4-3 CONSERVATION OF MASS

The first physical principle to which we apply the relation between system and control volume formulations is conservation of mass. It is intuitive that mass can be neither created nor destroyed; if the flow rate of mass into a control volume exceeds the rate of flow out, mass will accumulate within the CV.

Recall that conservation of mass states simply that the mass of a system is constant,

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0 \tag{4.1a}$$

where

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\maltese(\text{system})} \rho \, d\maltese \tag{4.1b}$$

The system and control volume formulations are related by Eq. 4.11,

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta \rho \, d\maltese + \int_{\text{CS}} \eta \rho \vec{V} \cdot d\vec{A} \tag{4.11}$$

[1] Equation 4.11 has been derived for a control volume fixed in space relative to coordinates xyz. For the case of a *deformable* control volume, whose shape varies with time, Eq. 4.11 may be applied provided that the velocity, $\vec{V}$, in the flux integral is measured relative to the local control surface through which the flux occurs.

where

$$N_{\text{system}} = \int_{M(\text{system})} \eta \, dm = \int_{V(\text{system})} \eta \rho \, dV \tag{4.6}$$

To derive the control volume formulation of conservation of mass, we set

$$N = M \qquad \text{and} \qquad \eta = 1$$

With this substitution, we obtain

$$\left.\frac{dM}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \rho \, dV + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} \tag{4.12}$$

Comparing Eqs. 4.1a and 4.12, we arrive at the control volume formulation of the conservation of mass:

$$0 = \frac{\partial}{\partial t}\int_{\text{CV}} \rho \, dV + \int_{\text{CS}} \rho \vec{V} \cdot d\vec{A} \tag{4.13}$$

In Eq. 4.13 the first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass flux out through the control surface. Conservation of mass requires that the sum of the rate of change of mass within the control volume and the net rate of mass outflow through the control surface be zero.

We emphasize that the velocity, $\vec{V}$, in Eq. 4.13 is measured relative to the control surface. Furthermore, the dot product, $\rho \vec{V} \cdot d\vec{A}$, is a scalar product. The sign depends on the direction of the velocity vector, $\vec{V}$, relative to the area vector, $d\vec{A}$. Referring back to the derivation of Eq. 4.11, we see that the dot product, $\rho \vec{V} \cdot d\vec{A}$, is positive where flow is out through the control surface, negative where flow is in through the control surface, and zero where flow is tangent to the control surface.

4-3.1 Special Cases

In special cases it is possible to simplify Eq. 4.13. Consider first the case of incompressible flow, in which density remains constant. When ρ is constant, it is not a function of space or time. Consequently, for incompressible flow, Eq. 4.13 may be written as

$$0 = \rho \frac{\partial}{\partial t}\int_{\text{CV}} dV + \rho \int_{\text{CS}} \vec{V} \cdot d\vec{A} \tag{4.14a}$$

The integral of dV over the control volume is simply the volume of the control volume. Thus, on dividing through by ρ, we write Eq. 4.14a as

$$0 = \frac{\partial V}{\partial t} + \int_{\text{CS}} \vec{V} \cdot d\vec{A} \tag{4.14b}$$

For a nondeformable control volume of fixed size and shape, V = constant. The conservation of mass for incompressible flow through a fixed control volume becomes

$$0 = \int_{\text{CS}} \vec{V} \cdot d\vec{A} \tag{4.14c}$$

Note that we have not assumed the flow to be steady in reducing Eq. 4.13 to the form 4.14c. We have only imposed the restriction of incompressible flow. Thus Eq. 4.14c is a statement of conservation of mass for an incompressible flow that may be steady or unsteady.

The dimensions of the integrand in Eq. 4.14c are L^3/t. The integral of $\vec{V} \cdot d\vec{A}$ over a section of the control surface is commonly called the *volume flow rate* or *volume rate of flow*. Thus, for incompressible flow, the volume flow rate into a fixed control volume must be equal to the volume flow rate out of the control volume. The volume flow rate Q, through a section of a control surface of area A, is given by

$$Q = \int_A \vec{V} \cdot d\vec{A} \tag{4.15a}$$

The average velocity magnitude, $\bar{V}$, at a section is defined as

$$\bar{V} = \frac{Q}{A} = \frac{1}{A}\int_A \vec{V} \cdot d\vec{A} \tag{4.15b}$$

Consider now the general case of steady, compressible flow through a fixed control volume. Since the flow is steady, this means that at most $\rho = \rho(x, y, z)$. By definition, no fluid property varies with time in a steady flow. Consequently, the first term of Eq. 4.13 must be zero and, hence, for steady flow, the statement of conservation of mass reduces to

$$0 = \int_{CS} \rho\vec{V} \cdot d\vec{A} \tag{4.16}$$

Thus, for steady flow, the mass flow rate into a control volume must be equal to the mass flow rate out of the control volume.

As we noted in our previous discussion of velocity fields in Section 2-2, the idealization of uniform flow at a section frequently provides an adequate flow model. Uniform flow at a section implies the velocity is constant across the entire area at a section. When the density also is constant at a section, the flux integral in Eq. 4.13 may be replaced by a product. Thus, when uniform flow at section n is assumed,

$$\int_{A_n} \rho\vec{V} \cdot d\vec{A} = \rho_n \vec{V}_n \cdot \vec{A}_n$$

or using scalar magnitudes

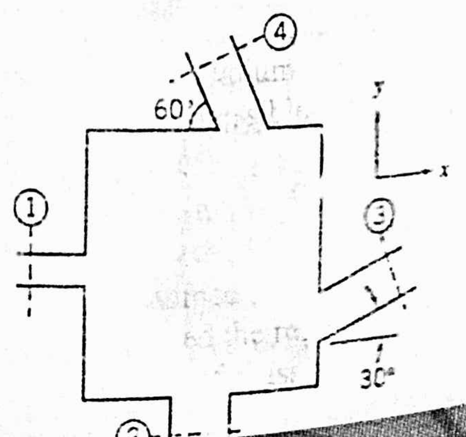
$$\int_{A_n} \rho\vec{V} \cdot d\vec{A} = \pm|\rho_n V_n A_n \cos\alpha|$$

Again note that when $\rho\vec{V} \cdot d\vec{A}$ is negative, mass flows in through the control surface. Mass flows out through the control surface in regions where $\rho\vec{V} \cdot d\vec{A}$ is positive. This fact provides a quick check of the signs on the various flux terms in an analysis.

The simplest choice of control surface is normal to the velocity vector so that $\cos\alpha = \pm 1$.

EXAMPLE 4.1 Mass Flow through Multiport Device

Consider steady flow of water through the device shown in the diagram. The areas are: $A_1 = 0.2\ \text{ft}^2$, $A_2 = 0.5\ \text{ft}^2$, and $A_3 = A_4 = 0.4\ \text{ft}^2$. The mass flow rate out through section ③ is 3.88 slug/s. The volume flow rate in through section ④ is 1 ft^3/s, and $\vec{V}_1 = 10\hat{i}$ ft/s. Determine the flow velocity at section ②.



EXAMPLE PROBLEM 4.1

GIVEN: Steady flow of water through the device.

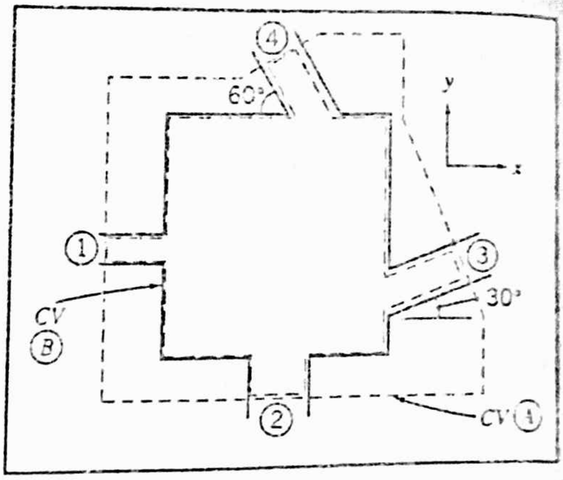
$A_1 = 0.2$ ft^2 $\qquad A_2 = 0.5$ ft^2

$A_3 = A_4 = 0.4$ ft^2 $\qquad \rho = 1.94$ slug/ft^3

$\dot{m}_3 = 3.88$ slug/s (outflow)

$\vec{V}_1 = 10\hat{i}$ ft/s

Volume flow rate in at ④ = 1.0 ft^3/s



FIND: Velocity at section ②.

SOLUTION:

Choose a fixed control volume. Two possibilities are shown by dashed lines.

Basic equation: $$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow (given)
(2) Incompressible flow
(3) Uniform properties at each section where fluid crosses the CV boundaries

For steady flow, the first term is zero by definition, so

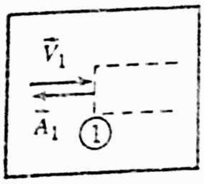
$$0 = \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

In looking at either control volume, we see that there are four sections where mass flows across the control surface. Thus we write

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = \int_{A_1} \rho \vec{V} \cdot d\vec{A} + \int_{A_2} \rho \vec{V} \cdot d\vec{A} + \int_{A_3} \rho \vec{V} \cdot d\vec{A} + \int_{A_4} \rho \vec{V} \cdot d\vec{A} = 0 \qquad (1)$$

Let us look at these integrals one at a time, under the assumptions of uniform properties over each area and ρ = constant.

$$\int_{A_1} \rho \vec{V} \cdot d\vec{A} = -\int_{A_1} |\rho V\, dA| = -|\rho V_1 A_1|$$

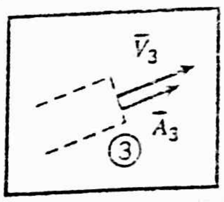


{Sign of $\vec{V} \cdot d\vec{A}$ is negative at surface ①.}

{With the absolute value signs indicated, we have accounted for the directions of $\vec{V}$ and $d\vec{A}$ in taking the dot product.}

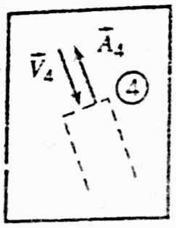
Since we do not know the direction of $\vec{V}_2$, we shall leave section ② for the moment.

$$\int_{A_3} \rho \vec{V} \cdot d\vec{A} = \int_{A_3} |\rho V\, dA| = |\rho V_3 A_3| = \dot{m}_3$$



{Sign of $\vec{V} \cdot d\vec{A}$ is positive at surface ③, since flow is out.}

$$\int_{A_4} \rho \vec{V} \cdot d\vec{A} = -\int_{A_4} |\rho V\, dA| = -|\rho V_4 A_4|$$
$$= -\rho|V_4 A_4| = -\rho|Q_4|$$



{Sign of $\vec{V} \cdot d\vec{A}$ is negative at surface ④.}

where Q is the volume flow rate.

