BASIC EQUATIONS IN INTEGRAL FORM FOR A CONTROL VOLUME

We shall begin our study of fluids in motion by developing the basic equations in integral form for application to control volumes. Why the control volume formulation rather than the system formulation? There are two basic reasons. First, since fluid media are capable of distortion and deformation that continually increase with time, often it is extremely difficult to identify and follow the same mass of fluid at all times (as must be done to apply the system formulation). Second, we often are interested, not in the motion of a given mass of fluid, but rather in the effect of the fluid motion on some device or structure. Thus it is more convenient to apply the basic laws to a defined volume in space, using a control volume analysis.

The basic laws for a system should be familiar from your earlier studies in physics, mechanics, and thermodynamics. Our approach to developing the mathematical formulation of these laws for a control volume will be to develop a general formulation that will allow us to convert from a system analysis to a control volume analysis.

4-1 BASIC LAWS FOR A SYSTEM

The basic laws for a system are summarized briefly; for convenient use, each of the basic equations for a system is written as a rate equation.

4-1.1 Conservation of Mass

Since a system is, by definition, an arbitrary collection of matter of fixed identity, a system is composed of the same quantity of matter at all times. Conservation of mass requires that the mass, M, of the system be constant. On a rate basis, we have

$$\left.\frac{dM}{dt}\right)_{\text{system}} = 0 \tag{4.1a}$$

where

$$M_{\text{system}} = \int_{M(\text{system})} dm = \int_{\mathcal{V}(\text{system})} \rho \, d\mathcal{V} \tag{4.1b}$$

4-1.2 Newton's Second Law

For a system moving relative to an inertial reference frame, Newton's second law states that the sum of all external forces acting on the system is equal to the time rate of change of linear momentum of the system,

$$\vec{F} = \left.\frac{d\vec{P}}{dt}\right)_{system} \tag{4.2a}$$

where the linear momentum of the system is given by

$$\vec{P}_{system} = \int_{M(system)} \vec{V}\, dm = \int_{V(system)} \vec{V} \rho\, d\forall \tag{4.2b}$$

4-1.3 The Angular Momentum Principle

The angular momentum principle for a system states that the rate of change of angular momentum is equal to the sum of all torques acting on the system,

$$\vec{T} = \left.\frac{d\vec{H}}{dt}\right)_{system} \tag{4.3a}$$

where the angular momentum of the system is given by

$$\vec{H}_{system} = \int_{M(system)} \vec{r} \times \vec{V}\, dm = \int_{V(system)} \vec{r} \times \vec{V} \rho\, d\forall \tag{4.3b}$$

Torque can be produced by surface and body forces, and also by shafts that cross the system boundary,

$$\vec{T} = \vec{r} \times \vec{F}_s + \int_{M(system)} \vec{r} \times \vec{g}\, dm + \vec{T}_{shaft} \tag{4.3c}$$

4-1.4 The First Law of Thermodynamics

The first law of thermodynamics is a statement of conservation of energy for a system,

$$\delta Q - \delta W = dE$$

The equation can be written in rate form as

$$\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{system} \tag{4.4a}$$

where the total energy of the system is given by

$$E_{system} = \int_{M(system)} e\, dm = \int_{V(system)} e \rho\, d\forall \tag{4.4b}$$

and

$$e = u + \frac{V^2}{2} + gz \tag{4.4c}$$

In Eq. 4.4a, $\dot{Q}$ (the rate of heat transfer) is positive when heat is added to the system from the surroundings; $\dot{W}$ (the rate of work) is positive when work is done by the system on its surroundings. In Eq. 4.4c, u is the specific internal energy, V the speed, and z the height relative to a convenient datum of a particle of substance having mass dm.

4-1.5 The Second Law of Thermodynamics

If an amount of heat, δQ, is transferred to a system at temperature T, the second law of thermodynamics states that the change in entropy, dS, of the system satisfies

$$dS \geq \frac{\delta Q}{T}$$

On a rate basis we can write

$$\left.\frac{dS}{dt}\right)_{\text{system}} \geq \frac{1}{T}\dot{Q} \tag{4.5a}$$

where the total entropy of the system is given by

$$S_{\text{system}} = \int_{M(\text{system})} s\,dm = \int_{V\!\!\!\!-(\text{system})} s\rho\,dV\!\!\!\!- \tag{4.5b}$$

