*4-6 MOMENTUM EQUATION FOR CONTROL VOLUME WITH ARBITRARY ACCELERATION

In Section 4-5 we formulated the momentum equation for a control volume with rectilinear acceleration. The purpose of this section is to extend the formulation to include rotation and angular acceleration of the control volume, in addition to translation and rectilinear acceleration.

First, we develop an expression for Newton's second law in an arbitrary, noninertial coordinate system. Then we use Eq. 4.26 to complete the formulation for a control volume. Newton's second law for a system moving relative to an inertial coordinate system is given by

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{\text{system}}$$

Since

$$\left.\vec{P}_{XYZ}\right)_{\text{system}} = \int_{M(\text{system})} \vec{V}_{XYZ}\, dm$$

and M(system) is constant,

$$\vec{F} = \frac{d}{dt}\int_{M(\text{system})} \vec{V}_{XYZ}\, dm = \int_{M(\text{system})} \frac{d\vec{V}_{XYZ}}{dt}\, dm$$

or

$$\vec{F} = \int_{M(\text{system})} \vec{a}_{XYZ}\, dm \tag{4.36}$$

The basic problem is to relate $\vec{a}_{XYZ}$ to the acceleration $\vec{a}_{xyz}$, measured relative to a noninertial coordinate system. For this purpose, consider the noninertial reference frame, xyz, shown in Fig. 4.5.

The noninertial frame, xyz, is located by position vector $\vec{R}$ relative to the fixed frame. The noninertial frame rotates with angular velocity $\vec{\omega}$.[2] A particle is located relative to the moving frame by position vector $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$. Relative to inertial reference frame XYZ, the position of the particle is denoted by position vector $\vec{X}$. From the geometry of the figure, $\vec{X} = \vec{R} + \vec{r}$.

* This section may be omitted without loss of continuity in the text material.

[2] Note that any arbitrary rigid-body motion can be decomposed into a translation plus a rotation.

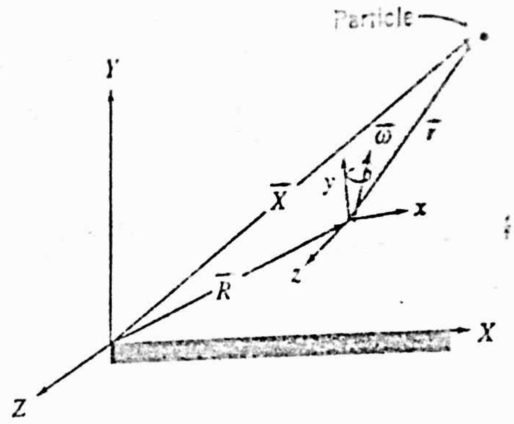


Fig. 4.5 Location of a particle in inertial (XYZ) and noninertial (xyz) reference frames.

The velocity of the particle, relative to an observer in the XYZ system, is

$$\vec{V}_{XYZ} = \frac{d\vec{X}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt} = \vec{V}_{rf} + \frac{d\vec{r}}{dt} \tag{4.37}$$

We must be careful in evaluating $d\vec{r}/dt$ because both the magnitude, $|\vec{r}|$, and the orientation of the unit vectors, $\hat{i}$, $\hat{j}$, and $\hat{k}$, are functions of time. Thus

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k}) = \hat{i}\frac{dx}{dt} + x\frac{d\hat{i}}{dt} + \hat{j}\frac{dy}{dt} + y\frac{d\hat{j}}{dt} + \hat{k}\frac{dz}{dt} + z\frac{d\hat{k}}{dt} \tag{4.38a}$$

The terms dx/dt, dy/dt, and dz/dt are the velocity components of the particle relative to xyz. Thus

$$\vec{V}_{xyz} = \hat{i}\frac{dx}{dt} + \hat{j}\frac{dy}{dt} + \hat{k}\frac{dz}{dt} \tag{4.38b}$$

You may recall from dynamics (see Example Problem 4.13) that for a rotating coordinate system,

$$\vec{\omega} \times \vec{r} = x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt} \tag{4.38c}$$

Combining Eqs. 4.38a, 4.38b, and 4.38c, we obtain

$$\frac{d\vec{r}}{dt} = \vec{V}_{xyz} + \vec{\omega} \times \vec{r} \tag{4.38d}$$

Substituting into Eq. 4.37 gives

$$\vec{V}_{XYZ} = \vec{V}_{rf} + \vec{V}_{xyz} + \vec{\omega} \times \vec{r} \tag{4.39}$$

