**3-7 FLUIDS IN RIGID-BODY MOTION

A fluid in rigid-body motion moves without deformation as though it were a solid body. Since there is no deformation, there can be no shear stress. Consequently, the only surface stress on each element of fluid is that due to pressure.

A fluid particle retains its identity in rigid-body motion because the fluid does not deform. As in the case of the static fluid, we may apply Newton's second law of motion to determine the pressure field that results from a specified rigid-body motion.

In Section 3-1 we derived an expression for the total force due to pressure and gravity acting on a fluid particle of volume, $d\forall$. We obtained

$$d\vec{F} = (-\text{grad}\, p + \rho\vec{g})\, d\forall$$

or

$$\frac{d\vec{F}}{d\forall} = -\text{grad}\, p + \rho\vec{g} \tag{3.2}$$

Newton's second law was written

$$d\vec{F} = \vec{a}\, dm = \vec{a}\rho\, d\forall$$

or

$$\frac{d\vec{F}}{d\forall} = \rho\vec{a}$$

Substituting from Eq. 3.2, we obtain

$$-\text{grad}\, p + \rho\vec{g} = \rho\vec{a} \tag{3.16}$$

** This section may be omitted without loss of continuity in the text material.

The physical significance of each term in this equation is

$$-\text{grad}\, p \quad + \quad \rho\vec{g} \quad = \quad \rho\vec{a}$$

$$\left\{\begin{matrix}\text{pressure force}\\ \text{per unit volume}\\ \text{at a point}\end{matrix}\right\} + \left\{\begin{matrix}\text{body force}\\ \text{per unit volume}\\ \text{at a point}\end{matrix}\right\} = \left\{\begin{matrix}\text{mass per}\\ \text{unit}\\ \text{volume}\end{matrix}\right\} \times \left\{\begin{matrix}\text{acceleration}\\ \text{of fluid}\\ \text{particle}\end{matrix}\right\}$$

This vector equation consists of three component equations that must be satisfied individually. In rectangular coordinates the component equations are

$$\left.\begin{aligned} -\frac{\partial p}{\partial x} + \rho g_x &= \rho a_x \qquad x \text{ direction} \\ -\frac{\partial p}{\partial y} + \rho g_y &= \rho a_y \qquad y \text{ direction} \\ -\frac{\partial p}{\partial z} + \rho g_z &= \rho a_z \qquad z \text{ direction} \end{aligned}\right\} \tag{3.17}$$

Component equations for other coordinate systems can be written using the appropriate expression for grad p.

Example 3.8

As a result of a promotion, you are transferred from your present location. You must transport a fish tank in the back of your station wagon. The tank is 12 in. × 24 in. × 12 in. How much water should you leave in the tank to be reasonably sure that it will not spill over during the trip?

EXAMPLE PROBLEM 3.8

GIVEN: Fish tank 12 in. × 24 in. × 12 in. partially filled with water to be transported in an automobile.

FIND: Allowable depth of water for reasonable assurance that it will not spill during the trip.

SOLUTION:

The first step in the solution is to formulate the problem, by translating the general problem into a more specific one.

We recognize that there will be motion of the water surface as a result of the car's traveling over bumps in the road, going around corners, etc. However, we shall assume that the main effect on the water surface is due to linear accelerations (and decelerations) of the car; we shall neglect sloshing.

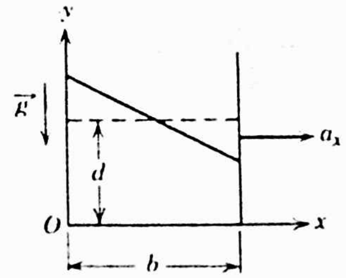
Thus we have reduced the problem to one of determining the effect of a linear acceleration on the free surface. We have not yet decided on the orientation of the tank relative to the direction of motion. Choosing the x coordinate in the direction of motion, should we align the tank with the long side parallel, or perpendicular, to the direction of motion?

If there will be no relative motion in the water, we must assume we are dealing with a constant acceleration, a_x. What is the shape of the free surface under these conditions?

Let us restate the problem to answer the original questions without making any restrictive assumptions at the outset.

GIVEN: Tank partially filled with water (to a depth d in.) subject to constant linear acceleration, a_x. Tank height is 12 in.; length parallel to direction of motion is b in. Width perpendicular to direction of motion is c in.