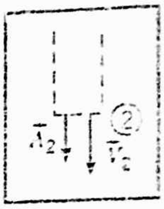
From Eq. 1 above,

$$\int_{A_2} \rho\vec{V}\cdot d\vec{A} = -\int_{A_1} \rho\vec{V}\cdot d\vec{A} - \int_{A_3} \rho\vec{V}\cdot d\vec{A} - \int_{A_4} \rho\vec{V}\cdot d\vec{A}$$

$$= +|\rho V_1 A_1| - \dot{m}_3 + \rho|Q_4|$$

$$= \left|1.94\frac{\text{slug}}{\text{ft}^3} \times 10\frac{\text{ft}}{\text{s}} \times 0.2\,\text{ft}^2\right| - 3.88\frac{\text{slug}}{\text{s}} + 1.94\frac{\text{slug}}{\text{ft}^3}\left|1.0\frac{\text{ft}^3}{\text{s}}\right|$$

$$\int_{A_2} \rho\vec{V}\cdot d\vec{A} = 1.94\ \text{slug/s}$$



Since this is positive, $\vec{V}\cdot d\vec{A}$ at section ② is positive. Flow is out, as shown in the sketch:

$$\int_{A_2} \rho\vec{V}\cdot d\vec{A} = \int_{A_2} |\rho V\,dA| = |\rho V_2 A_2| = 1.94\ \text{slug/s}$$

$$|V_2| = \frac{1.94\ \text{slug/s}}{\rho A_2} = 1.94\frac{\text{slug}}{\text{s}} \times \frac{\text{ft}^3}{1.94\ \text{slug}} \times \frac{1}{0.5\ \text{ft}^2} = 2\ \text{ft/s}$$

Since V_2 is in the negative y direction, then

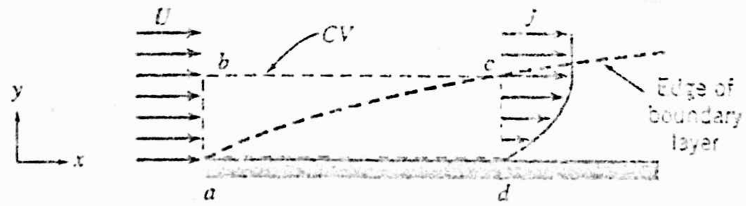
$$\vec{V}_2 = -2\hat{j}\ \text{ft/s} \qquad \vec{V}_2$$

{This problem illustrates the procedure recommended for evaluating $\int_{CS} \rho\vec{V}\cdot d\vec{A}$.}

EXAMPLE 4.2 Mass Flow Rate in Boundary Layer

The fluid in direct contact with a stationary solid boundary has zero velocity; there is no slip at the boundary. Thus the flow over a flat plate adheres to the plate surface and forms a boundary layer, as depicted below. The flow ahead of the plate is uniform with velocity, $\vec{V} = U\hat{i}$; $U = 30$ m/s. The velocity distribution within the boundary layer $(0 \le y \le \delta)$ along cd is approximated as $u/U = 2(y/\delta) - (y/\delta)^2$.

The boundary-layer thickness at location d is $\delta = 5$ mm. The fluid is air with density $\rho = 1.24\ \text{kg/m}^3$. Assuming the plate width perpendicular to the paper to be $w = 0.6$ m, calculate the mass flow rate across surface bc of control volume $abcd$.



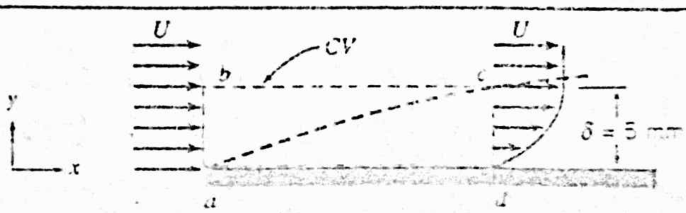
EXAMPLE PROBLEM 4.2

GIVEN: Steady, incompressible flow over a flat plate, $\rho = 1.24\ \text{kg/m}^3$.
Width of plate, $w = 0.6$ m.
Velocity ahead of plate is uniform: $\vec{V} = U\hat{i}$, $U = 30$ m/s.

At $x = x_d$:

$\delta = 5$ mm

$$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$$



FIND: Mass flow rate across surface bc.

SOLUTION:

The fixed control volume is shown by the dashed lines.

Basic equation:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow (given)
(2) Incompressible flow (given)
(3) Two-dimensional flow, properties are independent of z

For steady flow,

$$\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall = 0 \qquad \text{and hence} \qquad \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Assuming that there is no flow in the z direction, then $\begin{pmatrix}\text{no flow}\\ \text{across } da\end{pmatrix}$

$$0 = \int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} + \int_{A_{da}} \rho \vec{V} \cdot d\vec{A}$$

$$\therefore \dot{m}_{bc} = \int_{A_{bc}} \rho \vec{V} \cdot d\vec{A} = -\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} - \int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} \qquad (1)$$

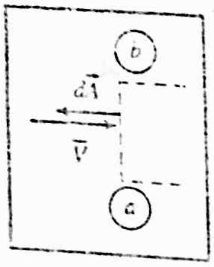
{We need to evaluate the integrals on the right side of the equation.}

For depth w in the z direction, we obtain

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} = -\int_{A_{ab}} |\rho u \, dA| = -\int_{y_a}^{y_b} |\rho u w \, dy|$$

$$= -\int_0^{\delta} |\rho u w \, dy| = -\left| \int_0^{\delta} \rho U w \, dy \right|$$

$$\int_{A_{ab}} \rho \vec{V} \cdot d\vec{A} = -|[\rho U w y]_0^{\delta}| = -\rho U w \delta$$



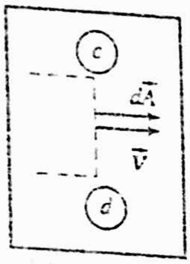
$\left\{ \begin{matrix} \vec{V} \cdot d\vec{A} \text{ is negative} \\ dA = w\, dy \end{matrix} \right\}$

$\{u = U \text{ over area } ab\}$

$$\int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} = \int_{A_{cd}} |\rho u \, dA| = \int_{y_d}^{y_c} |\rho u w \, dy|$$

$$= \int_0^{\delta} |\rho u w \, dy| = \int_0^{\delta} \left| \rho w U \left[2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2 \right] dy \right|$$

$$\int_{A_{cd}} \rho \vec{V} \cdot d\vec{A} = \left| \rho w U \left[\frac{y^2}{\delta} - \frac{y^3}{3\delta^2} \right]_0^{\delta} \right| = \left| \rho w U \delta \left[1 - \frac{1}{3} \right] \right| = \frac{2\rho U w \delta}{3}$$



$\left\{ \begin{matrix} \vec{V} \cdot d\vec{A} \text{ is positive} \\ dA = w\, dy \end{matrix} \right\}$

Substituting into Eq. 1, we obtain

$$\therefore \dot{m}_{bc} = \rho U w \delta - \frac{2\rho U w \delta}{3} = \frac{\rho U w \delta}{3}$$

$$= \frac{1}{3} \times 1.24 \frac{\text{kg}}{\text{m}^3} \times 30 \frac{\text{m}}{\text{s}} \times 0.6 \text{ m} \times 5 \text{ mm} \times \frac{\text{m}}{1000 \text{ mm}}$$

{Positive sign indicates flow out across surface bc.}

$\dot{m}_{bc} = 0.0372$ kg/s

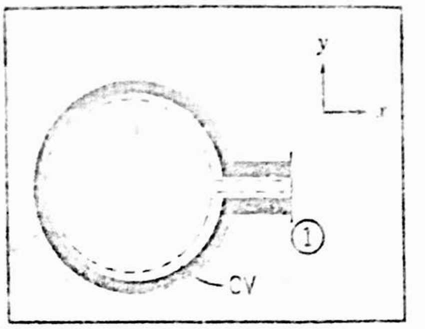
{This problem illustrates the application of the control volume formulation of conservation of mass to the case of nonuniform flow at a section.}

EXAMPLE 4.3 Density Change in Venting Tank

A tank of 0.05 m³ volume contains air at 800 kPa (absolute) and 15°C. At $t = 0$, air begins escaping from the tank through a valve with a flow area of 65 mm². The air passing through the valve has a speed of 311 m/s and a density of 6.13 kg/m³. Determine the instantaneous rate of change of density in the tank at $t = 0$.

EXAMPLE PROBLEM 4.3

GIVEN: Tank of volume $\forall = 0.05$ m³ contains air at $p = 800$ kPa (absolute), $T = 15°$C. At $t = 0$, air escapes through a valve. Air leaves with speed $V = 311$ m/s and density $\rho = 6.13$ kg/m³ through area $A = 65$ mm².



