSIS OF FLUID MOTION

In Chapter 4, we developed the basic equations in integral form for a control volume. The integral equations are particularly useful when we are interested in the gross behavior of a flow field and its effect on various devices. However, the integral approach does not provide detailed point by point knowledge of the flow field.

To obtain this detailed knowledge, we must apply the equations of fluid motion in differential form. In this chapter we shall develop differential equations for the conservation of mass and Newton's second law of motion. Since we are interested in formulating differential equations, our analysis will be in terms of infinitesimal systems and control volumes.

5-1 CONSERVATION OF MASS

In Chapter 2, we found that the continuum assumption—the assumption that a fluid could be treated as a continuous distribution of matter—led directly to a field representation of fluid properties. The property fields are defined by continuous functions of the space coordinates and time. The density and velocity fields are related through conservation of mass. We shall derive the differential equation for conservation of mass in rectangular and in cylindrical coordinates. In both cases the derivation is carried out by applying conservation of mass to a differential control volume.

5-1.1 Rectangular Coordinate System

In rectangular coordinates, the control volume chosen is an infinitesimal cube with sides of length dx, dy, dz as shown in Fig. 5.1. The density at the center, O, of the control volume is ρ and the velocity there is $\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$.

To evaluate the properties at each of the six faces of the control surface, we use a Taylor series expansion about point O. For example, at the right face,

$$\rho\Big)_{x+dx/2} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2} + \left(\frac{\partial^2 \rho}{\partial x^2}\right)\frac{1}{2!}\left(\frac{dx}{2}\right)^2 + \cdots$$

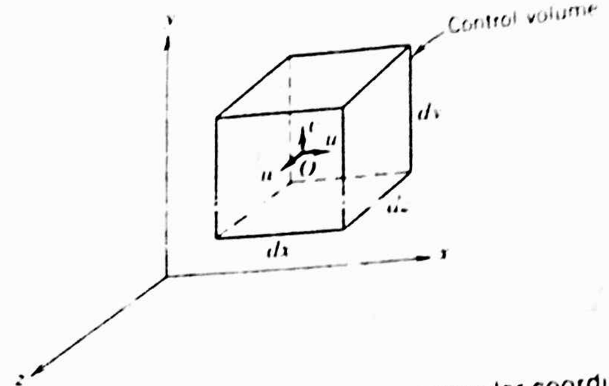


Fig. 5.1 Differential control volume in rectangular coordinates.

Neglecting higher order terms, we can write

$$\rho\Big)_{x+dx/2} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

and

$$u\Big)_{x+dx/2} = u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$$

The corresponding terms at the left face are

$$\rho\Big)_{x-dx/2} = \rho + \left(\frac{\partial \rho}{\partial x}\right)\left(-\frac{dx}{2}\right) = \rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}$$

$$u\Big)_{x-dx/2} = u + \left(\frac{\partial u}{\partial x}\right)\left(-\frac{dx}{2}\right) = u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}$$

A word statement of conservation of mass is

$$\begin{bmatrix}\text{Net rate of mass efflux}\\ \text{through the control surface}\end{bmatrix} + \begin{bmatrix}\text{Rate of change of mass}\\ \text{inside the control volume}\end{bmatrix} = 0$$

To evaluate the first term in this equation, we must consider the mass flux through each of the six surfaces of the control surface; we must evaluate $\int_{CS} \rho \vec{V} \cdot d\vec{A}$. The details of this evaluation are shown in Table 5.1.