FIND: (a) Shape of free surface under constant a_x.
(b) Allowable water height, d, to avoid spilling as a function of a_x and tank orientation.
(c) Optimum tank orientation and allowable depth.



SOLUTION:

Basic equation:

$$-\nabla p + \rho\vec{g} = \rho\vec{a}$$

$$-\left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) + \rho(\hat{i}g_x + \hat{j}g_y + \hat{k}g_z) = \rho(\hat{i}a_x + \hat{j}a_y + \hat{k}a_z)$$

Since p is not a function of z, $\partial p/\partial z = 0$. Also, $g_x = 0$, $g_y = -g$, $g_z = 0$ and $a_y = a_z = 0$.

$$\therefore -\hat{i}\frac{\partial p}{\partial x} - \hat{j}\frac{\partial p}{\partial y} - \hat{j}\rho g = \hat{i}\rho a_x$$

The component equations are:

$$\frac{\partial p}{\partial x} = -\rho a_x$$

$$\frac{\partial p}{\partial y} = -\rho g$$

{Recall that a partial derivative means that all other independent variables are held constant in the differentiation.}

The problem now is to find an expression for $p = p(x, y)$. This would enable us to find the equation of the free surface. But perhaps we do not have to do that.

Since the pressure, $p = p(x, y)$, the difference in pressure between two points (x, y) and $(x + dx, y + dy)$ is

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Since the free surface is a line of constant pressure, then along the free surface, p = constant, so $dp = 0$ and

$$0 = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = -\rho a_x\,dx - \rho g\,dy$$

Therefore,

$$\left.\frac{dy}{dx}\right)_{\text{free surface}} = -\frac{a_x}{g}$$

{The free surface is a straight line.}

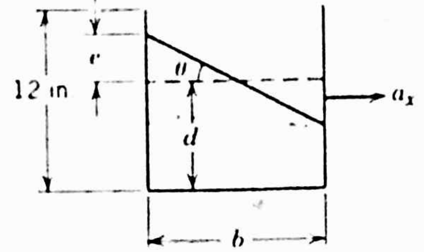
In the diagram below,

d = original depth

e = height above the original depth

b = tank length parallel to direction of motion

$$e = \frac{b}{2}\tan\theta = \frac{b}{2}\left(-\frac{dy}{dx}\right)_{\text{free surface}} = \frac{b}{2}\frac{a_x}{g} \qquad \left\{\text{Only valid for } d \geq \frac{b}{2}\right\}$$



Since we want e to be smallest for a given a_x, the tank should be aligned with b as small as possible. We should align the tank with the long side perpendicular to the direction of motion, that is, choose b = 12 in.

With b = 12 in.

$$e = 6\frac{a_x}{g} \text{ in.}$$

The maximum allowable value of $e = 12 - d$ in. Thus

$$12 - d = 6\frac{a_x}{g} \qquad \text{and} \qquad d_{max} = 12 - 6\frac{a_x}{g}$$

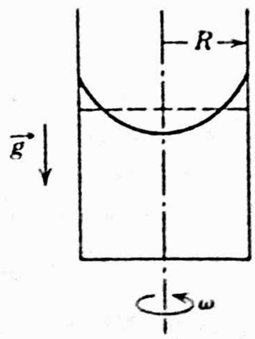
If the maximum a_x is assumed to be $\frac{2}{3}g$, then allowable d = 8 in.
To allow a margin of safety, perhaps we should select d = 6 in.

Recall that a steady acceleration was assumed in this problem. The car would have to be driven *very* carefully.

{This problem has been included to demonstrate:
(i) not all problems are clearly defined, nor do they have unique answers, and
(ii) the application of the equation, $-\nabla p + \rho\vec{g} = \rho\vec{a}$.}

Example 3.9

A cylindrical container, partially filled with liquid, is rotated at a constant angular velocity, ω, about its axis as shown in the diagram. After a short time there is no relative motion; the liquid rotates with the cylinder as if the system were a rigid body. Determine the shape of the free surface.