4-2 RELATION OF SYSTEM DERIVATIVES TO THE CONTROL VOLUME FORMULATION

In the previous section we summarized the basic equations for a system. We found that, when written on a rate basis, each equation involved the time derivative of an extensive property of the system (the total mass, momentum, angular momentum, energy, or entropy of the system). To develop the control volume formulation of each basic law from the system formulation, we shall use the symbol N to designate any arbitrary extensive property of the system. The corresponding intensive property (extensive property per unit mass) will be designated by η. Thus

$$N_{\text{system}} = \int_{M(\text{system})} \eta\,dm = \int_{V\!\!\!\!-(\text{system})} \eta\rho\,dV\!\!\!\!- \tag{4.6}$$

Comparing Eq. 4.6 with Eqs. 4.1b, 4.2b, 4.3b, 4.4b, and 4.5b, we see that if:

$$
\begin{aligned}
N &= M, &\quad \text{then } \eta &= 1 \\
N &= \vec{P}, &\quad \text{then } \eta &= \vec{V} \\
N &= \vec{H}, &\quad \text{then } \eta &= \vec{r} \times \vec{V} \\
N &= E, &\quad \text{then } \eta &= e \\
N &= S, &\quad \text{then } \eta &= s
\end{aligned}
$$

The major task in going from the system to the control volume formulation of the basic laws is to express the rate of change of the arbitrary extensive property, N, for a system, in terms of variations of this property associated with a control volume. Since mass crosses the boundary of a control volume, time variations of the property N associated with the control volume involve the mass flux and the properties convected with it. A convenient way to account for mass flux is to use a limiting process involving a system and a control volume that coincide at a certain instant. Flux quantities in regions of overlap and regions surrounding the control volume are then formulated approximately, and the limiting process is applied to obtain exact results. The final equation relates the rate of change of the arbitrary extensive property N for a system to the time variations of this property associated with a control volume.

4-2.1 Derivation

The system and control volume to be used in the analysis are shown in Fig. 4.1. The flow field, $\vec{V}(x, y, z, t)$, is arbitrary relative to coordinates x, y, and z. The control volume is fixed in space relative to coordinate system xyz; by definition, the system always must consist of the same fluid particles, and consequently it must move with the flow field. In Fig. 4.1 the boundaries of the system are shown at two different instants, t_0 and $t_0 + \Delta t$. At t_0, the boundaries of the system and the control volume coincide; at $t_0 + \Delta t$, the system occupies regions II and III. The system has been chosen so that the mass within region I enters the control volume during interval Δt, and the mass in region III leaves the control volume during the same interval.

Recall that our objective is to relate the rate of change of any arbitrary extensive property, N, of the system to the time variations of this property associated with the control volume. From the definition of a derivative, the rate of change of N_{system} is given by

$$\left.\frac{dN}{dt}\right)_{system} = \lim_{\Delta t \to 0} \frac{N_s)_{t_0+\Delta t} - N_s)_{t_0}}{\Delta t} \tag{4.7}$$

For convenience, subscript s has been used to denote the system in the definition of a derivative in Eq. 4.7.

At $t_0 + \Delta t$, the system occupies regions II and III; at t_0, the system and the control volume coincide. Thus,

$$N_s)_{t_0+\Delta t} = (N_{II} + N_{III})_{t_0+\Delta t} = (N_{CV} - N_I + N_{III})_{t_0+\Delta t}$$

and

$$N_s)_{t_0} = (N_{CV})_{t_0}$$

Substituting into the definition of the system derivative, Eq. 4.7, we obtain

$$\left.\frac{dN}{dt}\right]_s = \lim_{\Delta t \to 0} \frac{(N_{CV} - N_I + N_{III})_{t_0+\Delta t} - N_{CV})_{t_0}}{\Delta t} \tag{4.8}$$

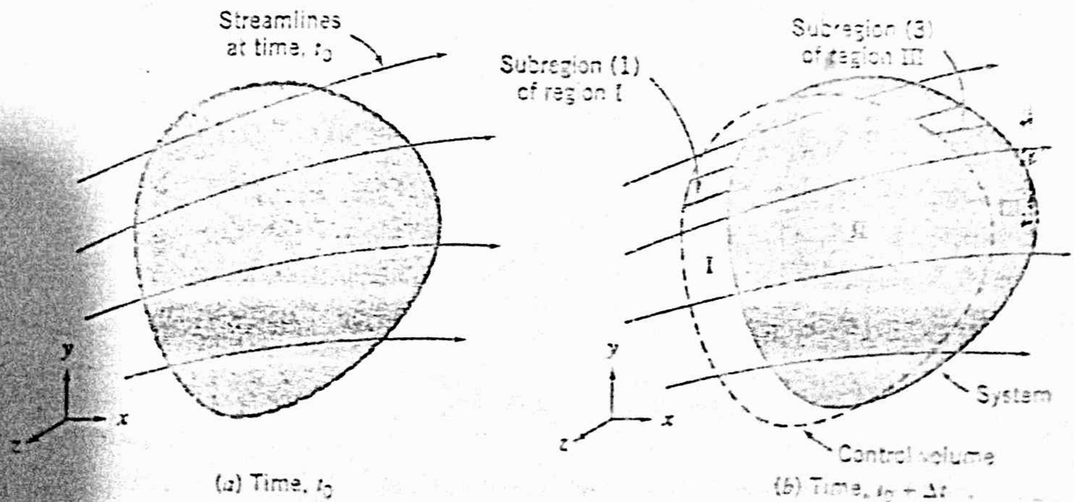

Fig. 4.1 System and control volume configuration.

