

*4-7 | THE ANGULAR-MOMENTUM PRINCIPLE

Next we develop a control volume expression for the angular-momentum principle. We begin with the mathematical statement for a system and use Eq. 4.11 to complete the formulation for a fixed (inertial) control volume (Section 4-7.1). To obtain the control volume formulation for a rotating (noninertial) control volume (Section 4-7.2), we first develop a suitable expression for the angular-momentum principle applied to a system in general motion. We then use Eq. 4.26 to complete the formulation for a control volume.

4-7.1 Equation for Fixed Control Volume

The angular-momentum principle for a system in an inertial frame is

$$\vec{T} = \frac{d\vec{H}}{dt}_{\text{system}} \quad (4.3a)$$

where \vec{T} = total torque exerted on the system by its surroundings, and
 \vec{H} = angular momentum of the system,

$$\vec{H} = \int_{M(\text{system})} \vec{r} \times \vec{V} dm = \int_{V(\text{system})} \vec{r} \times \vec{V} \rho dV \quad (4.3b)$$

All quantities in the system equation must be formulated with respect to an inertial reference frame. Reference frames at rest, or translating with constant linear velocity, are inertial, and Eq. 4.3b can be used directly to develop the control volume form of the angular-momentum principle. (Rotating reference frames are noninertial and will be treated in Section 4-7.2.)

The position vector, \vec{r} , locates each mass or volume element of the system with respect to the coordinate system. The torque, \vec{T} , applied to a system may be written

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g} dm + \vec{T}_{\text{shaft}} \quad (4.3c)$$

where \vec{F}_s is the surface force exerted on the system.

The relation between the system and fixed control volume formulations is

$$\left. \frac{dN}{dt} \right|_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \eta \rho dV + \int_{CS} \eta \rho \vec{V} \cdot d\vec{A} \quad (4.11)$$

* This section may be omitted without loss of continuity in the text material.

where

$$N_{\text{system}} = \int_{M(\text{system})} \eta dm$$

If we set $N = \bar{H}$, then $\eta = \vec{r} \times \vec{V}$, and

$$\frac{d\bar{H}}{dt} \Big|_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \quad (4.46)$$

Combining Eqs. 4.3a, 4.3c, and 4.46, we obtain

$$\vec{r} \times \vec{F}_s + \int_{M(\text{system})} \vec{r} \times \vec{g} dm + \vec{T}_{\text{shaft}} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A}$$

Since the system and control volume coincide at time t_0 ,

$$\vec{T}_{\text{system}} = \vec{T}_{CV}$$

and

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g} \rho dV + \vec{T}_{\text{shaft}} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V} \rho dV + \int_{CS} \vec{r} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \quad (4.47)$$

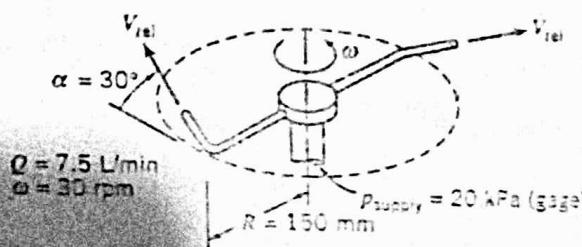
Equation 4.47 is a general formulation of the angular-momentum principle for an inertial control volume. The left side of the equation is an expression for all the torques that act on the control volume. Terms on the right express the rate of change of angular momentum within the control volume and the net rate of flux of angular momentum from the control volume. All velocities in Eq. 4.47 are measured relative to the fixed control volume.

For analysis of rotating machinery, Eq. 4.47 is often used in scalar form by considering only the component directed along the axis of rotation. This application is illustrated in Chapter 10.

The application of Eq. 4.47 to the analysis of a simple lawn sprinkler is illustrated in Example Problem 4.14. This same problem is considered in Example Problem 4.15 using the formulation of the angular-momentum principle for a rotating control volume.

EXAMPLE 4.14 Lawn Sprinkler: Analysis Using Fixed Control Volume

A small lawn sprinkler is shown in the sketch below. At an inlet gage pressure of 20 kPa, the total volume flow rate of water through the sprinkler is 7.5 liters per minute and it rotates at 30 rpm. The diameter of each jet is 4 mm. Calculate the jet speed relative to each sprinkler nozzle. Evaluate the friction torque at the sprinkler pivot.