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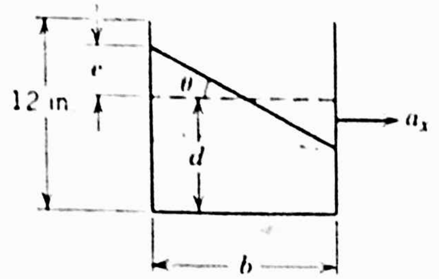

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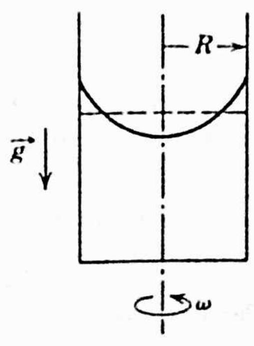
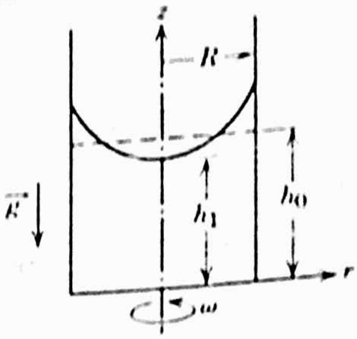


EXAMPLE PROBLEM 3.9

GIVEN: A cylinder of liquid in solid-body rotation with angular velocity, ω, about its axis.



FIND: The shape of the free surface.

SOLUTION:
It is convenient to use a cylindrical coordinate system, r, θ, z. Since there is circumferential symmetry in this problem, the pressure will not be a function of θ. Then $p = p(r, z)$. The free surface is a surface of constant pressure; the problem is to find the equation of the surface.

Since $p = p(r, z)$, the differential change, dp, in pressure between two points with coordinates (r, θ, z) and $(r + dr, \theta, z + dz)$ is given by

$$dp = \left.\frac{\partial p}{\partial r}\right)_z dr + \left.\frac{\partial p}{\partial z}\right)_r dz$$

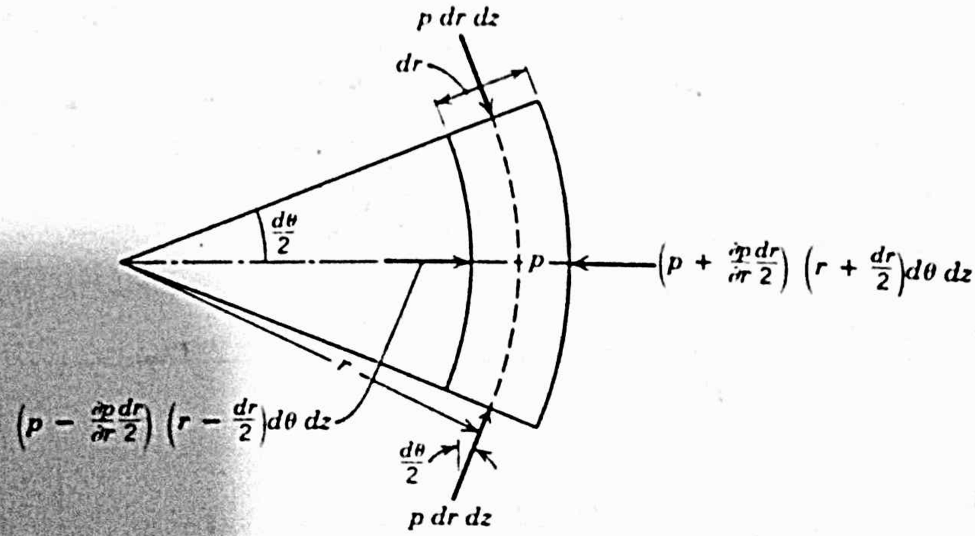
Consequently, we see that we need to obtain expressions for $\partial p/\partial z)_r$ and $\partial p/\partial r)_z$. This we can do by writing Newton's second law in the z and r directions, respectively, for an infinitesimal fluid element.

From Eq. 3.17 we have, for the z direction

$$-\left.\frac{\partial p}{\partial z}\right)_r + \rho g_z = \rho a_z$$

Since $g_z = -g$ and $a_z = 0$, then $\partial p/\partial z)_r = -\rho g$.

To obtain an expression for $\partial p/\partial r)_z$, we apply Newton's second law in the r direction to a suitable differential element.



