2-3 STRESS FIELD

Surface and body forces are encountered in the study of continuum fluid mecha *Surface forces* include all forces acting on the boundaries of a medium through d contact. Forces developed without physical contact, and distributed over the volun the fluid, are termed *body forces*. Gravitational and electromagnetic forces examples of body forces arising in a fluid.

The gravitational body force acting on an element of volume, dV, is given by $\rho \vec{g}$ where ρ is the density (mass per unit volume) and $\vec{g}$ is the local gravitati acceleration. Thus the gravitational body force per unit volume is $\rho\vec{g}$ and gravitational body force per unit mass is $\vec{g}$.

Stresses in a medium result from forces acting on some portion of the medium. concept of stress provides a convenient means to describe the manner in which fo acting on the boundaries of the medium are transmitted through the medium. S force and area are both vector quantities, we might anticipate that the stress field not be a vector field. We shall show that, in general, nine quantities are require specify the state of stress in a fluid. (Stress is a tensor quantity of second order.)

In a flowing fluid, consider a portion, $d\vec{A}$, of the surface passing through the poi The orientation of $d\vec{A}$ is given by the unit vector, $\hat{n}$, as shown in Fig. 2.5. The direc of $\hat{n}$ is normal to the surface.

The force, $\delta\vec{F}$, acting on $\delta\vec{A}$ may be resolved into two components, one normal the other tangential to the area. A normal stress σ_n and a shear stress τ_n are then def

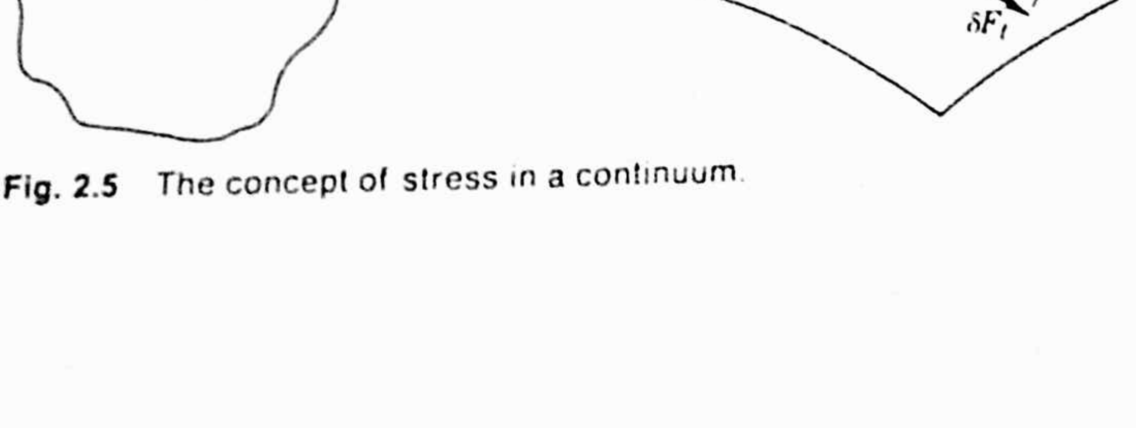


Fig. 2.5 The concept of stress in a continuum.

as

$$\sigma_n = \lim_{\delta A_n \to 0} \frac{\delta F_n}{\delta A_n} \tag{2.6}$$

and

$$\tau_n = \lim_{\delta A_n \to 0} \frac{\delta F_t}{\delta A_n} \tag{2.7}$$

The subscript, n, on the stress is included as a reminder that the stresses are associated with a particular surface $\delta \vec{A}$ through C, namely, the one having an outer normal in the $\hat{n}$ direction through C. For any other surface through C the values of the stresses could be different.

For purposes of analysis we usually reference the area to some coordinate system. In rectangular coordinates we might consider the stresses acting on planes whose outward drawn normals are in the x, y, or z directions. In Fig. 2.6 we consider the stress on the element, δA_x, whose outward drawn normal is in the x direction. The force, $\delta \vec{F}$, has been resolved into components along each of the coordinate directions. Dividing the magnitude of each force component by the area and taking the limit as δA_x approaches zero, we define the three stress components shown in Fig. 2.6b:

$$\sigma_{xx} = \lim_{\delta A_x \to 0} \frac{\delta F_x}{\delta A_x} \tag{2.8}$$

$$\tau_{xy} = \lim_{\delta A_x \to 0} \frac{\delta F_y}{\delta A_x} \qquad \tau_{xz} = \lim_{\delta A_x \to 0} \frac{\delta F_z}{\delta A_x}$$

We have used a double subscript notation to label the stresses. The first subscript (in this case, x) indicates the plane on which the stress acts (in this case, a surface perpendicular to the x axis). The second subscript indicates the direction in which the stress acts.