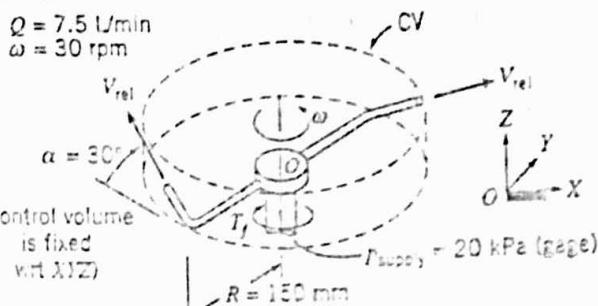
EXAMPLE PROBLEM 4.14

GIVEN: Small lawn sprinkler as shown.

FIND: (a) Jet speed relative to each nozzle.
 (b) Friction torque at pivot.

SOLUTION:

Apply continuity and angular momentum equations using fixed control volume enclosing sprinkler arms.



$$= 0(1)$$

$$\text{Basic equations: } 0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

$$\vec{F} \times \vec{F}_s + \int_{CV} \vec{F} \times \vec{g} \rho dV + \vec{T}_{shaft} = \frac{\partial}{\partial t} \int_{CV} \vec{F} \times \vec{V} \rho dV + \int_{CS} \vec{F} \times \vec{V} \rho \vec{V} \cdot d\vec{A} \quad (1)$$

where all velocities are measured relative to the inertial coordinates XYZ.

- Assumptions:** (1) Incompressible flow
 (2) Uniform flow at each section
 (3) $\vec{g} = \text{constant}$

From continuity, the jet speed relative to the nozzle is given by

$$V_{rel} = \frac{Q}{2A_{jet}} = \frac{Q}{2 \pi D_{jet}^2} \\ = \frac{1}{2} \times \frac{7.5 \text{ L}}{\text{min}} \times \frac{4}{\pi (4)^2 \text{ mm}^2} \times \frac{\text{m}^3}{1000 \text{ L}} \times \frac{10^6 \text{ mm}^2}{\text{m}^2} \times \frac{\text{min}}{60 \text{ s}}$$

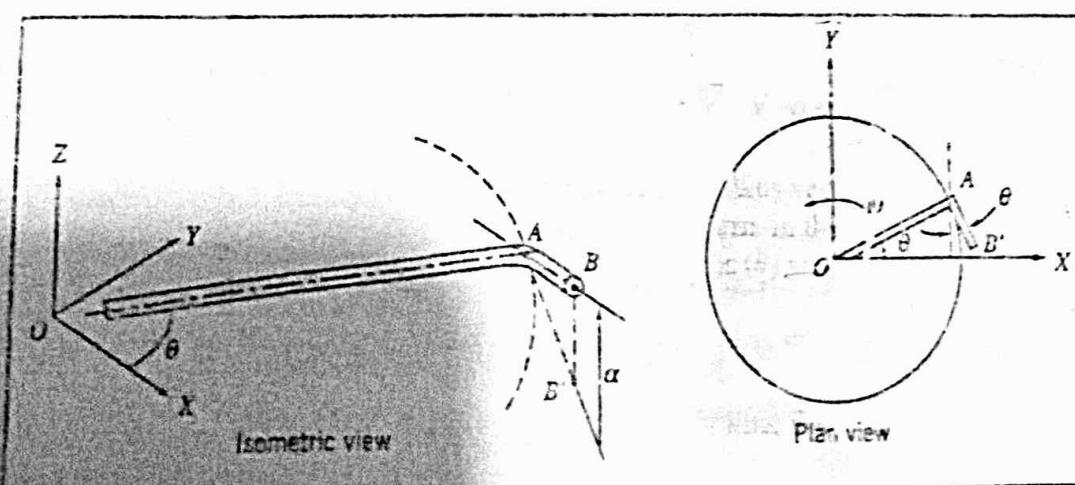
$$V_{rel} = 4.97 \text{ m/s}$$

$$V_{rel}$$

Consider terms in the angular momentum equation separately. Since atmospheric pressure acts on the entire control surface, and the pressure force at the inlet causes no moment about O, $\vec{F} \times \vec{F}_s = 0$. The moments of the body forces in the two arms are equal and opposite and hence the second term on the left side of the equation is zero. The only external torque acting on the CV is friction in the pivot. It opposes the motion, so

$$\vec{T}_{shaft} = -T_f \hat{K} \quad (2)$$

Before we can evaluate the control volume integral on the right side of Eq. 1, we need to develop expressions for the position vector, \vec{r} , and velocity vector, \vec{V} , (measured relative to the fixed coordinate system XYZ) of each element of fluid in the control volume.