FIND: Rate of change of air density in the tank at $t = 0$.

SOLUTION:

Choose a fixed control volume as shown by the dashed line.

Basic equation:
$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Properties in the tank are uniform, but time-dependent
(2) Uniform flow at section ①

Since properties are assumed uniform in the tank at any instant, we can take ρ out from within the integral of the first term,

$$\frac{\partial}{\partial t}\left[\rho_{CV}\int_{CV} d\forall\right] + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

Now, $\int_{CV} d\forall = \forall$, and hence

$$\frac{\partial}{\partial t}(\rho\forall)_{CV} + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0$$

The only place where mass crosses the boundary of the control volume is at surface ①. Hence

$$\int_{CS} \rho \vec{V} \cdot d\vec{A} = \int_{A_1} \rho \vec{V} \cdot d\vec{A} \quad \text{and} \quad \frac{\partial}{\partial t}(\rho\forall) + \int_{A_1} \rho \vec{V} \cdot d\vec{A} = 0$$

At surface ① the sign of $\rho\vec{V} \cdot d\vec{A}$ is positive, so

$$\frac{\partial}{\partial t}(\rho\forall) + \int_{A_1} |\rho V\, dA| = 0$$

Since flow is assumed uniform over surface ①, then

$$\frac{\partial}{\partial t}(\rho\forall) + |\rho_1 V_1 A_1| = 0 \quad \text{or} \quad \frac{\partial}{\partial t}(\rho\forall) = -|\rho_1 V_1 A_1|$$

Since the volume, $\forall$, of the tank is not a function of time,

$$\forall \frac{\partial \rho}{\partial t} = -|\rho_1 V_1 A_1|$$

and

$$\frac{\partial \rho}{\partial t} = -\frac{|\rho_1 V_1 A_1|}{\forall}$$

At $t = 0$,

$$\frac{\partial \rho}{\partial t} = -6.13 \frac{\text{kg}}{\text{m}^3} \times 311 \frac{\text{m}}{\text{s}} \times 65 \text{ mm}^2 \times \frac{1}{0.05 \text{ m}^3} \times \frac{\text{m}^2}{10^6 \text{ mm}^2}$$

$$\frac{\partial \rho}{\partial t} = -2.48 \text{ (kg/m}^3\text{)/s} \quad \{\text{The density is decreasing.}\}$$

$\frac{\partial \rho}{\partial t}$

{This problem illustrates the application of the control volume formulation of conservation of mass to an unsteady flow.}

4-4 MOMENTUM EQUATION FOR INERTIAL CONTROL VOLUME

We wish to develop a mathematical formulation of Newton's second law suitable for application to a control volume. In this section our derivation will be restricted to an inertial control volume fixed in space relative to coordinate system xyz that is not accelerating relative to stationary reference frame XYZ.

In deriving the control volume formulation of Newton's second law, the procedure is analogous to the procedure followed in deriving the mathematical formulation for conservation of mass applied to a control volume. We begin with the mathematical formulation for a system and then use Eq. 4.11 to go from the system to the control volume formulation.

Recall that Newton's second law for a system moving relative to an inertial coordinate system was given by Eq. 4.2a as

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{\text{system}} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{\text{system}} = \int_{M(\text{system})} \vec{V}\, dm = \int_{\forall(\text{system})} \vec{V} \rho\, d\forall \tag{4.2b}$$

and the resultant force, $\vec{F}$, includes all surface and body forces acting on the system,

$$\vec{F} = \vec{F}_S + \vec{F}_B$$

The system and control volume formulations are related by Eq. 4.11,

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta \rho\, d\forall + \int_{\text{CS}} \eta \rho \vec{V} \cdot d\vec{A} \tag{4.11}$$

To derive the control volume formulation of Newton's second law, we set

$$N = \vec{P} \qquad \text{and} \qquad \eta = \vec{V}$$

From Eq. 4.11, with this substitution, we obtain

$$\left.\frac{d\vec{P}}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}\rho\, d\forall + \int_{\text{CS}} \vec{V}\rho\vec{V}\cdot d\vec{A} \tag{4.17}$$

From Eq. 4.2a

$$\left.\frac{d\vec{P}}{dt}\right)_{\text{system}} = \left.\vec{F}\right)_{\text{on system}} \tag{4.2a}$$

Since, in deriving Eq. 4.11, the system and the control volume coincided at t_0, then

$$\left.\vec{F}\right]_{\text{on system}} = \left.\vec{F}\right]_{\text{on control volume}}$$

In light of this, Eqs. 4.2a and 4.17 may be combined to yield the control volume formulation of Newton's second law for a nonaccelerating control volume

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{\text{CV}} \vec{V}\rho\, d\forall + \int_{\text{CS}} \vec{V}\rho\vec{V}\cdot d\vec{A} \tag{4.18}$$

This equation states that the sum of all forces (surface and body forces) acting on a nonaccelerating control volume is equal to the sum of the rate of change of momentum inside the control volume and the net rate of flux of momentum out through the control surface.

The derivation of the momentum equation for a control volume was straightforward. Application of this basic equation to the solution of problems will not be difficult if you exercise care in using the equation.

In using any basic equation for a control volume analysis, the first step must be to draw the boundaries of the control volume and label appropriate coordinate directions. In Eq. 4.18, the force, $\vec{F}$, represents all forces acting on the control volume. It includes both surface forces and body forces. As in the case of the free-body diagram of basic mechanics, all forces (and moments) acting on the control volume should be shown so that they can be systematically accounted for in the application of the basic equations. If we denote the body force per unit mass as $\vec{B}$, then

$$\vec{F}_B = \int \vec{B}\, dm = \int_{\text{CV}} \vec{B}\rho\, d\forall$$

When the force of gravity is the only body force, then the body force per unit mass is $\vec{g}$. The surface force due to pressure is given by

$$\vec{F}_S = \int_A -p\, d\vec{A}$$

The nature of the forces acting on the control volume undoubtedly will influence the choice of the control volume boundaries.

All velocities, $\vec{V}$, in Eq. 4.18 are measured relative to the control volume. The momentum flux, $\vec{V}\rho\vec{V}\cdot d\vec{A}$, through an element of the control surface area, $d\vec{A}$, is a vector. The sign of the scalar product, $\rho\vec{V}\cdot d\vec{A}$, depends on the direction of the velocity vector, $\vec{V}$, relative to the area vector, $d\vec{A}$. The signs of the components of the velocity $\vec{V}$, depend on the coordinate system chosen.

The momentum equation is a vector equation. As with all vector equations, it may be written as three scalar component equations. The scalar components of Eq. 4.18, relative to an xyz coordinate system, are

$$F_x = F_{S_x} + F_{B_x} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\mathcal{V} + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \tag{4.19a}$$

$$F_y = F_{S_y} + F_{B_y} = \frac{\partial}{\partial t}\int_{CV} v\rho\, d\mathcal{V} + \int_{CS} v\rho\vec{V}\cdot d\vec{A} \tag{4.19b}$$

$$F_z = F_{S_z} + F_{B_z} = \frac{\partial}{\partial t}\int_{CV} w\rho\, d\mathcal{V} + \int_{CS} w\rho\vec{V}\cdot d\vec{A} \tag{4.19c}$$

To use the scalar equations, it again is necessary to select a coordinate system at the outset. The positive directions of the velocity components, u, v, and w, and the force components, F_x, F_y, F_z, are then established relative to the selected coordinate system. As we have previously pointed out, the sign of the scalar product, $\rho\vec{V}\cdot d\vec{A}$, depends on the direction of the velocity vector, $\vec{V}$, relative to the area vector, $d\vec{A}$. Thus the flux term in either Eq. 4.18 or Eqs. 4.19 is a product of two quantities, both of which have algebraic signs. We suggest that you proceed in two steps to determine the momentum flux through any portion of a control surface:

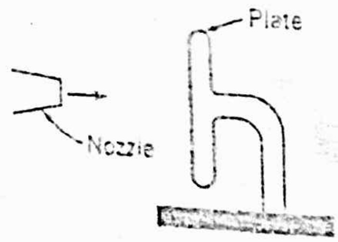
1. The first step is to determine the sign of $\rho\vec{V}\cdot d\vec{A}$,

$$\rho\vec{V}\cdot d\vec{A} = \rho|V\,dA|\cos\alpha = \pm|\rho V\,dA\cos\alpha|$$

2. The second step is to determine the sign for each velocity component, u, v, and w. The sign, which depends on the choice of coordinate system, should be accounted for when substituting numerical values into the terms $u\rho\vec{V}\cdot d\vec{A} = u\{\pm|\rho V\,dA\cos\alpha|\}$, and so on.

EXAMPLE 4.4 Choice of Control Volume for Momentum Analysis

Water from a stationary nozzle strikes a flat plate as shown. The water leaves the nozzle at 15 m/s; the nozzle area is 0.01 m². Assuming the water is directed normal to the plate, and flows along the plate, determine the horizontal force on the support.

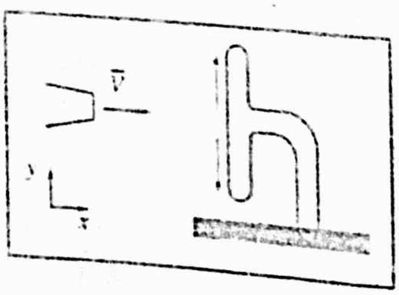


EXAMPLE PROBLEM 4.4

GIVEN: Water from a stationary nozzle is directed normal to the plate; subsequent flow is parallel to plate.

