Chapter 3

FLUID STATICS

By definition, a fluid must deform continuously when a shear stress of any magnitude is applied. The absence of relative motion (and thus, angular deformation) implies the absence of shear stresses. Therefore, fluids either at rest or in "rigid-body" motion are able to sustain only normal stresses. Analysis of hydrostatic cases is thus appreciably simpler than for fluids undergoing angular deformation (see Section 5–3.3).

Mere simplicity does not justify our study of a subject. Normal forces transmitted by fluids are important in many practical situations. Using the principles of hydrostatics, we can compute forces on submerged objects, develop instruments for measuring pressures, and deduce properties of the atmosphere and oceans. The principles of hydrostatics also may be used to determine forces developed by hydraulic systems in applications such as industrial presses or automobile brakes.

In a static fluid, or in a fluid undergoing rigid-body motion, a fluid particle retains its identity for all time. Since there is no relative motion within the fluid, a fluid element does not deform. We may apply Newton's second law of motion to evaluate the reaction of the particle to the applied forces.

3-1 THE BASIC EQUATION OF FLUID STATICS

Our primary objective is to obtain an equation that will enable us to determine the pressure field within the fluid. To do this, we choose a differential element of mass, dm, with sides dx, dy, and dz as shown in Fig. 3.1. The fluid element is stationary relative to the stationary rectangular coordinate system shown. (Fluids in rigid-body motion will be treated in Section 3–7.)

From our previous discussion, recall that two general types of forces may be applied to a fluid: body forces and surface forces. The only body force that must be considered in most engineering problems is due to gravity. In some situations body forces due to electric or magnetic fields might be present; they will not be considered in this text.

For a differential fluid element, the body force, $d\vec{F}_B$, is

$$d\vec{F}_B = \vec{g}\, dm = \vec{g}\, \rho\, d\forall$$

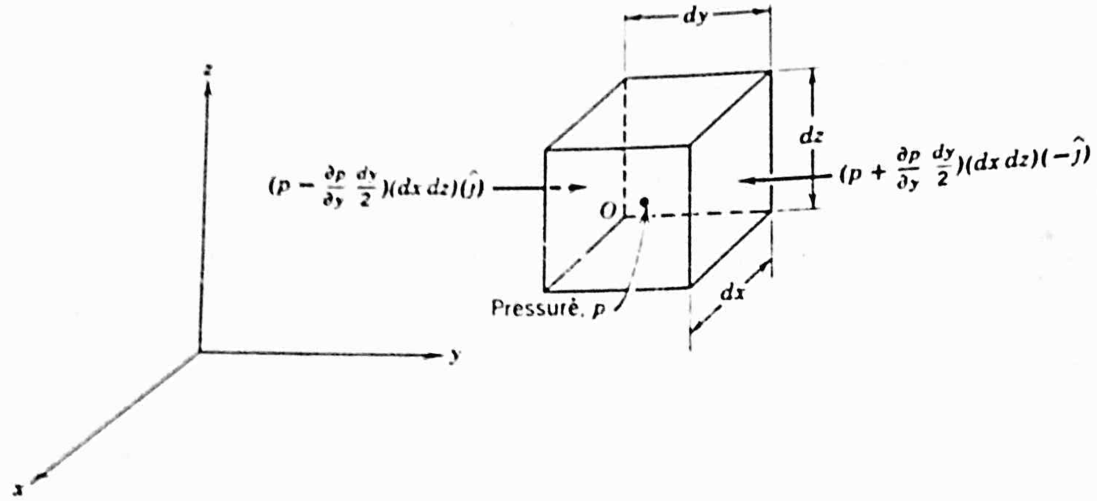


Fig. 3.1 Differential fluid element and pressure forces in the y direction.

where $\vec{g}$ is the local gravity vector, ρ is the density, and dV is the volume of the element. In Cartesian coordinates $dV = dx\,dy\,dz$, so

$$d\vec{F}_B = \rho\vec{g}\,dx\,dy\,dz$$

In a static fluid no shear stresses can be present. Thus the only surface force is the pressure force. Pressure is a field quantity, $p = p(x, y, z)$; the pressure varies with position within the fluid. The net pressure force that results from this variation can be evaluated by summing the forces that act on the six faces of the fluid element.