We see that the net rate of mass efflux through the control surface is given by

$$\left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right] dx\, dy\, dz$$

The mass inside the control volume at any instant is the product of the mass per unit volume, ρ, and the volume, $dx\, dy\, dz$. Thus the rate of change of mass inside the control volume is given by

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

Table 5.1 Mass Flux through the Control Surface of a Rectangular Differential Control Volume

Surface	$\int \rho \vec{V} \cdot d\vec{A}$
Left $(-x)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}\right]\left[u - \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right]dy\,dz = -\rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right]dx\,dy\,dz$
Right $(+x)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial x}\right)\frac{dx}{2}\right]\left[u + \left(\frac{\partial u}{\partial x}\right)\frac{dx}{2}\right]dy\,dz = \rho u\,dy\,dz + \frac{1}{2}\left[u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right]dx\,dy\,dz$
Bottom $(-y)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial y}\right)\frac{dy}{2}\right]\left[v - \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right]dx\,dz = -\rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right]dx\,dy\,dz$
Top $(+y)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial y}\right)\frac{dy}{2}\right]\left[v + \left(\frac{\partial v}{\partial y}\right)\frac{dy}{2}\right]dx\,dz = \rho v\,dx\,dz + \frac{1}{2}\left[v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right]dx\,dy\,dz$
Back $(-z)$	$= -\left[\rho - \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[w - \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right]dx\,dy = -\rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right]dx\,dy\,dz$
Front $(+z)$	$= \left[\rho + \left(\frac{\partial \rho}{\partial z}\right)\frac{dz}{2}\right]\left[w + \left(\frac{\partial w}{\partial z}\right)\frac{dz}{2}\right]dx\,dy = \rho w\,dx\,dy + \frac{1}{2}\left[w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right]dx\,dy\,dz$

Then,

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = \left[\left\{u\left(\frac{\partial \rho}{\partial x}\right) + \rho\left(\frac{\partial u}{\partial x}\right)\right\} + \left\{v\left(\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial v}{\partial y}\right)\right\} + \left\{w\left(\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial w}{\partial z}\right)\right\}\right]dx\,dy\,dz$$

or

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right]dx\,dy\,dz$$

In rectangular coordinates the differential equation for the conservation of mass is then

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \tag{5.1a}$$

Since the vector operator, ∇, in rectangular coordinates, is given by

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

then

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = \nabla \cdot \rho \vec{V}$$

and the conservation of mass may be written as

$$\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0 \tag{5.1b}$$

Two flow cases for which the differential continuity equation may be simplified are worthy of note. For incompressible flow, ρ = constant; density is neither a function of space coordinates nor time. For incompressible flow, the continuity equation simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or

$$\nabla \cdot \vec{V} = 0$$

Thus the velocity field, $\vec{V}(x, y, z, t)$, for incompressible flow must satisfy $\nabla \cdot \vec{V} = 0$.

For steady flow, all fluid properties are, by definition, independent of time. Thus at most $\rho = \rho(x, y, z)$, and for steady flow, the continuity equation can be written as

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

or

$$\nabla \cdot \rho \vec{V} = 0$$

Example 5.1

For a two-dimensional flow in the xy plane, the x component of velocity is given by $u = Ax$. Determine a possible y component for steady, incompressible flow. How many possible y components are there?

EXAMPLE PROBLEM 5.1

GIVEN: Two-dimensional flow in the xy plane for which $u = Ax$.

FIND: (a) Possible v component for steady, incompressible flow.
(b) How many v components are possible?

SOLUTION:

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$

For steady, incompressible flow, $\frac{\partial \rho}{\partial t} = 0$, and $\rho =$ constant; thus $\nabla \cdot \vec{V} = 0$. In rectangular coordinates

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For two-dimensional flow in the xy plane, $\vec{V} = \vec{V}(x, y)$. Then partial derivatives with respect to z are zero, and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Then

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = -A$$

which gives an expression for the rate of change of v holding x constant. This equation can be integrated to obtain an expression for v. The result is

$$v = \int \frac{\partial v}{\partial y}\, dy + f(x) = -Ay + f(x)$$

{The function of x appears because we had the partial derivative of v with respect to y.}

Any function $f(x)$ is allowable, since $\frac{\partial}{\partial y} f(x) = 0$. Thus any number of expressions for v could satisfy the differential continuity equation under the given conditions. The simplest expression for v would be obtained by setting $f(x) = 0$. Then

$$v = -Ay$$

and

$$\vec{V} = Ax\hat{i} - Ay\hat{j}$$

{This problem illustrates use of the differential continuity equation for steady, incompressible flow to evaluate a possible velocity component and introduces the integration of a partial derivative.}

