4-5 MOMENTUM EQUATION FOR CONTROL VOLUME WITH RECTILINEAR ACCELERATION

For an inertial control volume (having no acceleration relative to a stationary frame of reference), the appropriate formulation of Newton's second law is given by Eq. 4.27,

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A} \tag{4.27}$$

Not all control volumes are inertial; for example, a rocket must accelerate if it is to get off the ground. Since we are interested in analyzing control volumes that may accelerate relative to inertial coordinates, it is logical to ask whether Eq. 4.27 can be used for an accelerating control volume. To answer this question, let us briefly review the two major elements used in developing Eq. 4.27.

First, in relating the system derivatives to the control volume formulation (Eq. 4.26 or 4.11), the control volume was fixed relative to xyz; the flow field, $\vec{V}(x, y, z, t)$, was specified relative to the coordinates x, y, and z. No restriction was placed on the motion of the xyz reference frame. Consequently, Eq. 4.26 (or Eq. 4.11) is valid at any instant for any arbitrary motion of the coordinates x, y, and z provided that all velocities in the equation are measured relative to the control volume.

Second, the system equation

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{\forall\,(\text{system})} \vec{V}\rho\, d\forall \tag{4.2b}$$

is valid only for velocities measured relative to an inertial reference frame. Thus, if we denote the inertial reference frame by XYZ, then Newton's second law states that

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{\text{system}} \tag{4.28}$$

Since the time derivatives of $\vec{P}_{XYZ}$ and $\vec{P}_{xyz}$ are not equal when reference frame xyz is accelerating relative to the inertial reference frame, Eq. 4.27 is not valid for an accelerating control volume.

To develop the momentum equation for a linearly accelerating control volume, it is necessary to relate $\vec{P}_{XYZ}$ of the system to $\vec{P}_{xyz}$ of the system. The system derivative $d\vec{P}_{xyz}/dt$ can be related to control volume variables through Eq. 4.26. We begin by writing Newton's second law for a system, remembering that the acceleration must be measured relative to an inertial reference frame that we have designated XYZ. We write

$$\vec{F} = \left.\frac{d\vec{P}_{XYZ}}{dt}\right)_{\text{system}} = \frac{d}{dt}\int_{M(\text{system})} \vec{V}_{XYZ}\, dm = \int_{M(\text{system})} \frac{d\vec{V}_{XYZ}}{dt}\, dm \qquad (4.29)$$

The velocities with respect to the inertial (XYZ) and the control volume coordinates (xyz) are related by the relative-motion equation

$$\vec{V}_{XYZ} = \vec{V}_{xyz} + \vec{V}_{rf} \qquad (4.30)$$

where $\vec{V}_{rf}$ is the velocity of the control volume reference frame.

Since the motion of xyz is pure translation, without rotation, relative to inertial reference frame XYZ, then

$$\frac{d\vec{V}_{XYZ}}{dt} = \vec{a}_{XYZ} = \frac{d\vec{V}_{xyz}}{dt} + \frac{d\vec{V}_{rf}}{dt} = \vec{a}_{xyz} + \vec{a}_{rf} \qquad (4.31)$$

where

$\vec{a}_{XYZ}$ is the rectilinear acceleration of the system relative to inertial reference frame XYZ.

$\vec{a}_{xyz}$ is the rectilinear acceleration of the system relative to noninertial reference frame xyz, and

$\vec{a}_{rf}$ is the rectilinear acceleration of noninertial reference frame xyz relative to inertial frame XYZ.

Substituting from Eq. 4.31 into Eq. 4.29 gives

$$\vec{F} = \int_{M(\text{system})} \vec{a}_{rf}\, dm + \int_{M(\text{system})} \frac{d\vec{V}_{xyz}}{dt}\, dm$$

or

$$\vec{F} - \int_{M(\text{system})} \vec{a}_{rf}\, dm = \left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} \qquad (4.32a)$$

where the linear momentum of the system is given by

$$\left.\vec{P}_{xyz}\right)_{\text{system}} = \int_{M(\text{system})} \vec{V}_{xyz}\, dm = \int_{\forall(\text{system})} \vec{V}_{xyz}\rho\, d\forall \qquad (4.32b)$$

and the force, $\vec{F}$, includes all surface and body forces acting on the system.