Let the pressure at the center, O, of the element be p. To determine the pressure at each of the six faces of the element, we use a Taylor series expansion of the pressure about the point O. The pressure at the left face of the differential element is

$$p_L = p + \frac{\partial p}{\partial y}(y_L - y) = p + \frac{\partial p}{\partial y}\left(-\frac{dy}{2}\right) = p - \frac{\partial p}{\partial y}\frac{dy}{2}$$

(Terms of higher order are omitted because they will vanish in the subsequent limiting process.) The pressure on the right face of the differential element is

$$p_R = p + \frac{\partial p}{\partial y}(y_R - y) = p + \frac{\partial p}{\partial y}\frac{dy}{2}.$$

The pressure *forces* acting on the two y surfaces of the differential element are shown in Fig. 3.1. Each pressure force is a product of three terms. The first is the magnitude of the pressure. The magnitude is multiplied by the area of the face to give the pressure force, and a unit vector is introduced to indicate direction. Note also in Fig. 3.1 that the pressure force on each face acts *against* the face. A positive pressure corresponds to a *compressive* stress.

Pressure forces on the other faces of the element are obtained in the same way. Combining all such forces gives the net surface force acting on the element. Thus

$$d\vec{F}_S = \left(p - \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dy\,dz)(\hat{i}) + \left(p + \frac{\partial p}{\partial x}\frac{dx}{2}\right)(dy\,dz)(-\hat{i})$$

$$+ \left(p - \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dx\,dz)(\hat{j}) + \left(p + \frac{\partial p}{\partial y}\frac{dy}{2}\right)(dx\,dz)(-\hat{j})$$

$$+ \left(p - \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dx\,dy)(\hat{k}) + \left(p + \frac{\partial p}{\partial z}\frac{dz}{2}\right)(dx\,dy)(-\hat{k})$$

Collecting and canceling terms, we obtain

$$d\vec{F}_S = \left(-\frac{\partial p}{\partial x}\hat{i} - \frac{\partial p}{\partial y}\hat{j} - \frac{\partial p}{\partial z}\hat{k}\right)dx\,dy\,dz$$

or,

$$d\vec{F}_S = -\left(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}\right)dx\,dy\,dz \tag{3.1a}$$

The term in parentheses is called the gradient of the pressure or simply the pressure gradient and may be written grad p or ∇p. In rectangular coordinates

$$\text{grad}\, p \equiv \nabla p \equiv \left(\hat{i}\frac{\partial p}{\partial x} + \hat{j}\frac{\partial p}{\partial y} + \hat{k}\frac{\partial p}{\partial z}\right) \equiv \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)p$$

The gradient can be viewed as a vector operator; taking the gradient of a scalar field gives rise to a vector field. Using the gradient designation, Eq. 3.1a can be written as

$$d\vec{F}_S = -\text{grad}\, p(dx\,dy\,dz) = -\nabla p\,dx\,dy\,dz \tag{3.1b}$$

From Eq. 3.1b,

$$\boxed{\text{grad}\, p = \nabla p = -\frac{d\vec{F}_S}{dx\,dy\,dz}}$$

Physically the gradient of pressure is the negative of the surface force per unit volume due to pressure. We note that the level of pressure is not important in evaluating the net pressure force. Instead, what matters is the rate at which pressure changes occur with distance, the *pressure gradient*. We shall find this term very useful throughout our study of fluid mechanics.

Since no other kinds of force may be present in a static fluid, we can combine the formulations for surface and body forces that we have developed to obtain the total force acting on a fluid element. Thus

$$d\vec{F} = d\vec{F}_S + d\vec{F}_B = (-\text{grad}\, p + \rho\vec{g})\,dx\,dy\,dz$$

or on a unit volume basis

$$\frac{d\vec{F}}{d\forall} = \frac{d\vec{F}}{dx\,dy\,dz} = -\text{grad}\, p + \rho\vec{g} \tag{3.2}$$

For a fluid particle, Newton's second law gives $d\vec{F} = \vec{a}\,dm = \vec{a}\rho\,dV$. For a static fluid, $\vec{a} = 0$. Thus

$$\frac{d\vec{F}}{dV} = \rho\vec{a} = 0$$

Substituting for $d\vec{F}/dV$ from Eq. 3.2, we obtain

$$-\operatorname{grad} p + \rho\vec{g} = 0 \tag{3.3}$$

Let us review briefly our derivation of this equation. The physical significance of each term is

$$\begin{matrix} -\operatorname{grad} p & + & \rho\vec{g} & = 0 \\ \left\{\begin{matrix}\text{pressure force}\\ \text{per unit volume}\\ \text{at a point}\end{matrix}\right\} & + & \left\{\begin{matrix}\text{body force per}\\ \text{unit volume}\\ \text{at a point}\end{matrix}\right\} & = 0\end{matrix}$$