Jet velocity, $\vec{V} = 15\hat{i}$ m/s

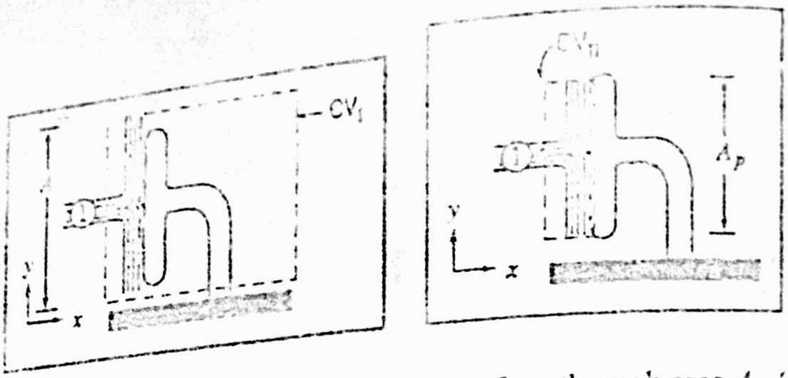
Nozzle area, $A_n = 0.01$ m²



FIND: Horizontal force on the support

SOLUTION:

We chose a coordinate system in defining the problem above. We must now choose a suitable control volume. Two possible choices are shown by the dashed lines below.



In both cases, water from the nozzle crosses the control surface through area A_1 (assumed equal to the nozzle area) and is assumed to leave the control volume tangent to the plate surface in the $+y$ or $-y$ direction. Before trying to decide which is the "best" control volume to use, let us write the basic equations.

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, dV + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A} \qquad \text{and} \qquad \frac{\partial}{\partial t}\int_{CV} \rho\, dV + \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

Assumptions: (1) Steady flow
(2) Incompressible flow
(3) Uniform flow at each section where fluid crosses the CV boundaries

Regardless of our choice of control volume, the flow is steady and the basic equations become

$$\vec{F} = \vec{F}_S + \vec{F}_B = \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A} \qquad \text{and} \qquad \int_{CS} \rho\vec{V}\cdot d\vec{A} = 0$$

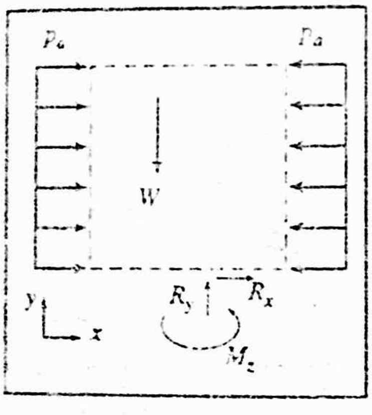
Evaluating the momentum flux term will lead to the same result for both control volumes. We should choose the control volume that allows the most straightforward evaluation of the forces.

Remember in applying the momentum equation that the force, $\vec{F}$, represents all forces acting *on* the control volume.

Let us solve the problem using each of the control volumes.

CV$_I$

The control volume has been selected so that the area of the left surface is equal to the area of the right surface. Denote this area by A.



The control volume cuts through the support. We denote the components of the reaction force of the support on the control volume as R_x and R_y and assume both to be positive. (The force of the control volume on the support is equal and opposite to R_x and R_y.) M_z is the reaction moment (about the z axis) from the support on the control volume.

Atmospheric pressure acts on all surfaces of the control volume. (The distributed force due to atmospheric pressure has been shown on the vertical faces only.)

The body force on the control volume is denoted as W.

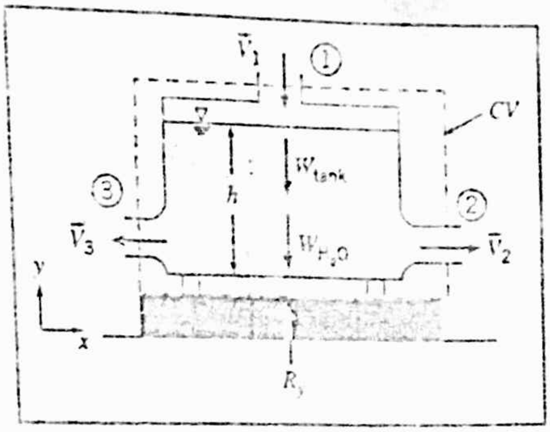
Since we are looking for the horizontal force, we write the x component of the steady flow momentum equation

$$F_{S_x} + F_{B_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

$A_1 = 0.1\ \text{ft}^2$

$\vec{V}_1 = -5\hat{j}\ \text{ft/s}$

$A_2 = A_3 = 0.1\ \text{ft}^2$



FIND: Scale reading.

SOLUTION:
Choose a control volume as shown; R_y is the force of the scale on the control volume (exerted on the control volume through the supports) and is assumed positive.

The weight of the tank is designated W_{tank}; the weight of the water in the tank is W_{H_2O}.

Atmospheric pressure acts uniformly on all surfaces of the control volume. (This pressure force has not been shown on the control volume.) There is no net force on the control volume due to atmospheric pressure.

Basic equations:

$$\vec{F}_S + \vec{F}_B = \overbrace{\frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall}^{=0(1)} + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A}$$

$$0 = \overbrace{\frac{\partial}{\partial t}\int_{CV} \rho\, d\forall}^{=0(1)} + \int_{CS} \rho\vec{V}\cdot d\vec{A}$$

{In the solution of problems it is our general practice to begin with the basic equations, simplifying on the basis of given information or appropriate assumptions and proceeding to obtain an algebraic expression before substituting numerical values. When terms are simplified, we indicate the basis for the simplification in parentheses. In the present case, each of the two terms set to zero follows directly from the given conditions of steady flow.}

Assumptions: (1) Steady flow (given)
(2) Incompressible flow
(3) Uniform flow at each section where fluid crosses the CV boundaries

We write the y component of the momentum equation

$$F_{S_y} + F_{B_y} = \int_{CS} v\rho\vec{V}\cdot d\vec{A} \qquad (1)$$

$F_{S_y} = R_y$ {There is no net force due to atmospheric pressure.}

$F_{B_y} = -W_{tank} - W_{H_2O}$ {Both body forces act in negative y direction.}

$$W_{H_2O} = \rho g \forall = \gamma A h$$

$$\int_{CS} v\rho\vec{V}\cdot d\vec{A} = \int_{A_1} v\rho\vec{V}\cdot d\vec{A} = \int_{A_1} v\{-|\rho_1 V_1\, dA|\}$$

{$\vec{V}\cdot d\vec{A}$ is negative at ①. $v = 0$ at sections ② and ③.}

$$= -v_1|\rho_1 V_1 A_1|$$

{We are assuming uniform properties at ①.}

Substituting into Eq. 1 gives

$$R_y - W_{tank} - \gamma A h = -v_1|\rho_1 V_1 A_1|$$

or

$$R_y = W_{tank} + \gamma A h - v_1 |\rho_1 V_1 A_1|$$

Substituting numbers with $v_1 = -5$ ft/s gives

$$R_y = 5\ \text{lbf} + \frac{62.4\ \text{lbf}}{\text{ft}^3} \times 1\ \text{ft}^2 \times 1.9\ \text{ft} - \left(-5\ \frac{\text{ft}}{\text{s}}\right)\left|1.94\ \frac{\text{slug}}{\text{ft}^3}\left(-5\ \frac{\text{ft}}{\text{s}}\right) 0.1\ \text{ft}^2\right| \frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}}$$

$$R_y = 128\ \text{lbf} \qquad \text{\{Force of scale on CV is upward.\}}$$

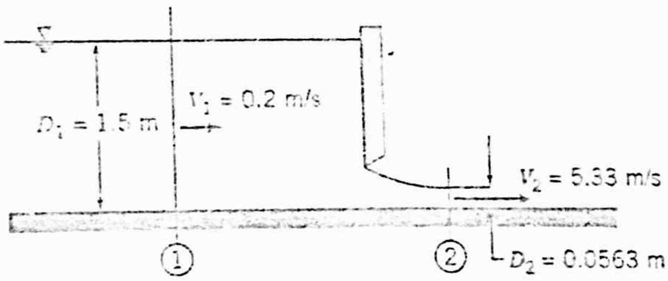
The force of the control volume on the scale is $K_y = -R_y = -128$ lbf.

The minus sign indicates that the force on the scale is downward in our coordinate system. Therefore, the scale reading is 128 lbf.

{This problem illustrates the application of the momentum equation to an inertial control volume with body forces included.}

EXAMPLE 4.6 Flow under Sluice Gate: Hydrostatic Pressure Force

Water in an open channel flows under a sluice gate as shown in the sketch. The flow is incompressible and uniform at sections ① and ②. Hydrostatic pressure distributions may be assumed at sections ① and ② because the flow streamlines are essentially straight there. Determine the magnitude and direction of the horizontal force per unit width exerted on the gate by the flow.



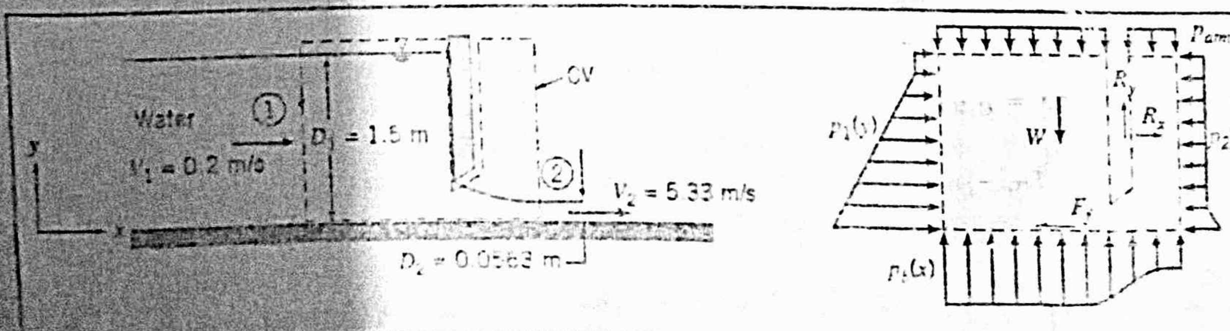
EXAMPLE PROBLEM 4.6

GIVEN: Flow under sluice gate. Width = w.