To derive the control volume formulation of Newton's second law, we set

$$N = \vec{P}_{xyz} \qquad \text{and} \qquad \eta = \vec{V}_{xyz}$$

From Eq. 4.26, with this substitution, we obtain

$$\left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\rho\, d\forall + \int_{\text{CS}} \vec{V}_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A} \qquad (4.33)$$

From the system equation,

$$\left.\frac{d\vec{P}_{xyz}}{dt}\right)_{\text{system}} = \vec{F}_{\text{on system}} - \int_{\mathcal{V}(\text{system})} \vec{a}_{rf}\,\rho\, d\mathcal{V} \tag{4.32a}$$

Since the system and the control volume coincide at t_0,

$$\vec{F}_{\text{on system}} - \int_{\mathcal{V}(\text{system})} \vec{a}_{rf}\,\rho\, d\mathcal{V} = \vec{F}_{\text{on CV}} - \int_{\text{CV}} \vec{a}_{rf}\,\rho\, d\mathcal{V}$$

In light of this, Eqs. 4.32a and 4.33 may be combined to yield the formulation of Newton's second law for a control volume accelerating, without rotation, relative to an inertial reference frame:

$$\vec{F} - \int_{\text{CV}} \vec{a}_{rf}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\,\rho\, d\mathcal{V} + \int_{\text{CS}} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Since $\vec{F} = \vec{F}_S + \vec{F}_B$, this equation becomes

$$\vec{F}_S + \vec{F}_B - \int_{\text{CV}} \vec{a}_{rf}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}_{xyz}\,\rho\, d\mathcal{V} + \int_{\text{CS}} \vec{V}_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \tag{4.34}$$

Comparing the momentum equation for a control volume with rectilinear acceleration, Eq. 4.34, to that for a nonaccelerating control volume, Eq. 4.27, we see that the only difference is the presence of one additional term in Eq. 4.34. When the control volume is not accelerating relative to inertial reference frame XYZ, then $\vec{a}_{rf} = 0$, and Eq. 4.34 reduces to Eq. 4.27.

The precautions concerning the use of Eq. 4.27 also apply to the use of Eq. 4.34. Before attempting to apply either equation, one must draw the boundaries of the control volume and label appropriate coordinate directions. For an accelerating control volume, one must label two coordinate systems: one (xyz) on the control volume and the other (XYZ) an inertial reference frame.

In Eq. 4.34, $\vec{F}_S$ represents all surface forces acting on the control volume. Since the mass within the control volume may vary with time, both the remaining terms on the left side of the equation may be functions of time. Furthermore, the acceleration, $\vec{a}_{rf}$, of the reference frame xyz relative to an inertial frame, will in general be a function of time.

All velocities in Eq. 4.34 are measured relative to the control volume. The momentum flux, $\vec{V}_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$, through an element of the control surface area, $d\vec{A}$, is a vector. The sign of the scalar product, $\rho\vec{V}_{xyz}\cdot d\vec{A}$, depends on the direction of the velocity vector, $\vec{V}_{xyz}$, relative to the area vector, $d\vec{A}$. The signs of the components of the vector velocity, $\vec{V}_{xyz}$, depend on the coordinate system chosen.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Eq. 4.34 are

$$F_{S_x} + F_{B_x} - \int_{\text{CV}} a_{rf_x}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{CV}} u_{xyz}\,\rho\, d\mathcal{V} + \int_{\text{CS}} u_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \tag{4.35a}$$

$$F_{S_y} + F_{B_y} - \int_{\text{CV}} a_{rf_y}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{CV}} v_{xyz}\,\rho\, d\mathcal{V} + \int_{\text{CS}} v_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \tag{4.35b}$$

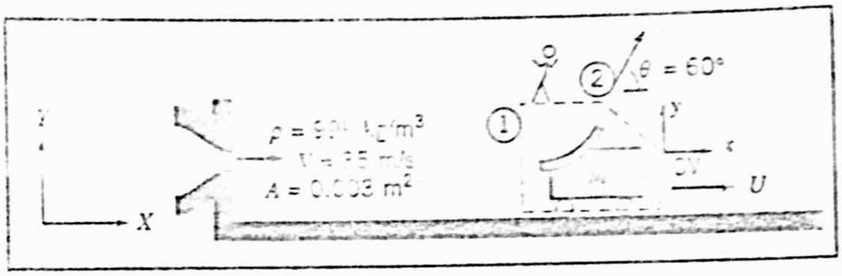
$$F_{S_z} + F_{B_z} - \int_{\text{CV}} a_{rf_z}\,\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{\text{CV}} w_{xyz}\,\rho\, d\mathcal{V} + \int_{\text{CS}} w_{xyz}\,\rho \vec{V}_{xyz}\cdot d\vec{A} \tag{4.35c}$$