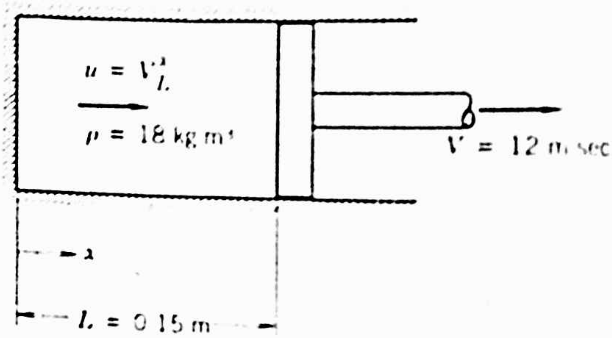
Example 5.2

A gas-filled pneumatic strut in an automobile suspension system behaves like a piston-cylinder apparatus. At one instant when the piston is $L = 0.15$ m away from the closed end of the cylinder, the gas density is uniform at $\rho = 18$ kg/m³. The piston suddenly begins to move away from the closed end at $V = 12$ m/sec. The gas motion is one-dimensional and proportional to distance from the closed end; it varies linearly from zero at the end to $u = V$ at the piston. Evaluate the rate of change of gas density at this instant. Obtain an expression for the average density as a function of time.

EXAMPLE PROBLEM 5.2

GIVEN: Piston-cylinder as shown.

FIND: (a) Rate of change of density.
(b) $\rho(t)$.



SOLUTION:

Basic equation: $\nabla \cdot \rho \vec{V} + \dfrac{\partial \rho}{\partial t} = 0$

In rectangular coordinates, $\dfrac{\partial \rho u}{\partial x} + \dfrac{\partial \rho v}{\partial y} + \dfrac{\partial \rho w}{\partial z} + \dfrac{\partial \rho}{\partial t} = 0$

Since $u = u(x)$, then partial derivatives with respect to y and z are zero, and

Then

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho}{\partial t} = 0$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u}{\partial x} = -\rho\frac{\partial u}{\partial x} - u\frac{\partial \rho}{\partial x}$$

Since ρ is assumed uniform in the volume, then $\dfrac{\partial \rho}{\partial x} = 0$, and $\dfrac{d\rho}{dt} = -\rho\dfrac{\partial u}{\partial x}$.

Since $u = V\dfrac{x}{L}$, then $\dfrac{\partial u}{\partial x} = \dfrac{V}{L}$, and $\dfrac{d\rho}{dt} = -\rho\dfrac{V}{L}$. However, note $L = L_0 + Vt$.

Separate variables and integrate,

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = -\int_0^t \frac{V}{L}\,dt = -\int_0^t \frac{V\,dt}{L_0 + Vt}$$

$$\ln\frac{\rho}{\rho_0} = \ln\frac{L_0}{L_0 + Vt} \quad \text{and} \quad \rho(t) = \rho_0\left[\frac{1}{1 + Vt/L_0}\right] \qquad \rho(t)$$

At $t = 0$

$$\frac{\partial \rho}{\partial t} = -\rho_0\frac{V}{L} = -\frac{18\text{ kg}}{\text{m}^3} \times \frac{12\text{ m}}{\text{sec}} \times \frac{1}{0.15\text{ m}} = -1440\text{ kg/m}^3\cdot\text{sec} \qquad \frac{\partial \rho}{\partial t}$$

{This problem illustrates use of the differential continuity equation to evaluate a density variation.}

5-1.2 Cylindrical Coordinate System

In cylindrical coordinates a suitable differential control volume is shown in Fig. 5.2. The density at the center, O, of the control volume is ρ and the velocity there is $\vec{V} = \hat{i}_r V_r + \hat{i}_\theta V_\theta + \hat{i}_z V_z$, where $\hat{i}_r$, $\hat{i}_\theta$, and $\hat{i}_z$ are unit vectors in the r, θ, and z directions, respectively, and V_r, V_θ, and V_z are the velocity components in the r, θ, and z directions, respectively. To evaluate $\int_{CS} \rho \vec{V} \cdot d\vec{A}$, we must consider the mass flux through each of the six faces of the control surface. The properties at each of the six faces of the control surface are obtained from a Taylor series expansion about point O. The details of the mass flux evaluation are shown in Table 5.2.