The acceleration of the particle relative to an observer in the XYZ system is

$$\vec{a}_{XYZ} = \frac{d\vec{V}_{XYZ}}{dt} = \frac{d\vec{V}_{rf}}{dt} + \frac{d\vec{V}_{xyz}}{dt} + \frac{d}{dt}(\vec{\omega} \times \vec{r})$$

or

$$\vec{a}_{XYZ} = \vec{a}_{rf} + \frac{d\vec{V}_{xyz}}{dt} + \frac{d}{dt}(\vec{\omega} \times \vec{r}) \tag{4.40}$$

Both $\vec{V}_{xyz}$ and $\vec{r}$ are measured relative to xyz, so the same caution observed in developing Eq. 4.38d applies. Thus

$$\frac{d\vec{V}_{xyz}}{dt} = \vec{a}_{xyz} + \vec{\omega} \times \vec{V}_{xyz} \tag{4.41a}$$

and

$$\begin{aligned}\frac{d}{dt}(\vec{\omega} \times \vec{r}) &= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \\ &= \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{V}_{xyz} + \vec{\omega} \times \vec{r})\end{aligned}$$

or

$$\frac{d}{dt}(\vec{\omega} \times \vec{r}) = \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \tag{4.41b}$$

Substituting Eqs. 4.41a and 4.41b into Eq. 4.40, we obtain

$$\vec{a}_{XYZ} = \vec{a}_{rf} + \vec{a}_{xyz} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \tag{4.42}$$

The physical meaning of each term in Eq. 4.42 is

$\vec{a}_{XYZ}$	: Absolute rectilinear acceleration of a particle relative to fixed reference frame XYZ
$\vec{a}_{rf}$	: Absolute rectilinear acceleration of origin of moving reference frame xyz relative to fixed frame XYZ
$\vec{a}_{xyz}$	: Rectilinear acceleration of a particle *relative* to moving reference frame xyz (this acceleration would be that seen by an observer on moving frame xyz)
$2\vec{\omega} \times \vec{V}_{xyz}$	: Coriolis acceleration due to motion of the particle *within* moving frame xyz
$\vec{\omega} \times (\vec{\omega} \times \vec{r})$	: Centripetal acceleration due to rotation of moving frame xyz
$\dot{\vec{\omega}} \times \vec{r}$	: Tangential acceleration due to angular acceleration of moving reference frame xyz

Substituting $\vec{a}_{XYZ}$, as given by Eq. 4.42, into Eq. 4.36, we obtain

$$\vec{F}_{\text{system}} = \int_{M(\text{system})} [\vec{a}_{rf} + \vec{a}_{xyz} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\, dm$$

or

$$\vec{F} - \int_{M(\text{system})} [\vec{a}_{rf} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\, dm = \int_{M(\text{system})} \vec{a}_{xyz}\, dm \tag{4.43a}$$

But

$$\int_{M(\text{system})} \vec{a}_{xyz}\, dm = \int_{M(\text{system})} \frac{d\vec{V}_{xyz}}{dt}\, dm = \frac{d}{dt}\int_{M(\text{system})} \vec{V}_{xyz}\, dm = \left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} \tag{4.43b}$$

Combining Eqs. 4.43a and 4.43b, we obtain

$$\vec{F} - \int_{M(\text{system})} [\vec{a}_{rf} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\, dm = \left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}}$$

or

$$\vec{F}_S + \vec{F}_B - \int_{V(\text{system})} [\vec{a}_{rf} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\rho\, dV = \left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} \tag{4.44}$$

Equation 4.44 is a statement of Newton's second law for a system. The system derivative, $d\vec{P}_{xyz}/dt$, represents the rate of change of momentum, $\vec{P}_{xyz}$, of the system measured relative to xyz, as seen by an observer in xyz. This system derivative can be related to control volume variables through Eq. 4.26,

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{CV} \eta\rho\, dV + \int_{CS} \eta\rho\, \vec{V}_{xyz} \cdot d\vec{A} \tag{4.26}$$

To obtain the control volume formulation, we set $N = \vec{P}_{xyz}$, and $\eta = \vec{V}_{xyz}$. Then Eqs. 4.26 and 4.44 may be combined to give

$$\vec{F}_S + \vec{F}_B - \int_{CV} [\vec{a}_{rf} + 2\vec{\omega} \times \vec{V}_{xyz} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}]\rho\, dV = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho\, dV + \int_{CS} \vec{V}_{xyz}\rho\vec{V}_{xyz} \cdot d\vec{A} \tag{4.45}$$

Equation 4.45 is the most general control volume formulation of Newton's second law. Comparing the momentum equation for a control volume moving with arbitrary acceleration, Eq. 4.45, with that for a control volume moving with rectilinear acceleration, Eq. 4.34, we see that the only difference is the presence of three additional terms on the left side of Eq. 4.45. These terms result from the angular motion of noninertial reference frame xyz. Note that Eq. 4.45 reduces to Eq. 4.34 when the angular terms are zero and to Eq. 4.27 for an inertial control volume.