OA lies in the XY plane; AB is inclined at angle α to the XY plane; point B' is the projection of point B on the XY plane.

We assume that the length, L , of the tip AB is small compared with the length, R , of the horizontal arm OA . Consequently we neglect the angular momentum of the fluid in the tips compared with the angular momentum in the horizontal arms.

Consider flow in the horizontal tube OA of length R . Denote the radial distance from O by r . At any point in the tube the fluid velocity relative to fixed coordinates XYZ is the sum of the velocity relative to the tube \vec{V}_t and the tangential velocity $r\omega \times \vec{r}$. Thus

$$\vec{V} = \hat{I}(V_t \cos \theta - r\omega \sin \theta) + \hat{J}(V_t \sin \theta + r\omega \cos \theta)$$

The position vector is

$$\vec{r} = \hat{I}r \cos \theta + \hat{J}r \sin \theta$$

$$\text{and } \vec{r} \times \vec{V} = \hat{K}(r^2 \omega \cos^2 \theta + r^2 \omega \sin^2 \theta) = R^2 \omega$$

Then

$$\int_{V_{OA}} \vec{r} \times \vec{V} \rho dV = \int_0^R R^2 \omega \rho A dr = R^3 \omega \rho A$$

and

$$\frac{\partial}{\partial t} \int_{V_{OA}} \vec{r} \times \vec{V} \rho dV = \frac{\partial}{\partial t} \left[R^3 \omega \rho A \right] = 0 \quad (3)$$

where A is the cross-sectional area of the horizontal tube. Identical results are obtained for the other horizontal tube in the control volume.

To evaluate the flux of angular momentum through the control surface we need an expression for $\vec{r}_{jet} = \vec{r}_B$ and the jet velocity, \vec{V}_j , measured relative to the fixed coordinate system XYZ . From the geometry of arm OAB ,

$$\vec{r}_B = \hat{I}(R \cos \theta + L \cos \alpha \sin \theta) + \hat{J}(R \sin \theta - L \cos \alpha \cos \theta) + \hat{K}L \sin \alpha$$

For $L \ll R$, then

$$\vec{r}_B = \hat{I}R \cos \theta + \hat{J}R \sin \theta$$

$$\vec{V}_j = \vec{V}_{rel} + \vec{V}_{sp} = \vec{V}_{rel} \cos \alpha \sin \theta - \hat{I}\vec{V}_{rel} \cos \alpha \cos \theta + \hat{K}\vec{V}_{rel} \sin \alpha - \hat{I}\omega R \sin \theta + \hat{J}\omega R \cos \theta$$

$$\vec{V}_j = \hat{I}(\vec{V}_{rel} \cos \alpha - \omega R) \sin \theta - \hat{J}(\vec{V}_{rel} \cos \alpha - \omega R) \cos \theta + \hat{K}\vec{V}_{rel} \sin \alpha$$

and

$$\vec{r}_B \times \vec{V}_j = \hat{I}R\vec{V}_{rel} \sin \alpha \sin \theta - \hat{J}R\vec{V}_{rel} \sin \alpha \cos \theta - \hat{K}R(\vec{V}_{rel} \cos \alpha - \omega R)(\sin^2 \theta + \cos^2 \theta)$$

$$\vec{r}_B \times \vec{V}_j = \hat{I}R\vec{V}_{rel} \sin \alpha \sin \theta - \hat{J}R\vec{V}_{rel} \sin \alpha \cos \theta - \hat{K}R(\vec{V}_{rel} \cos \alpha - \omega R)$$