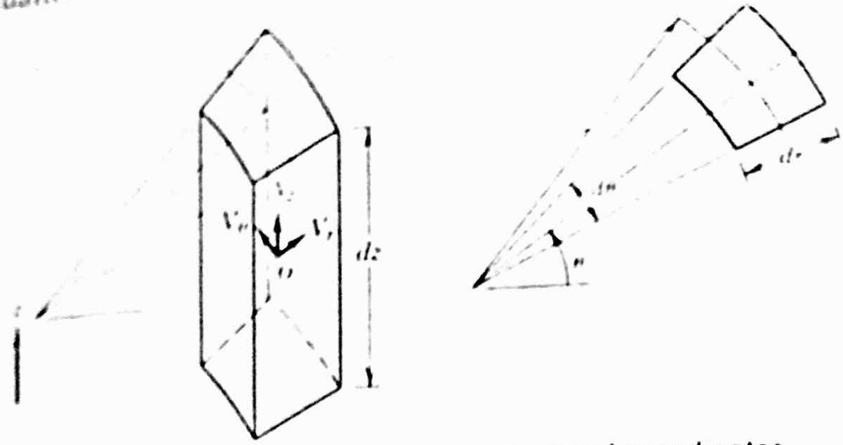


Fig. 5.2 Differential control volume in cylindrical coordinates

We see that the net rate of mass efflux through the control surface is given by

$$\left[\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z}\right] dr\, d\theta\, dz$$

The mass inside the control volume at any instant is the product of the mass per unit volume, ρ, and the volume, $r\, d\theta\, dr\, dz$. Thus the rate of change of mass inside the control volume is given by

$$\frac{\partial \rho}{\partial t} r\, d\theta\, dr\, dz$$

In cylindrical coordinates the differential equation for the conservation of mass is then

$$\rho V_r + r\frac{\partial \rho V_r}{\partial r} + \frac{\partial \rho V_\theta}{\partial \theta} + r\frac{\partial \rho V_z}{\partial z} + r\frac{\partial \rho}{\partial t} = 0$$

Dividing by r gives

$$\frac{\rho V_r}{r} + \frac{\partial \rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0$$

or

$$\frac{1}{r}\frac{\partial r \rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} + \frac{\partial \rho}{\partial t} = 0 \qquad (5.2)$$

Table 5.2 Mass Flux through the Control Surface of a Cylindrical Differential Control Volume

Surface	$\int \rho\vec{V}\cdot d\vec{A}$
Inside $(-r)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r - \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r - \frac{dr}{2}\right)d\theta\,dz = -\rho V_r r\,d\theta\,dz + \rho V_r\frac{dr}{2}\,d\theta\,dz + \rho\left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}\,d\theta\,dz + V_r\left(\frac{\partial\rho}{\partial r}\right)r\frac{dr}{2}\,d\theta\,dz$
Outside $(+r)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial r}\right)\frac{dr}{2}\right]\left[V_r + \left(\frac{\partial V_r}{\partial r}\right)\frac{dr}{2}\right]\left(r + \frac{dr}{2}\right)d\theta\,dz = \rho V_r r\,d\theta\,dz + \rho V_r\frac{dr}{2}\,d\theta\,dz + \rho\left(\frac{\partial V_r}{\partial r}\right)r\frac{dr}{2}\,d\theta\,dz + V_r\left(\frac{\partial\rho}{\partial r}\right)r\frac{dr}{2}\,d\theta\,dz$
Front $(-\theta)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\right]\left[V_\theta - \left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\right]dr\,dz = -\rho V_\theta\,dr\,dz + \rho\left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\,dr\,dz + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\,dr\,dz$
Back $(+\theta)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\right]\left[V_\theta + \left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\right]dr\,dz = \rho V_\theta\,dr\,dz + \rho\left(\frac{\partial V_\theta}{\partial\theta}\right)\frac{d\theta}{2}\,dr\,dz + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\frac{d\theta}{2}\,dr\,dz$
Bottom $(-z)$	$= -\left[\rho - \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z - \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r\,d\theta\,dr = -\rho V_z r\,d\theta\,dr + \rho\left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\,r\,d\theta\,dr + V_z\left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\,r\,d\theta\,dr$
Top $(+z)$	$= \left[\rho + \left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\right]\left[V_z + \left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\right]r\,d\theta\,dr = \rho V_z r\,d\theta\,dr + \rho\left(\frac{\partial V_z}{\partial z}\right)\frac{dz}{2}\,r\,d\theta\,dr + V_z\left(\frac{\partial\rho}{\partial z}\right)\frac{dz}{2}\,r\,d\theta\,dr$