FIND: Horizontal force exerted (per unit width) on the gate.

SOLUTION:

Choose the CV and coordinate system shown for analysis.



There are no body forces in the x direction, so $F_{B_x} = 0$, and

$$F_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

To evaluate F_{S_x}, we must include all surface forces acting on the control volume

$$F_{S_x} = \underbrace{p_a A}_{\substack{\text{force due to atmospheric}\\ \text{pressure acts to right}\\ \text{(positive direction) on}\\ \text{left surface}}} - \underbrace{p_a A}_{\substack{\text{force due to atmospheric}\\ \text{pressure acts to left}\\ \text{(negative direction) on}\\ \text{right surface}}} + \underbrace{R_x}_{\substack{\text{force of support on}\\ \text{control volume}\\ \text{(assumed positive)}}}$$

Consequently, $F_{S_x} = R_x$, and

$$R_x = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\rho\vec{V}\cdot d\vec{A}$$ {For mass crossing top and bottom surfaces, $u = 0$.}

$$= \int_{A_1} u\{-|\rho V_1\, dA|\}$$ {At ①, $\rho\vec{V}\cdot d\vec{A} = -|\rho V_1\, dA|$, since direction of $\vec{V}_1$ and $d\vec{A}_1$ are 180° apart.}

$$= -u_1|\rho V_1 A_1|$$ {properties uniform over A_1}

$$= -15\,\frac{\text{m}}{\text{s}} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 15\,\frac{\text{m}}{\text{s}} \times 0.01\,\text{m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$ {$u_1 = 15$ m/s}

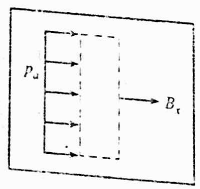
$$R_x = -2.25\ \text{kN}$$ {R_x acts opposite to positive direction assumed.}

The horizontal force on the support is

$$K_x = -R_x = 2.25\ \text{kN}$$ {force on support acts to the right} K_x

CV$_{II}$ with Horizontal Forces Shown

The control volume has been selected so the areas of the left surface and of the right surface are equal to the area of the plate. Denote this area by A_p.



The control volume is in contact with the plate over the entire plate surface. We denote the horizontal reaction force from the plate on the control volume as B_x (and assume it to be positive).

Atmospheric pressure acts on the left surface of the control volume (and on the two horizontal surfaces).

The body force on this control volume has no component in the x direction.

Then the x component of the momentum equation,

$$F_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A}$$

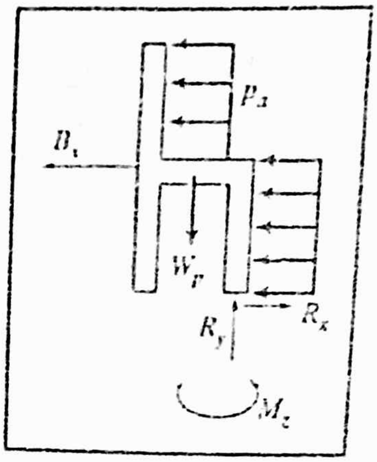
yields

$$F_{S_x} = p_a A_p + B_x = \int_{A_1} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\{-|\rho V_1\, dA|\} = -2.25\ \text{kN}$$

Then

$$B_x = -p_a A_p - 2.25\ \text{kN}$$

To determine the net force on the plate, we need a free-body diagram of the plate:



$$\sum F_x = 0 = -B_x - p_a A_p + R_x$$

$$R_x = p_a A_p + B_x$$

$$R_x = p_a A_p + (-p_a A_p - 2.25 \text{ kN}) = -2.25 \text{ kN}$$

Then the horizontal force on the support is $K_x = -R_x = 2.25$ kN.

Note that the choice of CV_{II} resulted in the need for an additional free-body diagram. In general it is advantageous to select the control volume so that the force sought acts explicitly on the control volume.

This problem illustrates the application of the momentum equation to an inertial control volume, with emphasis on choosing a suitable control volume.

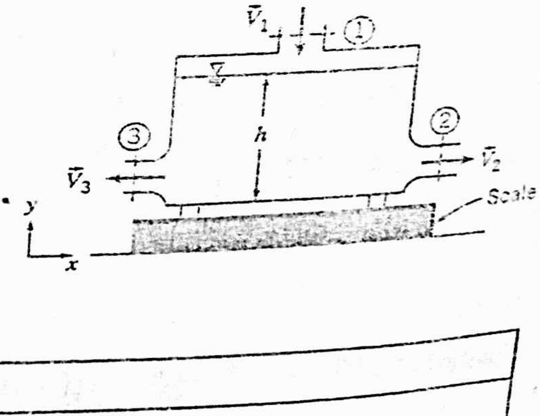
EXAMPLE 4.5 Tank on Scale: Body Force

A metal container 2 ft high, with an inside cross-sectional area of 1 ft², weighs 5 lbf when empty. The container is placed on a scale and water flows in through an opening in the top and out through the two equal-area openings in the sides, as shown in the diagram. Under steady flow conditions, the height of the water in the tank is $h = 1.9$ ft. Determine the reading on the scale.

$$A_1 = 0.1 \text{ ft}^2$$

$$\vec{V}_1 = -5\hat{j} \text{ ft/s}$$

$$A_2 = A_3 = 0.1 \text{ ft}^2$$



EXAMPLE PROBLEM 4.5

GIVEN: Metal container, of height 2 ft and cross-sectional area $A = 1$ ft², weighs 5 lbf when empty. Container rests on scale. Under steady flow conditions water depth is $h = 1.9$ ft. Water enters vertically at section ① and leaves horizontally through sections ② and ③.

The forces acting on the control volume include

- Force of gravity W
- Friction force F_f
- Components R_x and R_y of reaction force from gate
- Uniform atmospheric pressure p_a on top surface
- Hydrostatic pressure distribution on vertical surfaces (assumption 6)
- Pressure distribution $p_b(x)$ along bottom surface

Apply the x component of the momentum equation.

Basic equation:

$$F_{S_x} + \cancel{F_{B_x}} = \cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall} + \int_{CS} u\rho \vec{V}\cdot d\vec{A} \qquad = 0(2) \quad = 0(3)$$

Assumptions: (1) F_f negligible (neglect friction on channel bottom)
(2) $F_{B_x} = 0$
(3) Steady flow
(4) Incompressible flow (given)
(5) Uniform flow at each section (given)
(6) Hydrostatic pressure distributions at ① and ② (given)

Then

$$F_{S_x} = u_1\{-|\rho V_1 w D_1|\} + u_2\{|\rho V_2 w D_2|\}$$

The surface forces acting on the CV are due to pressure and the unknown force, R_x. From assumption (6),

$$\frac{dp}{dy} = -\rho g \qquad p = p_0 + \rho g(y_0 - y) = p_{atm} + \rho g(D - y)$$

Evaluating F_{S_x} gives

$$F_{S_x} = \int_0^{D_1} p_1\, dA_1 - \int_0^{D_2} p_2\, dA_2 - p_{atm}(D_1 - D_2)w + R_x$$

$$= \int_0^{D_1} [p_{atm} + \rho g(D_1 - y)]w\, dy - \int_0^{D_2} [p_{atm} + \rho g(D_2 - y)]w\, dy - p_{atm}(D_1 - D_2)w + R_x$$

$$F_{S_x} = \cancel{p_{atm} D_1 w} + \frac{\rho g D_1^2}{2}w - \cancel{p_{atm} D_2 w} - \frac{\rho g D_2^2}{2}w - \cancel{p_{atm} D_1 w} + \cancel{p_{atm} D_2 w} + R_x$$

or

$$F_{S_x} = R_x + \frac{\rho g w}{2}(D_1^2 - D_2^2)$$

Substituting into the momentum equation, with $u_1 = V_1$ and $u_2 = V_2$, gives

$$R_x + \frac{\rho g w}{2}(D_1^2 - D_2^2) = -V_1|\rho V_1 w D_1| + V_2|\rho V_2 w D_2|$$

or

$$R_x = \rho w(V_2^2 D_2 - V_1^2 D_1) - \frac{\rho g w}{2}(D_1^2 - D_2^2)$$

and

$$\frac{R_x}{w} = \rho(V_2^2 D_2 - V_1^2 D_1) - \frac{\rho g}{2}(D_1^2 - D_2^2)$$

$$= 999\,\frac{\text{kg}}{\text{m}^3}\left[(5.33)^2(0.0563) - (0.2)^2(1.5)\right]\frac{\text{m}^2}{\text{s}^2}\,\text{m} \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$- \frac{1}{2} \times 999\,\frac{\text{kg}}{\text{m}^3} \times 9.81\,\frac{\text{m}}{\text{s}^2}\left[(1.5)^2 - (0.0563)^2\right]\text{m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$\frac{R_x}{w} = -9.47\ \text{kN/m}$$

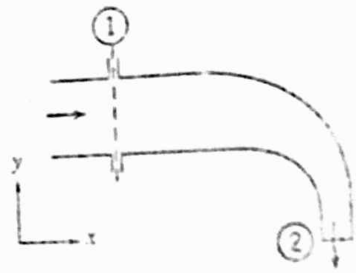
R_x is the unknown external force acting *on* the control volume. It is applied to the CV by the gate. Therefore, the force from all fluids *on* the gate is K_x, where $K_x = -R_x$. Thus

$$\frac{K_x}{w} = -\frac{R_x}{w} = 9.47 \text{ kN/m} \quad \{\text{applied to the right}\} \qquad \frac{K_x}{w}$$

{This problem illustrates application of the momentum equation to a control volume in which the pressure is not uniform over the entire control surface.}

EXAMPLE 4.7 Flow through Elbow: Use of Gage Pressures

Water flows steadily through the 90° reducing elbow shown in the diagram. At the inlet to the elbow, the absolute pressure is 221 kPa and the cross-sectional area is 0.01 m². At the outlet, the cross-sectional area is 0.0025 m² and the velocity is 16 m/s. The elbow discharges to the atmosphere. Determine the force required to hold the elbow in place.