The precautions concerning the use of Eqs. 4.27 and 4.34 also apply to the use of Eq. 4.45. Before attempting to apply this equation, one must draw the boundaries of the control volume and label appropriate coordinate directions. For a control volume moving with arbitrary acceleration, one must label a coordinate system (xyz) on the control volume and an inertial reference frame (XYZ).

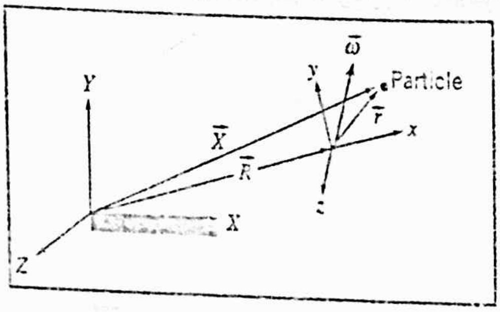
EXAMPLE 4.13 Velocity in Fixed and Noninertial Reference Frames

A reference frame, xyz, moves arbitrarily with respect to a fixed frame, XYZ. A particle moves with velocity $\vec{V}_{xyz} = (dx/dt)\hat{i} + (dy/dt)\hat{j} + (dz/dt)\hat{k}$, relative to frame xyz. Show that the absolute velocity of the particle is given by

$$\vec{V}_{XYZ} = \vec{V}_{rf} + \vec{V}_{xyz} + \vec{\omega} \times \vec{r}$$

EXAMPLE PROBLEM 4.13

GIVEN: Fixed and noninertial frames as shown.



FIND: $\vec{V}_{XYZ}$ in terms of $\vec{V}_{xyz}$, $\vec{\omega}$, $\vec{r}$, and $\vec{V}_{rf}$.

SOLUTION:

From the geometry of the sketch, $\vec{X} = \vec{R} + \vec{r}$, so

$$\vec{V}_{XYZ} = \frac{d\vec{X}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}}{dt} = \vec{V}_{rf} + \frac{d\vec{r}}{dt}$$

Since

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

we have

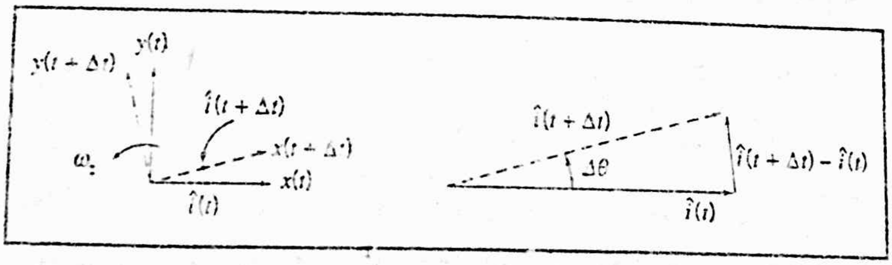
$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

or

$$\frac{d\vec{r}}{dt} = \vec{V}_{xyz} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

The problem now is to evaluate $d\hat{i}/dt$, $d\hat{j}/dt$, and $d\hat{k}/dt$ due to the angular motion of frame xyz. To evaluate these derivatives, we must consider the rotation of each unit vector due to the three components of the angular velocity, $\vec{\omega}$, of frame xyz.

Consider the unit vector $\hat{i}$. It will rotate in the xy plane because of ω_z, as follows:



Now from the diagram

$$\hat{i}(t + \Delta t) - \hat{i}(t) = (1)\sin\Delta\theta\,\hat{j} + (1)(1 - \cos\Delta\theta)(-\hat{i})$$

But for small angles $\cos\Delta\theta \approx 1 - [(\Delta\theta)^2/2]$ and $\sin\Delta\theta \approx \Delta\theta$, so