The pressure at the center of the element is p. Using a Taylor series expansion, we express forces acting in the $r\theta$ plane on the element as shown in the diagram. Writing Newton's second law in the r direction, we have

$$\sum dF_r = a_r\,dm = a_r \rho\,d\forall = -\omega^2 r \rho\,d\forall = -\omega^2 r \rho r\,d\theta\,dr\,dz$$

From the figure

$$\sum dF_r = \left(p - \frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r - \frac{dr}{2}\right) d\theta\, dz - \left(p + \frac{\partial p}{\partial r}\frac{dr}{2}\right)\left(r + \frac{dr}{2}\right) d\theta\, dz + 2p\, dr\, dz \sin\frac{d\theta}{2}$$

Expanding and canceling like terms, recognizing $\sin d\theta/2 = d\theta/2$ (small angles) gives

$$\sum dF_r = d\theta\, dz \left\{\cancel{pr} - \cancel{p\frac{dr}{2}} - r\frac{\partial p}{\partial r}\frac{dr}{2} + \cancel{\frac{\partial p}{\partial r}\left(\frac{dr}{2}\right)^2} - \cancel{pr} - \cancel{p\frac{dr}{2}} - r\frac{\partial p}{\partial r}\frac{dr}{2} - \cancel{\frac{\partial p}{\partial r}\left(\frac{dr}{2}\right)^2} + \cancel{p\,dr}\right\}$$

$$\sum dF_r = d\theta\, dz \left\{-r\frac{\partial p}{\partial r}\, dr\right\}$$

Then

$$-r\frac{\partial p}{\partial r}\, dr\, d\theta\, dz = -\omega^2 r \rho r\, d\theta\, dr\, dz$$

Dividing both sides by $-r\, dr\, d\theta\, dz$ results in

$$\frac{\partial p}{\partial r} = \rho\omega^2 r$$

Since

$$dp = \left.\frac{\partial p}{\partial r}\right)_z dr + \left.\frac{\partial p}{\partial z}\right)_r dz$$

Then

$$dp = \rho\omega^2 r\, dr - \rho g\, dz$$

To obtain the pressure difference between a reference point (r_1, z_1), where the pressure is p_1, and the arbitrary point (r, z), where the pressure is p, we must integrate

$$\int_{p_1}^{p} dp = \int_{r_1}^{r} \rho\omega^2 r\, dr - \int_{z_1}^{z} \rho g\, dz$$

$$p - p_1 = \frac{\rho\omega^2}{2}(r^2 - r_1^2) - \rho g(z - z_1)$$

Taking the reference point on the cylinder axis at the free surface gives

$$p_1 = p_{atm} \qquad r_1 = 0 \qquad z_1 = h_1$$

Then

$$p - p_{atm} = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

Since the free surface is a surface of constant pressure ($p = p_{atm}$), the equation of the free surface is given by

$$0 = \frac{\rho\omega^2 r^2}{2} - \rho g(z - h_1)$$

or

$$z = h_1 + \frac{(\omega r)^2}{2g}$$

The equation of the free surface is a parabola with vertex on the axis at $z = h_1$.

We can solve for the height h_1 under conditions of rotation in terms of the original surface height, h_0, in the absence of rotation. To do this, we use the fact that the volume of fluid must remain constant. With no rotation

$$V = \pi R^2 h_0$$

With rotation

$$V = \int_0^R \int_0^z 2\pi r \, dz \, dr = \int_0^R 2\pi z r \, dr = \int_0^R 2\pi \left(h_1 + \frac{\omega^2 r^2}{2g} \right) r \, dr$$

$$V = 2\pi \left[h_1 \frac{r^2}{2} + \frac{\omega^2 r^4}{8g} \right]_0^R = \pi \left[h_1 R^2 + \frac{\omega^2 R^4}{4g} \right]$$

Then

$$\pi R^2 h_0 = \pi \left[h_1 R^2 + \frac{\omega^2 R^4}{4g} \right]$$

and

$$h_1 = h_0 - \frac{(\omega R)^2}{4g}$$

Finally,

$$z = h_0 - \frac{(\omega R)^2}{4g} + \frac{(\omega r)^2}{2g}$$

$$z = h_0 - \frac{(\omega R)^2}{2g} \left[\frac{1}{2} - \left(\frac{r}{R} \right)^2 \right]$$

$z(r)$

This problem illustrates the application of Newton's second law to a differential element **and the physical behavior** of a liquid with a free surface undergoing solid-body rotation.