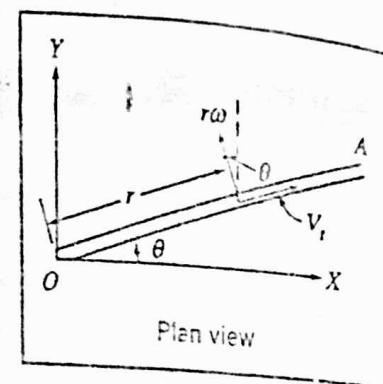
The flux integral is evaluated for flow crossing the control surface. For arm OAB ,

$$\int_{CS} \vec{r} \times \vec{V}_j \rho \vec{V} \cdot d\vec{A} = \left[\hat{I}R\vec{V}_{rel} \sin \alpha \sin \theta - \hat{J}R\vec{V}_{rel} \sin \alpha \cos \theta - \hat{K}R(\vec{V}_{rel} \cos \alpha - \omega R) \right] \rho \frac{Q}{2}$$

The velocity and radius vectors for flow in the left arm must be described in terms of the same unit vectors used for the right arm. In the left arm the \hat{I} and \hat{J} components of the cross product are of opposite sign, since $\sin(\theta + \pi) = -\sin(\theta)$ and $\cos(\theta + \pi) = -\cos(\theta)$. Thus for the complete CV,

$$\int_{CS} \vec{r} \times \vec{V}_j \rho \vec{V} \cdot d\vec{A} = -\hat{K}R(\vec{V}_{rel} \cos \alpha - \omega R) \rho Q \quad (4)$$

In the supply line, $\vec{r} \times \vec{V} = 0$ and hence term (4) represents the total angular momentum flux through the control surface.



Substituting terms (2), (3), and (4) into Eq. 1, we obtain

$$-T_f \hat{R} = -\hat{R} R(V_{rel} \cos \alpha - \omega R) \rho Q$$

$$T_f = R(V_{rel} \cos \alpha - \omega R) \rho Q$$

From the data given,

$$\omega R = \frac{10 \text{ rev}}{\text{min}} \times \frac{150 \text{ mm}}{\text{rev}} \times \frac{2\pi \text{ rad}}{\text{rev}} \times \frac{\text{min}}{60 \text{ s}} \times \frac{\text{m}}{1000 \text{ mm}} = 0.471 \frac{\text{m}}{\text{s}}$$

Substituting gives

$$T_f = 150 \text{ mm} \left(\frac{4.97 \text{ m}}{\text{s}} \times \cos 30^\circ - 0.471 \frac{\text{m}}{\text{s}} \right) 999 \frac{\text{kg}}{\text{m}^3} \times 7.5 \frac{\text{L}}{\text{min}} \\ \times \frac{\text{m}^3}{1000 \text{ L}} \times \frac{\text{min}}{60 \text{ s}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \times \frac{\text{m}}{1000 \text{ mm}}$$

$$T_f = 0.0718 \text{ N} \cdot \text{m}$$

T_f

This problem has been included to illustrate use of the angular-momentum principle for an inertial control volume. Note that in using Eq. 4.47, the angular momentum, $\vec{r} \times \vec{V}$, must be measured relative to an inertial reference frame. The problem is solved again using a noninertial control volume as Example Problem 4.15.

4-7.2 Equation for Rotating Control Volume

In problems involving rotating components, such as the rotating sprinkler of Example Problem 4.14, it is often convenient to express all fluid velocities relative to the rotating component. The most convenient control volume is a noninertial one that rotates with the component. In this section we develop a formulation of the angular-momentum principle for a noninertial control volume rotating about an axis fixed in space.

Inertial and noninertial reference frames were related in Section 4-6. Figure 4.5 showed the notation used. For a system in an inertial frame,

$$\vec{T}_{\text{system}} = \frac{d\vec{H}}{dt} \Big|_{\text{system}} \quad (4.3a)$$

The angular momentum of a system in general motion must be specified relative to an inertial reference frame. Using the notation of Fig. 4.5,

$$\vec{H}_{\text{system}} = \int_{M(\text{system})} (\vec{R} + \vec{r}) \times \vec{V}_{XYZ} d\vec{n} = \int_{V(\text{system})} (\vec{R} + \vec{r}) \times \vec{V}_{XYZ} \rho dV$$