Since the limit of a sum is equal to the sum of the limits, we can write

$$\left.\frac{dN}{dt}\right]_s = \underset{①}{\lim_{\Delta t \to 0} \frac{N_{CV})_{t_0+\Delta t} - N_{CV})_{t_0}}{\Delta t}} + \underset{②}{\lim_{\Delta t \to 0} \frac{N_{III})_{t_0+\Delta t}}{\Delta t}} - \underset{③}{\lim_{\Delta t \to 0} \frac{N_I)_{t_0+\Delta t}}{\Delta t}} \tag{4.9}$$

Our task now is to evaluate each of the three terms in Eq. 4.9.

Term ① in Eq. 4.9 simplifies to

$$\lim_{\Delta t \to 0} \frac{N_{CV})_{t_0+\Delta t} - N_{CV})_{t_0}}{\Delta t} = \frac{\partial N_{CV}}{\partial t} = \frac{\partial}{\partial t}\int_{CV} \eta \rho \, d\mathcal{V}$$

To evaluate term ② we first develop an expression for $N_{III})_{t_0+\Delta t}$ by looking at the enlarged view of a typical subregion of region III shown in Fig. 4.2. Vector $d\vec{A}$ has magnitude equal to the element of area dA of the control surface, the direction of $d\vec{A}$ is that of the normal drawn outward from the element of control surface area. Angle α is the angle between $d\vec{A}$ and the velocity vector, $\vec{V}$. Since the mass in region III is that which flows *out* of the control volume during interval Δt, angle α will always be less than $\pi/2$ over the entire area of the control surface bounding region III.

For subregion (3) we can write

$$dN_{III})_{t_0+\Delta t} = (\eta \rho \, d\mathcal{V})_{t_0+\Delta t} = [\eta \rho (\Delta l \cos \alpha \, dA)]_{t_0+\Delta t}$$

since $d\mathcal{V} = \Delta l \cos \alpha \, dA$. Then for the entire region III,

$$N_{III})_{t_0+\Delta t} = \left[\int_{CS_{III}} \eta \rho \, \Delta l \cos \alpha \, dA\right]_{t_0+\Delta t}$$

where CS_{III} is the surface common to region III and the control volume. In this expression, Δl is the distance traveled during interval Δt, along a streamline that existed at t_0, by a particle that was on the system surface at time t_0.

Now that we have an expression for $N_{III})_{t_0+\Delta t}$, we can evaluate term ② in Eq. 4.9:

$$\lim_{\Delta t \to 0} \frac{N_{III})_{t_0+\Delta t}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\int_{CS_{III}} \eta \rho \, \Delta l \cos \alpha \, dA}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \int_{CS_{III}} \eta \rho \frac{\Delta l}{\Delta t} \cos \alpha \, dA = \int_{CS_{III}} \eta \rho |\vec{V}| \cos \alpha |d\vec{A}|$$

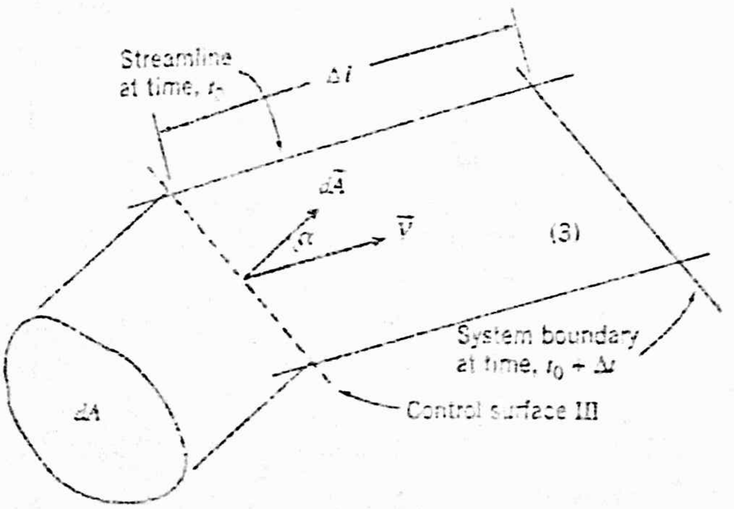

Fig. 4.2 Enlarged view of subregion (3) from Fig. 4.1.