EXAMPLE PROBLEM 4.7

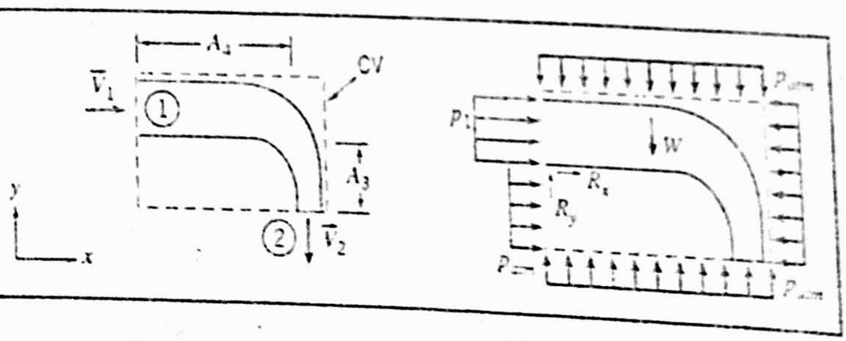
GIVEN: Steady flow of water through 90° reducing elbow.

$p_1 = 221$ kPa (abs) $\quad A_1 = 0.01 \text{ m}^2 \quad \vec{V}_2 = -16\hat{j}$ m/s $\quad A_2 = 0.0025 \text{ m}^2$

FIND: Force required to hold elbow in place.

SOLUTION:

Choose control volume as shown by the dashed line.



A_3 is the area of vertical sides of CV excluding A_1; $A_{\text{vertical sides}} = A_1 + A_3$.
A_4 is the area of horizontal sides of CV excluding A_2; $A_{\text{horizontal sides}} = A_2 + A_4$.

The forces acting on the control volume include those due to

- Pressure p_1 acting on area A_1.
- Atmospheric pressure acting over remainder of control surface.
- Reaction force components R_x and R_y from base support acting on control volume. (These force components, required to hold the elbow in place, are assumed positive.)

Note that since the elbow is anchored to the supply pipe, there would also be a reaction moment from the base support acting on the control volume.

Basic equations:

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho\, d\forall + \int_{CS} \vec{V}\rho\vec{V}\cdot d\vec{A} \quad (=0(4))$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho\vec{V}\cdot d\vec{A} \quad (=0(4))$$

Assumptions:
(1) Uniform flow at each section
(2) Atmospheric pressure, $p_a = 101$ kPa (abs)
(3) Incompressible flow
(4) Steady flow (given)
(5) Neglect weight of elbow and water in elbow

Writing the x component of the momentum equation results in

$$F_{S_x} = \int_{CS} u\rho\vec{V}\cdot d\vec{A} = \int_{A_1} u\rho\vec{V}\cdot d\vec{A} \qquad \{F_{B_x} = 0 \text{ and } u_2 = 0\}.$$

$$p_1A_1 + p_aA_2 - p_a(A_1 + A_2) + R_x = \int_{A_1} u\rho\vec{V}\cdot d\vec{A}$$

{Pressure over right side is p_a. Pressure over left side is p_1 on A_1 and p_a on A_2.}

$$(p_1 - p_a)A_1 + R_x = \int_{A_1} u\{-|\rho V_1\, dA|\} \qquad \{\vec{V}\cdot d\vec{A} \text{ is negative at } A_1.\}$$

$$R_x = -p_{1_g}A_1 - u_1|\rho V_1 A_1| \qquad \{p_1 - p_a = p_{1_{gage}}\}$$

To find V_1, use the continuity equation:

$$\int_{CS} \rho\vec{V}\cdot d\vec{A} = 0 = \int_{A_1} \rho\vec{V}\cdot d\vec{A} + \int_{A_2} \rho\vec{V}\cdot d\vec{A}$$

$$\therefore 0 = -\int_{A_1} |\rho V\, dA| + \int_{A_2} |\rho V\, dA| = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

and

$$|V_1| = |V_2|\frac{A_2}{A_1} = \frac{16\text{ m}}{\text{s}} \times \frac{0.0025}{0.01} = 4 \text{ m/s} \qquad \therefore \vec{V}_1 = 4\hat{i}\text{ m/s}$$

$$R_x = -p_{1_g}A_1 - u_1|\rho V_1 A_1|$$

$$= -1.20\times 10^5 \frac{\text{N}}{\text{m}^2} \times 0.01\text{ m}^2 - 4\frac{\text{m}}{\text{s}}\left|999\frac{\text{kg}}{\text{m}^3} \times 4\frac{\text{m}}{\text{s}} \times 0.01\text{ m}^2 \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}\right|$$

$$R_x = -1.36 \text{ kN}$$ {R_x to hold elbow acts to left} R_x

Writing the y component of the momentum equation gives

$$F_{S_y} - F_{B_y} = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = \int_{A_2} v\rho\vec{V}\cdot d\vec{A} \qquad \{v_1 = 0\}$$

$$p_aA_1 + p_aA_2 - p_aA_1 - p_aA_2 + F_{B_y} + R_y = \int_{A_2} v\{|\rho V\, dA|\}$$

{Pressure is p_a over top and bottom of CV. $\vec{V}\cdot d\vec{A}$ is positive at ②.}

$$F_{B_y} + R_y = v_2|\rho V_2 A_2|$$

$$R_y = -F_{B_y} + v_2|\rho V_2 A_2|$$

{Since we do not know the volume or mass of the elbow, we cannot evaluate F_{B_y}.}

Substituting numbers, recognizing $\vec{V}_2 = -16\hat{j}$ m/s, so $v_2 = -16$ m/s

$$R_y = -F_{B_y} + \left(-16\,\frac{\text{m}}{\text{s}}\right)\left|999\,\frac{\text{kg}}{\text{m}^3}\left(-16\,\frac{\text{m}}{\text{s}}\right)0.0025\,\text{m}^2\right| \times \frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_y = -F_{B_y} - 639\text{ N}$$

Neglecting F_{B_y} gives

$$R_y = -639\text{ N} \qquad \{R_y \text{ to hold elbow acts down}\}\ R_y$$

This problem illustrates the application of the momentum equation to an inertial control volume in which the pressure is not atmospheric across the entire control surface.

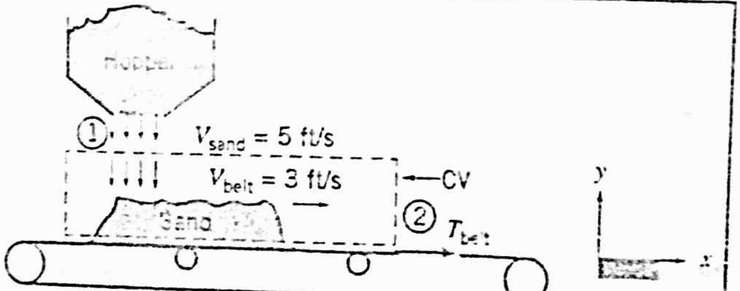
Since the pressure forces over the entire control surface must be included in the analysis, use of gage pressures on all surfaces gives correct (and often more direct) results.

EXAMPLE 4.8 Conveyor Belt Filling: Rate of Change of Momentum in Control Volume

A horizontal conveyor belt moving at 3 ft/s receives sand from a hopper. The sand falls vertically from the hopper to the belt at a speed of 5 ft/s and a flow rate of 500 lbm/s (the density of sand is approximately 2700 lbm/cubic yard). The conveyor belt is initially empty but begins to fill with sand. If friction in the drive system and rollers is negligible, find the tension required to pull the belt while the conveyor is filling.

EXAMPLE PROBLEM 4.8

GIVEN: Conveyor and hopper shown in sketch.



FIND: T_{belt} at the instant shown.

SOLUTION:

Use the control volume and coordinates shown. Apply the x component of the momentum equation.

Basic equations:

$$F_{S_x} + \cancel{F_{B_x}}^{=0(2)} = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\forall + \int_{CS} u\rho\vec{V}\cdot d\vec{A} \qquad 0 = \frac{\partial}{\partial t}\int_{CV}\rho\, d\forall + \int_{CS}\rho\vec{V}\cdot d\vec{A}$$

Assumptions:
(1) $F_{S_x} = T_{belt} = T$
(2) $F_{B_x} = 0$
(3) Uniform flow at section ①
(4) All sand on belt moves with $V_{belt} = V_b$

Then

$$T = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\mathcal{V} + u_1\{-|\rho V_1 A_1|\} + u_2\{|\rho V_2 A_2|\}$$

Since $u_1 = 0$, and there is no flow at section ②, then

$$T = \frac{\partial}{\partial t}\int_{CV} u\rho\, d\mathcal{V}$$

From assumption (4), inside the CV, $u = V_b$ = constant, and hence

$$T = V_b \frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} = V_b \frac{\partial M_s}{\partial t}$$

where M_s is the mass of sand on the belt (inside the control volume). From the continuity equation,

$$\frac{\partial}{\partial t}\int_{CV} \rho\, d\mathcal{V} = \frac{\partial}{\partial t} M_s = -\int_{CS} \rho \vec{V} \cdot d\vec{A} = \dot{m}_s = 500 \text{ lbm/s}$$

Then

$$T = V_b \dot{m}_s = \frac{3 \text{ ft}}{\text{s}} \times \frac{500 \text{ lbm}}{\text{s}} \times \frac{\text{slug}}{32.2 \text{ lbm}} \times \frac{\text{lbf} \cdot \text{s}^2}{\text{slug} \cdot \text{ft}}$$

$$T = 46.6 \text{ lbf}$$ T

{This problem illustrates an application of the momentum equation to a problem in which the rate of change of momentum within the control volume is not equal to zero.}

*4-4.1 Differential Control Volume Analysis

We have considered a number of examples in which conservation of mass and the momentum equation have been applied to finite control volumes. The control volume chosen for analysis need not be finite in size.