With $\vec{R} = 0$ the xyz frame is restricted to rotation within XYZ , and the equation becomes

$$\vec{H}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times \vec{V}_{XYZ} d\vec{n} = \int_{V(\text{system})} \vec{r} \times \vec{V}_{XYZ} \rho dV$$

so that

$$\vec{T}_{\text{system}} = \frac{d}{dt} \int_{M(\text{system})} \vec{r} \times \vec{V}_{XYZ} d\vec{n}$$

Since the mass of a system is constant,

$$\bar{T}_{\text{system}} = \int_{M(\text{system})} \frac{d}{dt} (\vec{r} \times \vec{V}_{XYZ}) dm$$

or

$$\bar{T}_{\text{system}} = \int_{M(\text{system})} \left(\frac{d\vec{r}}{dt} \times \vec{V}_{XYZ} + \vec{r} \times \frac{d\vec{V}_{XYZ}}{dt} \right) dm \quad (4.48)$$

From the analysis of Section 4-6,

$$\vec{V}_{XYZ} = \vec{V}_{rf} + \frac{d\vec{r}}{dt} \quad (4.37)$$

With xyz restricted to pure rotation, $\vec{V}_{rf} = 0$. The first term under the integral on the right side of Eq. 4.48 is then

$$\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0$$

Thus Eq. 4.48 reduces to

$$\bar{T}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times \frac{d\vec{V}_{XYZ}}{dt} dm = \int_{M(\text{system})} \vec{r} \times \vec{\alpha}_{XYZ} dm \quad (4.49)$$

From Eq. 4.42 with $\vec{\alpha}_{rf} = 0$ (since xyz does not translate),

$$\vec{\alpha}_{XYZ} = \vec{\alpha}_{xyz} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}$$

Substituting into Eq. 4.49, we obtain

$$\bar{T}_{\text{system}} = \int_{M(\text{system})} \vec{r} \times [\vec{\alpha}_{xyz} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] dm$$

or

$$\begin{aligned} \bar{T}_{\text{system}} &= \int_{M(\text{system})} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] dm \\ &= \int_{M(\text{system})} \vec{r} \times \vec{\alpha}_{xyz} dm = \int_{M(\text{system})} \vec{r} \times \frac{d\vec{V}_{xyz}}{dt} dm \end{aligned} \quad (4.50)$$

We can write the last term as

$$\int_{M(\text{system})} \vec{r} \times \frac{d\vec{V}_{xyz}}{dt} dm = \frac{d}{dt} \int_{M(\text{system})} \vec{r} \times \vec{V}_{xyz} dm = \frac{d\bar{H}_{xyz}}{dt} \quad (4.51)$$

The torque on the system is given by

$$\bar{T}_{\text{system}} = \vec{r} \times \vec{F}_S + \int_{M(\text{system})} \vec{r} \times \vec{g} dm + \bar{T}_{\text{shaft}} \quad (4.52)$$

The relation between the system and control volume formulations is

$$\frac{dN}{dt} \Big|_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \eta \rho dV + \int_{CS} \eta \rho \vec{V}_{xyz} : d\vec{A} \quad (4.53)$$

where

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dn$$

Setting N equal to $\bar{H}_{xyz,\text{system}}$ and $\eta = \vec{r} \times \vec{V}_{xyz}$ yields

$$\frac{d\bar{H}_{xyz}}{dt} \Big|_{\text{system}} = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} \quad (4.52)$$

Combining Eqs. 4.50, 4.51, 4.52, and 4.3c, we obtain

$$\begin{aligned} \vec{r} \times \vec{F}_S + \int_{M(\text{system})} \vec{r} \times \vec{g} \, dn + \vec{T}_{\text{shaft}} \\ - \int_{M(\text{system})} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] \, dn \\ = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} \end{aligned}$$