EXAMPLE 4.11 Vane Moving with Rectilinear Acceleration

A vane, with turning angle $\theta = 60°$, is attached to a cart. The cart and vane, of mass $M = 75$ kg, roll on a level track. Friction and air resistance may be neglected. The vane receives a jet of water, which leaves a stationary nozzle horizontally at $V = 35$ m/s. The nozzle exit area is $A = 0.003$ m^2. Determine the velocity of the cart as a function of time and plot the results.

EXAMPLE PROBLEM 4.11

GIVEN: Vane and cart as sketched, with $M = 75$ kg.



FIND: $U(t)$ and plot results.

SOLUTION:

Choose the control volume and coordinate systems shown for the analysis. Note that XY is a fixed frame, while frame xy moves with the cart. Apply the x component of the momentum equation.

Basic equation:

$$\overset{=0(1)}{F_{S_x}} + \overset{=0(2)}{F_{B_x}} - \int_{CV} a_{rf_x}\rho\, d\forall = \overset{\approx 0(4)}{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) $F_{S_x} = 0$, since no resistance is present
(2) $F_{B_x} = 0$
(3) Neglect the mass of water in contact with the vane compared to the cart mass
(4) Neglect rate of change of momentum of liquid inside the CV

$$\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall \simeq 0$$

(5) Uniform flow at sections ① and ②
(6) Speed of water stream is not slowed by friction on the vane, so $|\vec{V}_{xyz_1}| = |\vec{V}_{xyz_2}|$
(7) $A_2 = A_1 = A$

Then

$$-\int_{CV} a_{rf_x}\rho\, d\forall = u_{xyz_1}\{-|\rho V_{xyz_1} A_1|\} + u_{xyz_2}\{|\rho V_{xyz_2} A_2|\}$$

where all velocities must be measured relative to the xyz frame. Dropping subscripts rf and xyz, we obtain

$$-\int_{CV} a_x\rho\, d\forall = u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\} \qquad (1)$$

Evaluating these terms separately gives

$$-\int_{CV} a_x\rho\, d\forall = -a_x M_{CV} = -a_x M = -\frac{dU}{dt}M$$

$$u_1\{-|\rho V_1 A_1|\} = (V-U)\{-|\rho(V-U)A|\} = -\rho(V-U)^2 A$$

$$u_2\{|\rho V_2 A_2|\} = (V-U)\cos\theta\{|\rho(V-U)A|\} = \rho(V-U)^2 A\cos\theta$$

Absolute value signs have been dropped from the flux terms, since $V \geq U$. Substitution into Eq. 1 gives

$$-M\frac{dU}{dt} = -\rho(V-U)^2A + \rho(V-U)^2A\cos\theta$$

or

$$-M\frac{dU}{dt} = (\cos\theta - 1)\rho(V-U)^2A$$

Separating variables, we obtain

$$\frac{dU}{(V-U)^2} = \frac{(1-\cos\theta)\rho A}{M}dt = b\,dt \qquad \text{where } b = \frac{(1-\cos\theta)\rho A}{M}$$

Note that since V = constant, $dU = -d(V-U)$. Integrating between limits $U = 0$ at $t = 0$, and $U = U$ at $t = t$,

$$\int_0^U \frac{dU}{(V-U)^2} = \int_0^U \frac{-d(V-U)}{(V-U)^2} = \frac{1}{(V-U)}\bigg]_0^U = \int_0^t b\,dt = bt$$

or

$$\frac{1}{(V-U)} - \frac{1}{V} = \frac{U}{V(V-U)} = bt$$

Solving for U, we obtain

$$\frac{U}{V} = \frac{Vbt}{1+Vbt}$$

Evaluating Vb gives

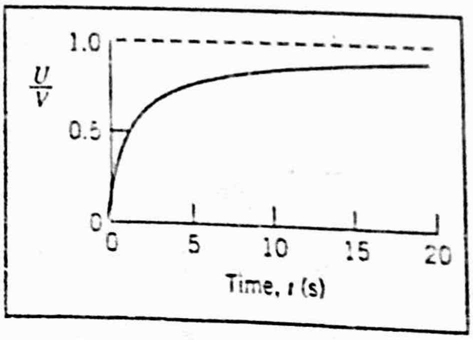
$$Vb = V\frac{(1-\cos\theta)\rho A}{M}$$

$$Vb = 35\,\frac{\text{m}}{\text{s}} \times \frac{(1-0.5)}{75\text{ kg}} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 0.003\text{ m}^2 = 0.699\text{ s}^{-1}$$