Application of the basic equations to a differential control volume leads to differential equations describing the relationships among properties in the flow field. In some cases, the differential equations can be solved to give detailed information about property variations in the flow field. For the case of steady, incompressible, frictionless

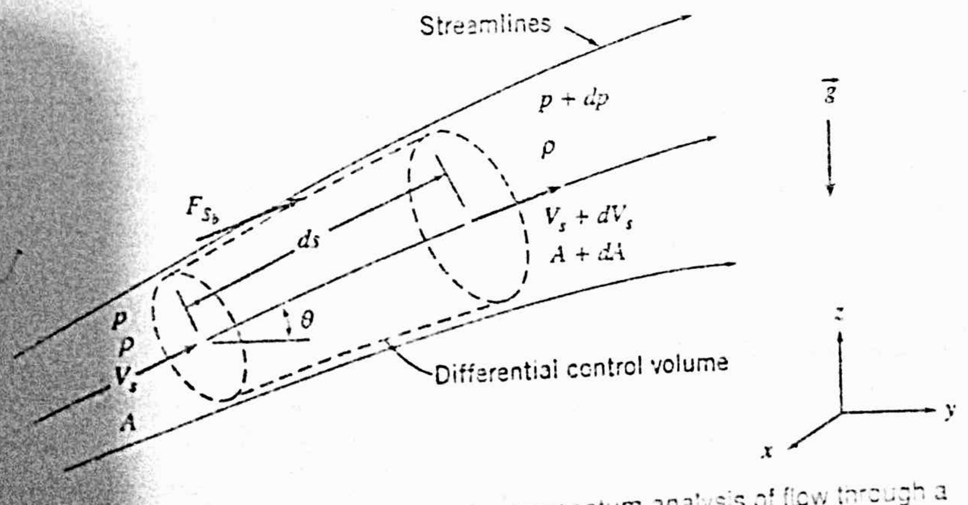


Fig. 4.4 Differential control volume for momentum analysis of flow through a stream tube.

* This section may be omitted without loss of continuity in the text material.

flow along a streamline, integration of one such differential equation leads to a useful relationship among speed, pressure, and elevation in a flow field. This case is presented to illustrate the use of differential control volumes.

Let us apply the continuity and momentum equations to a steady incompressible flow without friction, as shown in Fig. 4.4. The control volume chosen is fixed in space and bounded by flow streamlines, and is thus an element of a stream tube. The length of the control volume is ds.

Because the control volume is bounded by streamlines, flow across the bounding surfaces occurs only at the end sections. These are located at coordinates s and $s + ds$, measured along the central streamline.

Properties at the inlet section are assigned arbitrary symbolic values. Properties at the outlet section are assumed to increase by differential amounts. Thus at $s + ds$, the flow speed is assumed to be $V_s + dV_s$, and so forth. The differential changes, dp, dV_s, and dA, all are assumed to be positive in formulating the problem. (As in a free-body analysis in statics or dynamics, the actual algebraic sign of each differential change will be determined from the results of the analysis.)

Now let us apply the continuity equation and the s component of the momentum equation to the control volume of Fig. 4.4.

a. Continuity Equation

Basic equation:

$$0 = \overbrace{\frac{\partial}{\partial t}\int_{CV} \rho \, d\forall}^{=0(1)} + \int_{CS} \rho \vec{V} \cdot d\vec{A} \tag{4.13}$$

Assumptions: (1) Steady flow
(2) No flow across bounding streamlines
(3) Incompressible flow, ρ = constant

Then

$$0 = \{-|\rho V_s A|\} + \{|\rho (V_s + dV_s)(A + dA)|\}$$

and

$$\rho V_s A = \rho (V_s + dV_s)(A + dA) \tag{4.20a}$$

On expanding the right side and simplifying, we obtain

$$0 = V_s \, dA + A \, dV_s + dA \, dV_s$$

But $dA \, dV_s$ is a product of differentials, which may be neglected compared with $V_s \, dA$ or $A \, dV_s$. Thus

$$0 = V_s \, dA + A \, dV_s \tag{4.20b}$$

b. Streamwise Component of the Momentum Equation

Basic equation:

$$F_{S_s} + F_{B_s} = \overbrace{\frac{\partial}{\partial t}\int_{CV} u_s \rho \, d\forall}^{=0(1)} + \int_{CS} u_s \rho \vec{V} \cdot d\vec{A} \tag{4.21}$$

Assumption: (4) No friction, so F_{S_s} is due to pressure forces only.

The surface force (due only to pressure) will have three terms:

$$F_{S_s} = pA - (p + dp)(A + dA) + \left(p + \frac{dp}{2}\right)dA \tag{4.22a}$$

The first and second terms in Eq. 4.22a are the pressure forces on the end faces of the control surface. The third term is, F_{S_b}, the pressure force acting in the s direction on the bounding stream surface of the control volume. Its magnitude is the product of the average pressure acting on the stream surface, $p + \frac{1}{2}dp$, times the area component of the stream surface in the s direction, dA. Equation 4.22a simplifies to

$$F_{S_s} = -A\,dp - \tfrac{1}{2}dp\,dA \tag{4.22b}$$

The body force component in the s direction is

$$F_{B_s} = \rho g_s\,d\forall = \rho(-g\sin\theta)\left(A + \frac{dA}{2}\right)ds$$

But $\sin\theta\,ds = dz$, so that

$$F_{B_s} = -\rho g\left(A + \frac{dA}{2}\right)dz \tag{4.22c}$$

The momentum flux will be

$$\int_{CS} u_s\rho\vec{V}\cdot d\vec{A} = V_s\{-|\rho V_s A|\} + (V_s + dV_s)\{|\rho(V_s + dV_s)(A + dA)|\}$$

since there is no mass flux across the bounding stream surfaces. The factors in braces are equal from continuity, Eq. 4.20a, so

$$\int_{CS} u_s\rho\vec{V}\cdot d\vec{A} = V_s(-\rho V_s A) + (V_s + dV_s)(\rho V_s A) = \rho V_s A\,dV_s \tag{4.23}$$

Substituting Eqs. 4.22b, 4.22c, and 4.23 into the momentum equation gives

$$-A\,dp - \tfrac{1}{2}dp\,dA - \rho g A\,dz - \tfrac{1}{2}\rho g\,dA\,dz = \rho V_s A\,dV_s$$

Dividing by ρA and noting that products of differentials are negligible compared with the remaining terms, we obtain

$$-\frac{dp}{\rho} - g\,dz = V_s\,dV_s = d\left(\frac{V_s^2}{2}\right)$$

or

$$\frac{dp}{\rho} + d\left(\frac{V_s^2}{2}\right) + g\,dz = 0 \qquad \text{(4.24}$$

For incompressible flow, this equation may be integrated to obtain

$$\frac{p}{\rho} + \frac{V_s^2}{2} + gz = \text{constant}$$

or, dropping subscript s,

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant} \qquad \text{(4.2}$$

This equation is subject to the restrictions:

1. Steady flow
2. No friction
3. Flow along a streamline
4. Incompressible flow

By applying the momentum equation to an infinitesimal stream tube control volume, for steady incompressible flow without friction, we have derived a relation among pressure, speed, and elevation. This relationship is very powerful and useful. For example, it could have been used to evaluate the pressure at the inlet of the reducing elbow analyzed in Example Problem 4.7 or to determine the velocity of water leaving the sluice gate of Example Problem 4.6. In both of these flow situations the restrictions required to derive Eq. 4.25 are reasonable idealizations of the actual flow behavior. The restrictions must be emphasized heavily because they do not always form a realistic model for flow behavior; consequently, they must be justified carefully each time Eq. 4.25 is applied.

Equation 4.25 is a form of the Bernoulli equation. It will be derived again in detail in Chapter 6 because it is such a useful tool for flow analysis and because an alternative derivation will give added insight into the need for care in applying the equation.

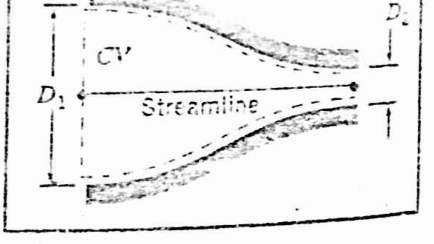
EXAMPLE 4.9 Nozzle Flow: Application of Bernoulli Equation

Water flows steadily through a horizontal nozzle, discharging to the atmosphere. At the nozzle inlet the diameter is D_1; at the nozzle outlet the diameter is D_2. Derive an expression for the minimum gage pressure required at the nozzle inlet to produce a given volume flow rate, Q. Evaluate the inlet gage pressure if $D_1 = 3.0$ in., $D_2 = 1.0$ in., and the desired flow rate is 0.7 ft³/s.

EXAMPLE PROBLEM 4.9

GIVEN: Steady flow of water through a horizontal nozzle, discharging to the atmosphere.

$D_1 = 3.0$ in. $\quad D_2 = 1.0$ in. $\quad p_2 = p_{atm}$



FIND: (a) p_1, as a function of volume flow rate, Q.
(b) p_1, for $Q = 0.7$ ft³/s.