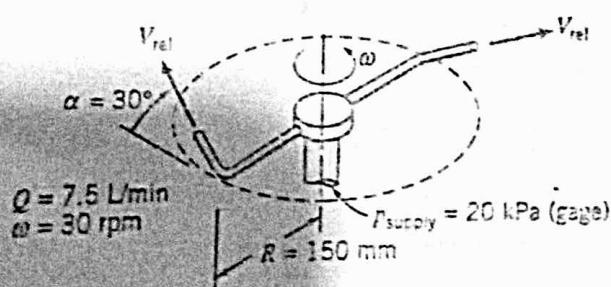
Since the system and control volume coincided at t_0 ,

$$\begin{aligned} \vec{r} \times \vec{F}_S + \int_{CV} \vec{r} \times \vec{g} \rho \, dV + \vec{T}_{\text{shaft}} \\ - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}] \rho \, dV \\ = \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho \, dV + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} \end{aligned} \quad (4.53)$$

Equation 4.53 is the formulation of the angular-momentum principle for a (noninertial) control volume rotating about an axis fixed in space. All fluid velocities in Eq. 4.53 are evaluated relative to the control volume. Application of the equation to a rotating sprinkler is illustrated in Example Problem 4.15.

EXAMPLE 4.15 Lawn Sprinkler: Analysis Using Rotating Control Volume

A small lawn sprinkler is shown in the sketch below. At an inlet gage pressure of 20 kPa, the total volume flow rate of water through the sprinkler is 7.5 liters per minute and it rotates at 30 rpm. The diameter of each jet is 4 mm. Calculate the jet speed relative to each sprinkler nozzle. Evaluate the friction torque at the sprinkler pivot.



EXAMPLE PROBLEM 4.15

GIVEN: Small lawn sprinkler as shown.

FIND: (a) Jet speed relative to each nozzle.
 (b) Friction torque at pivot.

SOLUTION:

Apply continuity and angular momentum equations using rotating control volume enclosing sprinkler arms.

$$= 0(1)$$

$$\text{Basic equations: } 0 = \frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V}_{xy} \cdot d\vec{A}$$

$$= 0(3)$$

$$\vec{r} \times \vec{F}_s + \int_{CV} \vec{r} \times \vec{g} \rho dV + \vec{T}_{\text{shaft}} - \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\rho} \times \vec{r}] \rho dV = 0(1)$$

$$= \frac{\partial}{\partial t} \int_{CV} \vec{r} \times \vec{V}_{xyz} \rho dV + \int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} \quad (4.53)$$

Assumptions: (1) Steady flow relative to the rotating CV
 (2) Uniform flow at each section
 (3) $\vec{\omega}$ = constant

From continuity

$$V_{\text{rel}} = \frac{Q}{2A_{\text{jet}}} = \frac{Q}{2} \frac{4}{\pi D_{\text{jet}}^2}$$

$$= \frac{1}{2} \times \frac{7.5}{\text{min}} \times \frac{4}{\pi (4)^2 \text{mm}^2} \times \frac{\text{m}^3}{1000 \text{L}} \times \frac{10^6 \text{mm}^2}{\text{m}^2} \times \frac{\text{min}}{60 \text{s}}$$

$$V_{\text{rel}} = 4.97 \text{ m/s}$$

Consider terms in the angular-momentum equation separately. As in Example Problem 4.14, the only external torque acting on the CV is friction in the pivot. It opposes the motion, so

$$\vec{T}_{\text{shaft}} = -T_f \hat{k}$$

$$V_{\text{rel}}$$

The integral on the left is evaluated for flow within the CV. Let the velocity and area within the sprinkler tubes be V_{cv} and A_{cv} , respectively. Then, for one side, the first term is

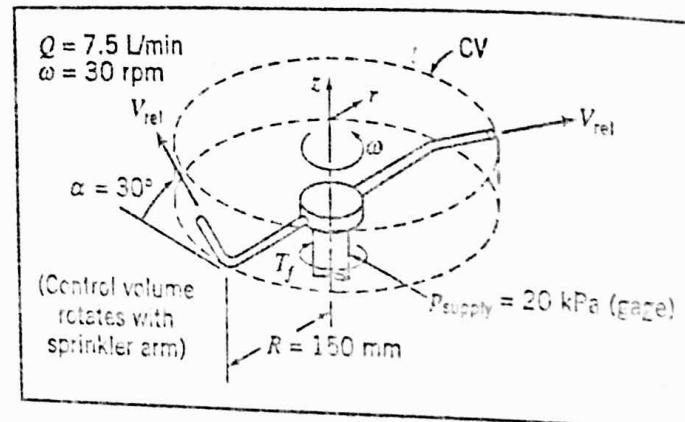
$$\begin{aligned} \int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz}] \rho dV &= \int_0^R r \hat{e}_r \times [2\omega \hat{k} \times V_{\text{cv}} \hat{e}_r] \rho A_{\text{cv}} dr \\ &= \int_0^R r \hat{e}_r \times 2\omega V_{\text{cv}} \hat{e}_s \rho A_{\text{cv}} dr \\ &= \int_0^R 2\omega V_{\text{cv}} \rho A_{\text{cv}} r dr \hat{k} = \omega R^2 \rho V_{\text{cv}} A_{\text{cv}} \hat{k} \end{aligned}$$