Thus

$$\frac{U}{V} = \frac{0.699t}{1+0.699t} \qquad (t \text{ in seconds}) \qquad U(t)$$

Plot:



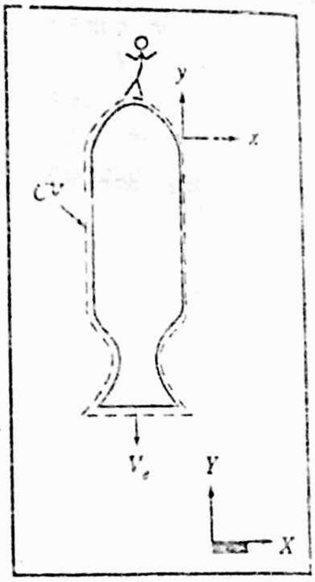
The plot shows an initial rapid rise in vane speed. The vane speed, U, approaches the jet speed, V, asymptotically as the mass flow rate crossing the control surface decreases toward zero.

EXAMPLE 4.12 Rocket Directed Vertically

A small rocket, with an initial mass of 400 kg, is to be launched vertically. Upon ignition the rocket consumes fuel at the rate of 5 kg/s and ejects gas at atmospheric pressure with a speed of 3500 m/s relative to the rocket. Determine the initial acceleration of the rocket and the rocket speed after 10 s, if air resistance is neglected.

EXAMPLE PROBLEM 4.12

GIVEN: Small rocket accelerates vertically from rest.
Initial mass, $M_0 = 400$ kg.
Air resistance may be neglected.
Rate of fuel consumption, $\dot{m}_{out} = 5$ kg/s.
Exhaust velocity, $V_e = 3500$ m/s, relative to rocket, leaving at atmospheric pressure.



FIND: (a) Initial acceleration of the rocket.
(b) Rocket velocity after 10 s.

SOLUTION:

Choose a control volume as shown by dashed lines. Because the control volume is accelerating, define inertial coordinate system XY and coordinate system xy attached to the CV. Apply the y component of the momentum equation.

Basic equation: $$F_{S_y} + F_{B_y} - \int_{CV} a_{rf_y}\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\mathcal{V} + \int_{CS} v_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Assumptions: (1) Atmospheric pressure acts on all surfaces of the CV; since air resistance is neglected, $F_{S_y} = 0$
(2) Gravity is the only body force; g is constant
(3) Flow leaving the rocket is uniform, and V_e is constant

Under these assumptions the momentum equation reduces to

$$F_{B_y} - \int_{CV} a_{rf_y}\rho\, d\mathcal{V} = \frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\mathcal{V} + \int_{CS} v_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A} \qquad (1)$$

Ⓐ Ⓑ Ⓒ Ⓓ

Let us look at the equation term by term:

Ⓐ $$F_{B_y} = -\int_{CV} g\rho\, d\mathcal{V} = -g\int_{CV}\rho\, d\mathcal{V} = -gM_{CV} \qquad \text{\{since } g \text{ is constant\}}$$

The mass of the CV will be a function of time because mass is leaving the CV at rate $\dot{m}_e$. To determine M_{CV} as a function of time, we use the conservation of mass equation

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} + \int_{CS}\rho\vec{V}\cdot d\vec{A} = 0$$

Then

$$\frac{\partial}{\partial t}\int_{CV}\rho\, d\mathcal{V} = -\int_{CS}\rho\vec{V}\cdot d\vec{A} = -\int_{A_e}\rho\vec{V}\cdot d\vec{A} = -\int_{A_e}\{|\rho V\, dA|\} = -|\dot{m}_e|$$