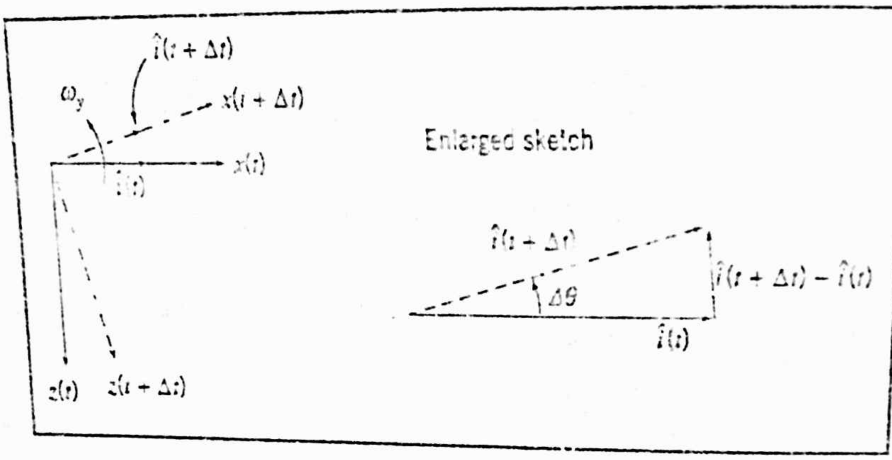
$$\hat{i}(t + \Delta t) - \hat{i}(t) = (1)\Delta\theta\,\hat{j} + (1)\frac{(\Delta\theta)^2}{2}(-\hat{i}) = (1)\Delta\theta\left[\hat{j} - \frac{\Delta\theta}{2}\hat{i}\right]$$

In the limit as $\Delta t \to 0$, since $\Delta\theta = \omega_z \Delta t$,

$$\left.\frac{d\hat{i}}{dt}\right]_{\text{due to } \omega_z} = \lim_{\Delta t \to 0}\left[\frac{\hat{i}(t+\Delta t) - \hat{i}(t)}{\Delta t}\right] = \lim_{\Delta t \to 0}\left[\frac{(1)\,\omega_z \Delta t\left[\hat{j} - \frac{\omega_z \Delta t}{2}\hat{i}\right]}{\Delta t}\right]$$

$$\left.\frac{d\hat{i}}{dt}\right]_{\text{due to } \omega_z} = \hat{j}\omega_z$$

Similarly, $\hat{i}$ will rotate in the xz plane because of ω_y.



Then from the diagram

$$\hat{i}(t+\Delta t) - \hat{i}(t) = (1)\sin\Delta\theta(-\hat{k}) + (1)(1-\cos\Delta\theta)(-\hat{i})$$

For small angles

$$\hat{i}(t+\Delta t) - \hat{i}(t) = (1)\Delta\theta(-\hat{k}) + (1)\frac{(\Delta\theta)^2}{2}(-\hat{i}) = (1)\Delta\theta\left(-\hat{k} - \frac{\Delta\theta}{2}\hat{i}\right)$$

In the limit as $\Delta t \to 0$, since $\Delta\theta = \omega_y \Delta t$,

$$\left.\frac{d\hat{i}}{dt}\right]_{\text{due to } \omega_y} = \lim_{\Delta t \to 0}\left[\frac{\hat{i}(t+\Delta t) - \hat{i}(t)}{\Delta t}\right] = \lim_{\Delta t \to 0}\left[\frac{(1)\,\omega_y \Delta t\left[-\hat{k} - \frac{\omega_y \Delta t}{2}\hat{i}\right]}{\Delta t}\right]$$

$$\left.\frac{d\hat{i}}{dt}\right]_{\text{due to } \omega_y} = -\hat{k}\omega_y$$

Rotation in the yz plane because of ω_x does not affect $\hat{i}$. Combining terms,

$$\frac{d\hat{i}}{dt} = \omega_z\hat{j} - \omega_y\hat{k}$$

By similar reasoning,

$$\frac{d\hat{j}}{dt} = \omega_x\hat{k} - \omega_z\hat{i} \quad \text{and} \quad \frac{d\hat{k}}{dt} = \omega_y\hat{i} - \omega_x\hat{j}$$

Thus

$$x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt} = (z\omega_y - y\omega_z)\hat{i} + (x\omega_z - z\omega_x)\hat{j} + (y\omega_x - x\omega_y)\hat{k}$$

But

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = (z\omega_y - y\omega_z)\hat{i} + (x\omega_z - z\omega_x)\hat{j} + (y\omega_x - x\omega_y)\hat{k}$$

Combining these results, we obtain

$$\vec{V}_{XYZ} = \vec{V}_{rf} + \vec{V}_{xyz} + \vec{\omega} \times \vec{r} \qquad \vec{V}_{XYZ}$$