This is a vector equation, which means that it consists of three component equations that must be satisfied individually. The components are

$$\left.\begin{matrix} -\dfrac{\partial p}{\partial x} + \rho g_x = 0 & x \text{ direction} \\ -\dfrac{\partial p}{\partial y} + \rho g_y = 0 & y \text{ direction} \\ -\dfrac{\partial p}{\partial z} + \rho g_z = 0 & z \text{ direction}\end{matrix}\right\} \tag{3.4}$$

Equations 3.4 describe the pressure variation in each of the three coordinate directions in a static fluid. To simplify further, it is logical to choose a coordinate system such that the gravity vector is aligned with one of the axes. If the coordinate system is chosen such that the z axis is directed vertically, then $g_x = 0$, $g_y = 0$, and $g_z = -g$. Under these conditions, the component equations become

$$\frac{\partial p}{\partial x} = 0, \qquad \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = -\rho g \tag{3.5}$$

Equations 3.5 indicate that under the assumptions made, the pressure is independent of coordinates x and y; it depends on z alone. Thus since p is a function of a single variable, a total derivative may be used instead of a partial derivative. With these simplifications, Eqs. 3.5 finally reduce to

$$\frac{dp}{dz} = -\rho g \equiv -\gamma \tag{3.6}$$

Restrictions: (1) Static fluid
(2) Gravity is the only body force
(3) The z axis is vertical

This equation is the basic pressure-height relation of fluid statics. It is subject to the restrictions noted. Therefore it must be applied only where these restrictions are reasonable for the physical situation. To determine the pressure distribution in a static fluid, Eq. 3.6 may be integrated and appropriate boundary conditions applied.

3-1.1 Pressure Variation in a Static Fluid

Although ρg may be defined as the *specific weight*, γ, it has been written as ρg in Eq. 3.6 to emphasize that *both* ρ and g must be considered variables. In order to integrate Eq. 3.6 to find the pressure distribution, assumptions must be made about variations in both ρ and g.

For most practical engineering situations, the variation in g will be negligible. Only for a situation such as computing very precisely the pressure change over a large elevation difference would the variation in g need to be included. For our purposes we shall assume g to be constant with elevation at any given location.

In many practical engineering problems the variation in ρ will be appreciable, and accurate results will require that it be accounted for. Several types of variation are easy to treat analytically. The simplest is the idealization of an incompressible fluid.

a. Incompressible Fluid

For an incompressible fluid, $\rho = \rho_0 =$ constant. Then for constant gravity,

$$\frac{dp}{dz} = -\rho_0 g = \text{constant}$$

To determine the pressure variation, we must integrate this equation and apply appropriate boundary conditions. If the pressure at the reference level, z_0, is designated as p_0, then the pressure, p, at location z is found by integration

$$\int_{p_0}^{p} dp = -\int_{z_0}^{z} \rho_0 g \, dz$$

or

$$p - p_0 = -\rho_0 g(z - z_0) = \rho_0 g(z_0 - z)$$

For liquids, it is often convenient to take the origin of the coordinate system at the free surface (reference level) and to measure distances as positive downward from the free surface, as shown in Fig. 3.2. With h measured positive downward, then

$$z_0 - z = h$$

and

$$p = p_0 + \rho_0 g h \tag{3.7}$$

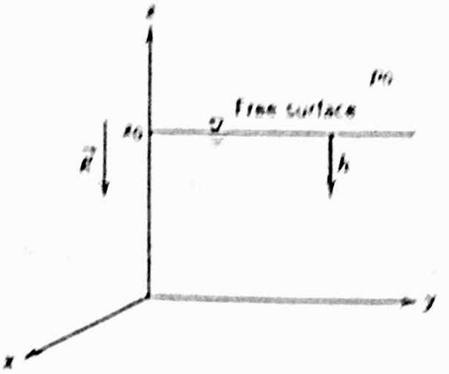


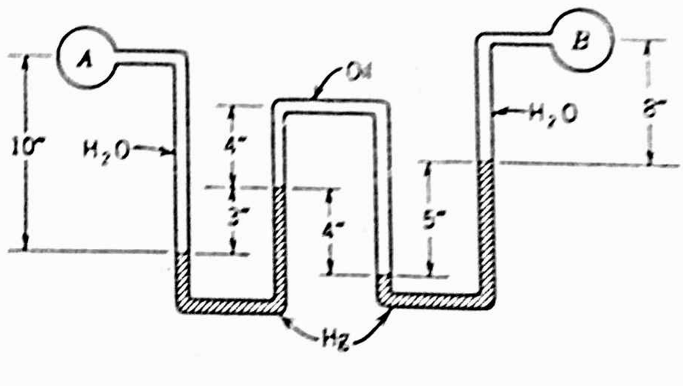
Fig. 3.2 Coordinates for determination of pressure variation in a static liquid.