The minus sign indicates that the mass of the CV is decreasing with time. Since the mass of the CV is only a function of time, we can write

$$\frac{dM_{CV}}{dt} = -|\dot{m}_e|$$

To find the mass of the CV at any time, t, we integrate

$$\int_{M_0}^{M} dM_{CV} = -\int_0^t |\dot{m}_e| dt \qquad \text{where at } t = 0,\ M_{CV} = M_0, \text{ and at } t = t,\ M_{CV} = M$$

Then, $M - M_0 = -|\dot{m}_e|t$, or $M = M_0 - \dot{m}_e t$.
{Since $\dot{m}_e$ is positive, we have dropped the absolute value sign.}

Substituting the expression for M into term Ⓐ, we obtain

$$F_{B_y} = -\int_{CV} g\rho \, dV = -gM_{CV} = -g(M_0 - \dot{m}_e t)$$

Ⓑ $-\int_{CV} a_{rf_y} \rho \, dV$

The acceleration, a_{rf_y}, of the CV is that seen by an observer in the XY coordinate system. Thus a_{rf_y} is not a function of the coordinates xyz, and

$$-\int_{CV} a_{rf_y} \rho \, dV = -a_{rf_y} \int_{CV} \rho \, dV = -a_{rf_y} M_{CV} = -a_{rf_y}(M_0 - \dot{m}_e t)$$

Ⓒ $\frac{\partial}{\partial t} \int_{CV} v_{xyz} \rho \, dV$

is the time rate of change of the y momentum of the fluid in the control volume measured relative to the control volume.

Even though the y momentum of the fluid inside the CV, measured relative to the CV, is a large number, it does not change appreciably with time. To see this, we must recognize that:

(1) The unburned fuel and the rocket structure have zero momentum relative to the rocket.
(2) The velocity of the gas at the nozzle exit remains constant with time as does the velocity at various points in the nozzle.

Consequently, it is reasonable to assume that

$$\frac{\partial}{\partial t} \int_{CV} v_{xyz} \rho \, dV \approx 0$$

Ⓓ $\int_{CS} v_{xyz} \rho \vec{V}_{xyz} \cdot d\vec{A} = \int_{A_e} v_{xyz} |\rho V_{xyz} \, dA| = v_{xyz} |\dot{m}_e|$

Since $\vec{V}_e = -V_e \hat{j}$,

$$v_{xyz} |\dot{m}_e| = -V_e |\dot{m}_e| = -V_e \dot{m}_e$$

Substituting terms Ⓐ through Ⓓ into Eq. 1, we obtain

$$-g(M_0 - \dot{m}_e t) - a_{rf_y}(M_0 - \dot{m}_e t) = -V_e \dot{m}_e$$

or

$$a_{rf_y} = \frac{V_e \dot{m}_e}{M_0 - \dot{m}_e t} - g \qquad (2)$$

At time $t = 0$,

$$a_{rf_y}\Big)_{t=0} = \frac{V_e \dot{m}_e}{M_0} - g = \frac{3500 \text{ m}}{\text{s}} \times \frac{5 \text{ kg}}{\text{s}} \times \frac{1}{400 \text{ kg}} - \frac{9.81 \text{ m}}{\text{s}^2}$$

$$a_{rf_y}\Big)_{t=0} = 33.9 \text{ m/s}^2 \qquad a_{rf_y}\Big)_{t=0}$$

The acceleration of the CV is by definition

$$a_{rf_y} = \frac{dV_{CV}}{dt}$$

Substituting from Eq. 2,

$$\frac{dV_{CV}}{dt} = \frac{V_e \dot{m}_e}{M_0 - \dot{m}_e t} - g$$

Separating variables and integrating gives

$$V_{CV} = \int_0^{V_{CV}} dV_{CV} = \int_0^t \frac{V_e \dot{m}_e \, dt}{M_0 - \dot{m}_e t} - \int_0^t g \, dt = -V_e \ln\left[\frac{M_0 - \dot{m}_e t}{M_0}\right] - gt$$

At $t = 10$ s,

$$V_{CV} = -3500\,\frac{\text{m}}{\text{s}} \times \ln\left[\frac{350\text{ kg}}{400\text{ kg}}\right] - 9.81\,\frac{\text{m}}{\text{s}^2} \times 10\text{ s}$$

$$V_{CV} = 369\text{ m/s}$$

$V_{CV})_{t=10\text{ sec}}$

{This problem illustrates the application of the momentum equation to a linearly accelerating control volume.}