SOLUTION:

Basic equations:

$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$= 0(1)$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A}$$

Assumptions: (1) Steady flow (given)
(2) Incompressible flow
(3) Frictionless flow

(4) Flow along a streamline
(5) $z_1 = z_2$
(6) Uniform flow at sections ① and ②

Apply the Bernoulli equation along a streamline between points ① and ② to evaluate p_1. Then

$$p_{1_g} = p_1 - p_{atm} = p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) = \frac{\rho}{2}V_1^2\left[\left(\frac{V_2}{V_1}\right)^2 - 1\right]$$

Apply the continuity equation

$$0 = \{-|\rho V_1 A_1|\} + \{|\rho V_2 A_2|\} \qquad \text{or} \qquad V_1 A_1 = V_2 A_2 = Q$$

so that

$$\frac{V_2}{V_1} = \frac{A_1}{A_2} \qquad \text{and} \qquad V_1 = \frac{Q}{A_1}$$

Then

$$p_{1_g} = \frac{\rho Q^2}{2A_1^2}\left[\left(\frac{A_1}{A_2}\right)^2 - 1\right]$$

Since $A = \pi D^2/4$, then

$$p_{1_g} = \frac{8\rho Q^2}{\pi^2 D_1^4}\left[\left(\frac{D_1}{D_2}\right)^4 - 1\right]$$ p_{1_g}

With $D_1 = 3.0$ in., $D_2 = 1.0$ in., and $\rho = 1.94$ slug/ft³,

$$p_{1_g} = \frac{8}{\pi^2} \times \frac{1.94 \text{ slug}}{\text{ft}^3} \times \frac{1}{(3)^4 \text{ in.}^4} \times Q^2\left[(3.0)^4 - 1\right]\frac{\text{lbf}\cdot\text{s}^2}{\text{slug}\cdot\text{ft}} \times \frac{144 \text{ in.}^2}{\text{ft}^2}$$

$$p_{1_g} = 224 Q^2 \frac{\text{lbf}\cdot\text{s}^2}{\text{in.}^2\cdot\text{ft}^6}$$

With $Q = 0.7$ ft³/s, then $\quad p_{1_g} = 110$ lbf/in.² $\quad p_{1_g}$

This problem illustrates the application of the Bernoulli equation to a flow where the restrictions of steady, incompressible, frictionless flow along a streamline are a reasonable flow model.

4-4.2 Control Volume Moving with Constant Velocity

In the preceding problems, which illustrate applications of the momentum equation to inertial control volumes, we have considered only stationary control volumes. A control volume (fixed relative to reference frame xyz) moving with constant velocity $\vec{V}_{rf}$ relative to a fixed (inertial) reference frame XYZ, is also inertial, since it has no acceleration with respect to XYZ.

Equation 4.11, which expresses system derivatives in terms of control volume variables, is valid for any motion of coordinate system xyz (fixed to the control volume), provided that all velocities are measured *relative* to the control volume. To emphasize this point, we rewrite Eq. 4.11 as

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{CV} \eta\rho\, d\forall + \int_{CS} \eta\rho \vec{V}_{xyz}\cdot d\vec{A} \qquad (4.26)$$

Since all velocities must be measured relative to the control volume, in using [...] tion to obtain the momentum equation for an inertial control volume from the sy[...] formulation, we must set

$$N = \vec{P}_{xyz} \quad \text{and} \quad \eta = \vec{V}_{xyz}$$

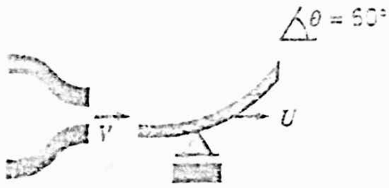
The control volume equation is then written as

$$\vec{F} = \vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\rho\vec{V}_{xyz} \cdot d\vec{A} \qquad (4.27)$$

Equation 4.27 is the formulation of Newton's second law applied to any inertial control volume (stationary or moving with a constant velocity). It is identical to Eq. 4.18 except that we have included subscript xyz to emphasize that quantities must be measured relative to the control volume. (It is helpful to imagine that the velocities are those that would be seen by an observer moving at constant speed with the control volume.) The momentum equation is applied to an inertial control volume moving with constant velocity in Example Problem 4.10.

EXAMPLE 4.10 Vane Moving with Constant Velocity

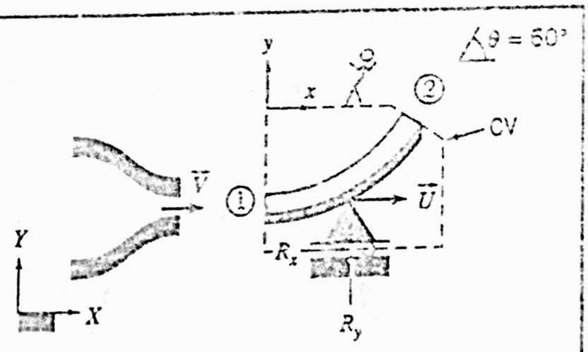
The sketch shows a vane with a turning angle of 60°. The vane moves at constant speed, $U = 10$ m/s, and receives a jet of water that leaves a stationary nozzle with speed $V = 30$ m/s. The nozzle has an exit area of 0.003 m². Determine the force components that act on the vane.



EXAMPLE PROBLEM 4.10

GIVEN: Vane, with turning angle $\theta = 60°$, moves with constant velocity, $\vec{U} = 10\hat{i}$ m/s. Water from a constant-area nozzle, $A = 0.003$ m², with velocity $\vec{V} = 30\hat{i}$ m/s, flows over the vane as shown.

FIND: Force components acting on the vane.



SOLUTION:

Select a control volume moving with the vane at constant velocity, $\vec{U}$, as shown by the dashed lines. R_x and R_y are the components of force required to maintain the velocity of the control volume at $10\hat{i}$ m/s.

The control volume is inertial, since it is not accelerating (U = constant). Remember that all velocities must be measured relative to the control volume in applying the basic equations.

Basic equations:

$$\vec{F}_S + \vec{F}_B = \frac{\partial}{\partial t}\int_{CV} \vec{V}_{xyz}\rho\, d\forall + \int_{CS} \vec{V}_{xyz}\rho\vec{V}_{xyz} \cdot d\vec{A}$$

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho\, d\forall + \int_{CS} \rho\vec{V}_{xyz} \cdot d\vec{A}$$

Assumptions. (1) Flow is steady relative to the vane
(2) Magnitude of relative velocity along the vane is constant: $|\vec{V}_1| = |\vec{V}_2| = V - U$
(3) Properties are uniform at sections ① and ②
(4) $F_{B_x} = 0$
(5) Incompressible flow

The x component of the momentum equation is

$$F_{S_x} + \cancel{F_{B_x}}^{=0(4)} = \cancel{\frac{\partial}{\partial t}\int_{CV} u_{xyz}\rho\, d\forall}^{=0(1)} + \int_{CS} u_{xyz}\rho \vec{V}_{xyz}\cdot d\vec{A}$$

There is no net pressure force, since p_{atm} acts on all sides of the CV. Thus

$$R_x = \int_{A_1} u\{-|\rho V\, dA|\} + \int_{A_2} u\{|\rho V\, dA|\} = -u_1|\rho V_1 A_1| + u_2|\rho V_2 A_2|$$

(All velocities are measured relative to xyz.) From the continuity equation

$$0 = \int_{A_1}\{-|\rho V\, dA|\} + \int_{A_2}|\rho V\, dA| = -|\rho V_1 A_1| + |\rho V_2 A_2|$$

or

$$|\rho V_1 A_1| = |\rho V_2 A_2|$$

Therefore,

$$R_x = (u_2 - u_1)|\rho V_1 A_1|$$

All velocities must be measured relative to the CV, so we note that

$$V_1 = V - U \qquad V_2 = V - U$$

$$u_1 = V - U \qquad u_2 = (V - U)\cos\theta$$

Substituting yields

$$R_x = [(V-U)\cos\theta - (V-U)]|\rho(V-U)A_1| = (V-U)(\cos\theta - 1)|\rho(V-U)A_1|$$

$$= (30-10)\frac{\text{m}}{\text{s}}(0.50-1)\times 999\frac{\text{kg}}{\text{m}^3}\,(30-10)\frac{\text{m}}{\text{s}}\times 0.003\ \text{m}^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_x = -599\ \text{N} \qquad \{\text{to the left}\}$$

Writing the y component of the momentum equation, we obtain

$$F_{S_y} + F_{B_y} = \cancel{\frac{\partial}{\partial t}\int_{CV} v_{xyz}\rho\, d\forall}^{=0(1)} + \int_{CS} v_{xyz}\rho\vec{V}_{xyz}\cdot d\vec{A}$$

Denoting the mass of the CV as M gives

$$R_y - Mg = \int_{CS} v\rho\vec{V}\cdot d\vec{A} = \int_{A_2} v\rho\vec{V}\cdot d\vec{A} \qquad \{v_1 = 0\} \qquad \left\{\text{All velocities are measured relative to } xyz.\right\}$$

$$= \int_{A_2} v|\rho V\, dA| = v_2|\rho V_2 A_2| = v_2|\rho V_1 A_1| \qquad \{\text{Recall } |\rho V_2 A_2| = |\rho V_1 A_1|\}$$

$$= (V-U)\sin\theta|\rho(V-U)A_1|$$

$$= (30-10)\frac{\text{m}}{\text{s}}(0.866)\times 999\frac{\text{kg}}{\text{m}^3}\,(30-10)\frac{\text{m}}{\text{s}}\times 0.003\ \text{m}^2\times\frac{\text{N}\cdot\text{s}^2}{\text{kg}\cdot\text{m}}$$

$$R_y - Mg = 1.04 \text{ kN} \qquad \{\text{upward}\}$$

Thus the vertical force is

$$R_y = 1.04 \text{ kN} + Mg \qquad \{\text{upward}\}$$

Then the net force on the vane (neglecting the weight of the vane and water within the CV) is

$$\vec{R} = -0.599\hat{i} + 1.04\hat{j} \text{ kN} \qquad \vec{R}$$

This problem illustrates the fact that in applying the momentum equation to an inertial control volume all velocities must be measured relative to the control volume.