{one side}

(The flow in the bent portion of the tube has no r component of velocity, so it does not contribute to the integral.)

From continuity, $Q = 2V_{\text{cv}} A_{\text{cv}}$, so for both sides the integral becomes

$$\int_{CV} \vec{r} \times [2\vec{\omega} \times \vec{V}_{xyz}] \rho dV = \omega R^2 \rho Q \hat{k} \quad (2)$$



The second term in the integral is evaluated as

$$\begin{aligned}\int_{CV} \vec{F} \times [\vec{\omega} \times (\vec{\omega} \times \vec{r})] \rho dV &= \int_{CV} r \hat{e}_r \times [\omega \hat{k} \times (\omega \hat{k} \times r \hat{e}_r)] \rho dV \\ &= \int_{CV} r \hat{e}_r \times [\omega \hat{k} \times \omega r \hat{e}_\theta] \rho dV = \int_{CV} r \hat{e}_r \times \omega^2 r (-\hat{e}_r) \rho dV = 0\end{aligned}$$

so it contributes no torque.

The integral on the right side of Eq. 4.53 is evaluated for flow crossing the control surface. For the right arm of the sprinkler,

$$\begin{aligned}\int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} &= R \hat{e}_r \times V_{rel} [\cos \alpha (-\hat{e}_\theta) + \sin \alpha \hat{k}] \{ + \rho V_{rel} A_{jet} \} \\ &= RV_{rel} [\cos \alpha (-\hat{k}) + \sin \alpha (-\hat{e}_\theta)] \rho \frac{Q}{2}\end{aligned}$$

The velocity and radius vectors for flow in the left arm must be described in terms of the same unit vectors used for the right arm. In the left sprinkler arm, the θ component has the same magnitude but opposite sign, so it cancels. For the complete CV,

$$\int_{CS} \vec{r} \times \vec{V}_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} = -RV_{rel} \cos \alpha \rho Q \hat{k} \quad (3)$$

Combining terms (1), (2), and (3), we obtain

$$-T_f \hat{k} - \omega R^2 \rho Q \hat{k} = -RV_{rel} \cos \alpha \rho Q \hat{k}$$

or

$$T_f = R(V_{rel} \cos \alpha - \omega R) \rho Q$$

From the data given,

$$\omega R = 30 \frac{\text{rev}}{\text{min}} \times 150 \text{ mm} \times \frac{2\pi \text{ rad}}{\text{rev}} \times \frac{\text{min}}{60 \text{ s}} \times \frac{\text{m}}{1000 \text{ mm}} = 0.471 \frac{\text{m}}{\text{s}}$$

Substituting gives

$$\begin{aligned}T_f &= 150 \text{ mm} \left(4.97 \frac{\text{m}}{\text{s}} \times \cos 30^\circ - 0.471 \frac{\text{m}}{\text{s}} \right) 999 \frac{\text{kg}}{\text{m}^3} \times 7.5 \frac{\text{L}}{\text{min}} \\ &\quad \times \frac{\text{m}^3}{1000 \text{ L}} \times \frac{\text{min}}{60 \text{ s}} \times \frac{\text{N} \cdot \text{s}^2}{\text{kg} \cdot \text{m}} \times \frac{\text{m}}{1000 \text{ mm}}\end{aligned}$$

$$T_f = 0.0713 \text{ N} \cdot \text{m}$$

This problem has been included to illustrate use of the angular-momentum equation for a (noninertial) rotating control volume. The result is identical to the result obtained using the fixed control volume analysis of Example Problem 4.14.