Now that we have obtained expressions for each of the three terms on the right side, Eq. 4.9 can be written

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\, d\forall + \int_{\text{CS}_{\text{I}}} \eta\rho|\vec{V}|\cos\alpha|d\vec{A}| + \int_{\text{CS}_{\text{III}}} \eta\rho|\vec{V}|\cos\alpha|d\vec{A}|$$

The entire control surface, CS, consists of three surfaces.

$$\text{CS} = \text{CS}_{\text{I}} + \text{CS}_{\text{III}} + \text{CS}_p$$

where CS_p is characterized by no flow across the surface, where either $\alpha = \pi/2$ or $\vec{V} = 0$.

Consequently, we can write

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\, d\forall + \int_{\text{CS}} \eta\rho|\vec{V}|\cos\alpha|d\vec{A}| \tag{4.10}$$

Since $|\vec{V}|\cos\alpha|d\vec{A}| = \vec{V}\cdot d\vec{A}$, Eq. 4.10 becomes

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\, d\forall + \int_{\text{CS}} \eta\rho\vec{V}\cdot d\vec{A} \tag{4.11}$$

Equation 4.11 is the relation we set out to obtain. It is the fundamental relation between the rate of change of any arbitrary extensive property, N, of a system and the variations of this property associated with a control volume. Some authors refer to Eq. 4.11 as the Reynolds Transport Theorem.

4-2.2 Physical Interpretation

We have taken several pages to derive Eq. 4.11. Recall that our objective was to obtain a general relation between the rate of change of any arbitrary extensive property, N, of a system and variations of this property associated with the control volume. The main reason for deriving it was to reduce the algebra required to obtain the control volume formulations of the basic equations. Because the working form of each basic equation for application to control volumes is developed from Eq. 4.11, we consider the equation itself to be "basic" and repeat it to emphasize its importance:

$$\left.\frac{dN}{dt}\right)_{\text{system}} = \frac{\partial}{\partial t}\int_{\text{CV}} \eta\rho\, d\forall + \int_{\text{CS}} \eta\rho\vec{V}\cdot d\vec{A} \tag{4.11}$$

It is important to recall that in deriving Eq. 4.11, the limiting process (taking the limit as $\Delta t \to 0$) ensured that the relation is valid at the instant when the system and the control volume coincide. In using Eq. 4.11 to go from the system formulations of the basic laws to the control volume formulations, we recognize that Eq. 4.11 relates the rate of change of any extensive property, N, of a system to variations of this property associated with a control volume at the instant when the system and the control volume coincide; this is true since, in the limit as $\Delta t \to 0$, the system and the control volume occupy the same volume and have the same boundaries.

Before using Eq. 4.11 to develop control volume formulations of the basic laws, let us make sure we understand each of the terms and symbols in the equation:

$\left.\frac{dN}{dt}\right)_{\text{system}}$ is the rate of change of any arbitrary extensive property of the system.

$\frac{\partial}{\partial t}\int_{CV} \eta \rho \, dV$ is the time rate of change of arbitrary extensive property N within the control volume.

: η is the intensive property corresponding to N; $\eta = N$ per unit mass.

: $\rho \, dV$ is an element of mass contained in the control volume.

: $\int_{CV} \eta \rho \, dV$ is the total amount of extensive property N contained within the control volume.

$\int_{CS} \eta \rho \vec{V} \cdot d\vec{A}$ is the net rate of flux of extensive property N out through the control surface.

: $\rho \vec{V} \cdot d\vec{A}$ is the rate of mass flux through area element $d\vec{A}$ per unit time (we recognize that the dot product is a scalar product; the sign of $\rho \vec{V} \cdot d\vec{A}$ depends on the direction of the velocity vector $\vec{V}$, relative to the area vector $d\vec{A}$).

: $\eta \rho \vec{V} \cdot d\vec{A}$ is the rate of flux of extensive property N through area $d\vec{A}$

An additional point should be made about Eq. 4.11. Velocity $\vec{V}$ is measured relative to the surface of the control volume. In developing Eq. 4.11, we considered a control volume fixed relative to coordinate system xyz. Since the velocity field was specified relative to the same reference coordinates, it follows that velocity $\vec{V}$ is measured relative to the control volume.

We shall further emphasize this point in deriving the control volume formulation of each of the basic laws. In each case, we begin with the familiar system formulation and use Eq. 4.11 to relate system derivatives to the time rates of change associated with a fixed control volume at the instant when the system and the control volume coincide.[1]