This form of the basic pressure-height relation often is used to solve manometer problems. Students sometimes have trouble analyzing multiple tube manometer situations. The following rules of thumb are useful:

1. Any two points at the same elevation in a continuous length of the same liquid are at the same pressure.
2. Pressure increases as one goes *down* a liquid column (remember the pressure change on diving into a swimming pool).

Example 3.1

Water flows through pipes A and B. Oil, with specific gravity 0.8, is in the upper portion of the inverted U. Mercury (specific gravity 13.6) is in the bottom of the manometer bends. Determine the pressure difference, $p_A - p_B$, in units of lbf/in.^{2}



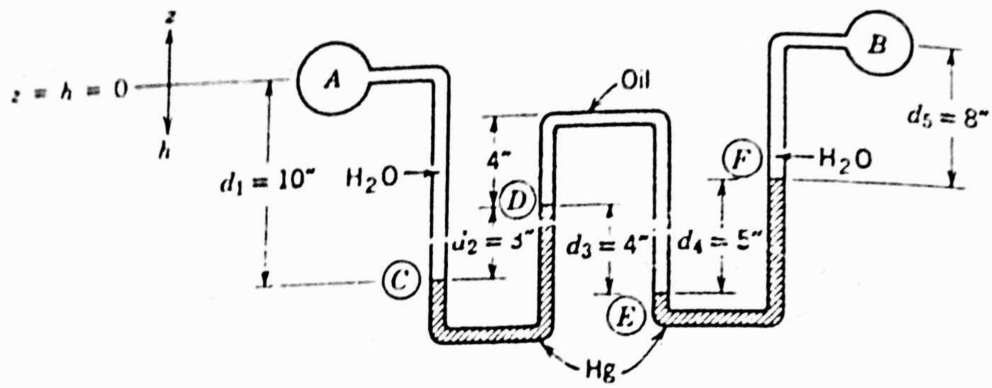
EXAMPLE PROBLEM 3.1

GIVEN: Multiple tube manometer as shown. Specific gravity of oil is 0.8; specific gravity of mercury is 13.6.

FIND: The pressure difference, $p_A - p_B$, in lbf/in.^{2}

SOLUTION:

Basic equations: $$\frac{dp}{dz} = -\frac{dp}{dh} = -\rho g = -\gamma \qquad SG = \frac{\rho}{\rho_{H_2O}} = \frac{\gamma}{\gamma_{H_2O}}$$



Then

$$dp = \gamma \, dh \quad \text{and} \quad \int_{p_1}^{p_2} dp = \int_{h_1}^{h_2} \gamma \, dh$$

For $\gamma =$ constant

$$p_2 - p_1 = \gamma(h_2 - h_1)$$

Beginning at point A and applying the equation between successive points around the manometer gives

$$p_C - p_A = +\gamma_{H_2O} d_1$$

$$p_D - p_C = -\gamma_{Hg} d_2$$

$$p_E - p_D = +\gamma_{oil} d_3$$

$$p_F - p_E = -\gamma_{Hg} d_4$$

$$p_B - p_F = -\gamma_{H_2O} d_5$$

$$p_A - p_B = (p_A - p_C) + (p_C - p_D) + (p_D - p_E) + (p_E - p_F) + (p_F - p_B)$$
$$= -\gamma_{H_2O} d_1 + \gamma_{Hg} d_2 - \gamma_{oil} d_3 + \gamma_{Hg} d_4 + \gamma_{H_2O} d_5$$

Substituting $\gamma = SG\gamma_{H_2O}$ yields

$$p_A - p_B = -\gamma_{H_2O} d_1 + 13.6\gamma_{H_2O} d_2 - 0.8\gamma_{H_2O} d_3 + 13.6\gamma_{H_2O} d_4 + \gamma_{H_2O} d_5$$
$$= \gamma_{H_2O}(-d_1 + 13.6 d_2 - 0.8 d_3 + 13.6 d_4 + d_5)$$
$$= \gamma_{H_2O}(-10 + 40.8 - 3.2 + 68 + 8) \text{ in.}$$
$$= \gamma_{H_2O} \times 103.6 \text{ in.}$$
$$= \frac{62.4 \text{ lbf}}{\text{ft}^3} \times 103.6 \text{ in.} \times \frac{\text{ft}}{12 \text{ in.}} \times \frac{\text{ft}^2}{144 \text{ in.}^2}$$