Then,

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\rho V_r + r\left\{\rho\left(\frac{\partial V_r}{\partial r}\right) + V_r\left(\frac{\partial\rho}{\partial r}\right)\right\} + \left\{\rho\left(\frac{\partial V_\theta}{\partial\theta}\right) + V_\theta\left(\frac{\partial\rho}{\partial\theta}\right)\right\} + r\left\{\rho\left(\frac{\partial V_z}{\partial z}\right) + V_z\left(\frac{\partial\rho}{\partial z}\right)\right\}\right]dr\,d\theta\,dz$$

or

$$\int_{CS}\rho\vec{V}\cdot d\vec{A} = \left[\rho V_r + r\frac{\partial\rho V_r}{\partial r} + \frac{\partial\rho V_\theta}{\partial\theta} + r\frac{\partial\rho V_z}{\partial z}\right]dr\,d\theta\,dz$$

In cylindrical coordinates the vector operator, ∇, is given by

$$\nabla = \hat{i}_r \frac{\partial}{\partial r} + \hat{i}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{i}_z \frac{\partial}{\partial z}$$

Thus in vector notation the conservation of mass may be written[1]

$$\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For incompressible flow, ρ = constant, and Eq. 5.2 reduces to

$$\frac{1}{r}\frac{\partial r V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

For steady flow, Eq. 5.2 reduces to

$$\frac{1}{r}\frac{\partial r \rho V_r}{\partial r} + \frac{1}{r}\frac{\partial \rho V_\theta}{\partial \theta} + \frac{\partial \rho V_z}{\partial z} = 0$$

Example 5.3
Consider a one-dimensional radial flow in the $r\theta$ plane, characterized by $V_r = f(r)$ and $V_\theta = 0$. Determine the conditions on $f(r)$ required for incompressible flow.

EXAMPLE PROBLEM 5.3

GIVEN: One-dimensional radial flow in the $r\theta$ plane.

$$V_r = f(r) \qquad \text{and} \qquad V_\theta = 0$$

FIND: Requirements on $f(r)$ for incompressible flow.

SOLUTION:

Basic equation: $\nabla \cdot \rho \vec{V} + \frac{\partial \rho}{\partial t} = 0$

For incompressible flow, ρ = constant, so $\frac{\partial \rho}{\partial t} = 0$. In cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) + \frac{1}{r}\frac{\partial}{\partial \theta} V_\theta + \frac{\partial V_z}{\partial z} = 0$$

For the given velocity field, $\vec{V} = \vec{V}(r)$. $V_\theta = 0$ and partial derivatives with respect to z are zero, so

$$\frac{1}{r}\frac{\partial}{\partial r}(r V_r) = 0$$

Integrating with respect to r gives

$$r V_r = \text{constant}$$

Thus the continuity equation shows that $V_r = \frac{C}{r}$.

[1] To evaluate the operation in cylindrical coordinates we must remember that

$$\frac{\partial \hat{i}_r}{\partial \theta} = \hat{i}_\theta \qquad \text{and} \qquad \frac{\partial \hat{i}_\theta}{\partial \theta} = -\hat{i}_r$$