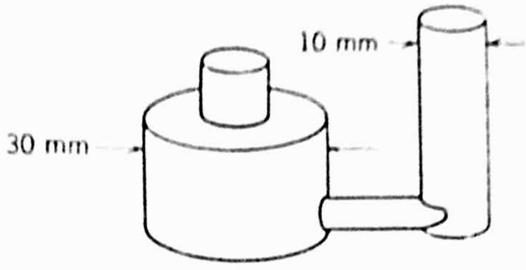
$$p_A - p_B = 3.74 \text{ lbf/in.}^2$$

$p_A - p_B$

Manometers are simple and inexpensive devices used frequently for pressure measurements. Because the liquid level change is small at low pressure differential, a U-tube manometer may be difficult to read accurately. The level change can be increased by changing the manometer design or by using two liquids of slightly different density. Analysis of a typical reservoir manometer design is illustrated in Example Problem 3.2.

Example 3.2

A reservoir manometer is built with a tube diameter of 10 mm and a reservoir diameter of 30 mm. The manometer liquid is Meriam red oil with SG = 0.827. Determine the manometer deflection in millimeters per millimeter of water applied pressure differential.

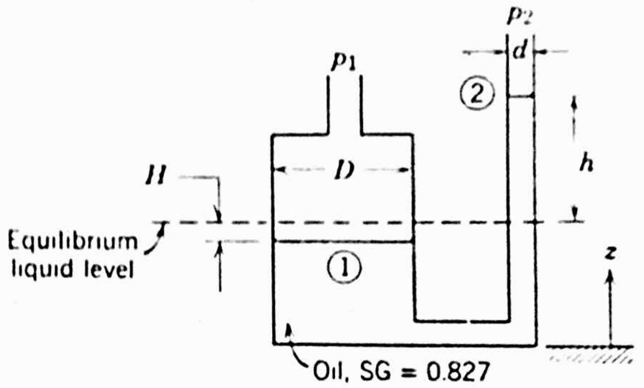


EXAMPLE PROBLEM 3.2

GIVEN: Reservoir manometer as shown.

$d = 10$ mm

$D = 30$ mm



FIND: Liquid deflection, h, in millimeters per millimeter of water applied pressure differential.

SOLUTION:

Basic equations: $\frac{dp}{dz} = -\rho g$, $\quad SG = \frac{\rho}{\rho_{H_2O}}$

Then

$$dp = -\rho g\, dz \quad \text{and} \quad \int_{p_1}^{p_2} dp = -\int_{z_1}^{z_2} \rho g\, dz$$

For ρ = constant

$$p_2 - p_1 = -\rho g(z_2 - z_1)$$

or

$$p_1 - p_2 = \rho g(z_2 - z_1) = \rho_{oil} g(h + H)$$

To eliminate H, note that the *volume* of manometer liquid must remain constant. Thus the volume displaced from the reservoir must be the same as that which rises into the tube,

$$\frac{\pi}{4} D^2 H = \frac{\pi}{4} d^2 h \quad \text{or} \quad H = \left(\frac{d}{D}\right)^2 h$$

Substituting gives

$$p_1 - p_2 = \rho_{oil} g h \left[1 + \left(\frac{d}{D}\right)^2\right]$$

This equation can be simplified by expressing the applied pressure differential as an equivalent water column of height Δh_e

$$p_1 - p_2 = \rho_{H_2O} g \, \Delta h_e$$

and noting that $\rho_{oil} = SG_{oil}\rho_{H_2O}$. Then

$$\rho_{H_2O} g \, \Delta h_e = SG_{oil}\rho_{H_2O} g h \left[1 + \left(\frac{d}{D}\right)^2\right]$$

or

$$\frac{h}{\Delta h_e} = \frac{1}{SG_{oil}[1 + (d/D)^2]}$$

Evaluating, we obtain

$$\frac{h}{\Delta h_e} = \frac{1}{0.827[1 + (10/30)^2]} = 1.09 \qquad \frac{h}{\Delta h_e}$$

This problem illustrates the effects of manometer design and choice of gage liquid on sensitivity.