Consideration of an area element, δA_y, would lead to the definitions of the stresses, σ_{yy}, τ_{yx}, and τ_{yz}; use of area element, δA_z, would similarly lead to the definitions of σ_{zz}, τ_{zx}, and τ_{zy}.

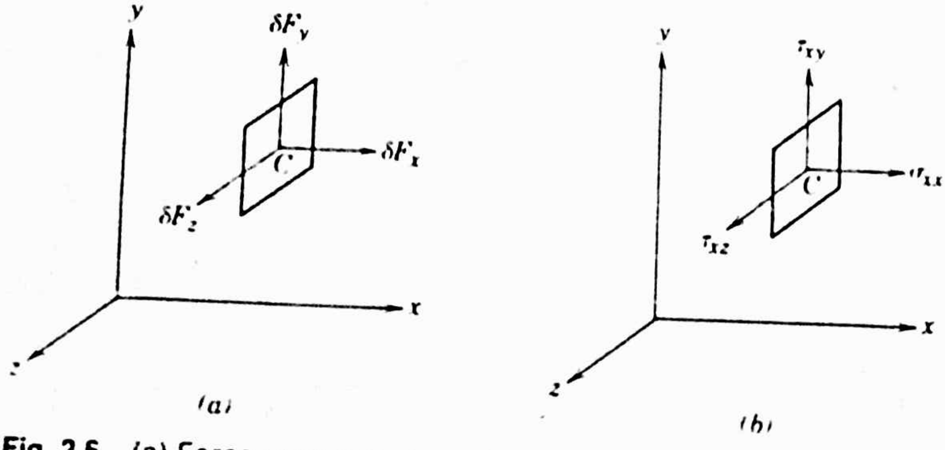


Fig. 2.6 (a) Force components, and (b) stress components, on the element of area, δA_x.

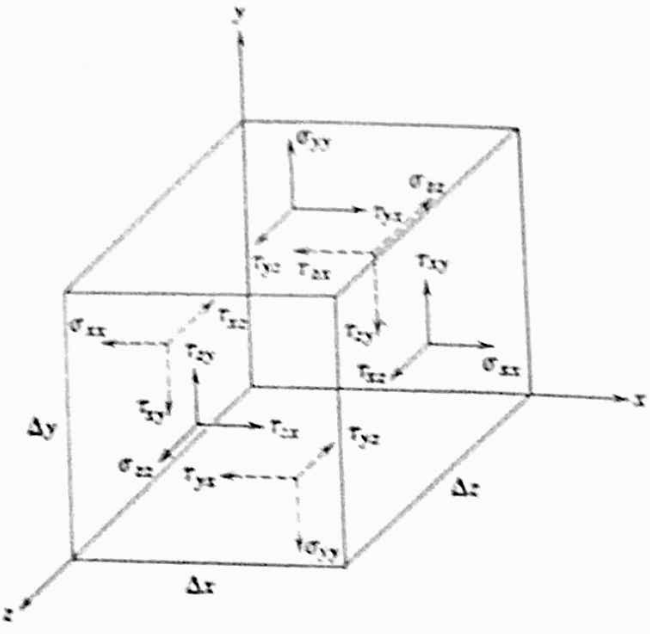


Fig. 2.7 Notation for stress.

An infinite number of planes can be passed through point C, resulting in an infinite number of stresses associated with that point. Fortunately, the state of stress at a point can be described completely by specifying the stresses acting on three mutually perpendicular planes through the point. The stress at a point is specified by the nine components

$$\begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix}.$$

where σ has been used to denote a normal stress and shear stresses are denoted by the symbol, τ. The notation for designating stress is shown in Fig. 2.7.

Referring to the infinitesimal element shown in Fig. 2.7, we see that there are six planes (two x planes, two y planes, and two z planes) on which stresses may act. In order to designate the plane of interest, we could use terms like front and back, top and bottom, or left and right. However, it is more logical to name the planes in terms of the coordinate axes. The planes are named and denoted as positive or negative according to the direction of the outward drawn normal to the plane. Thus the top plane, for example, is a positive y plane and the back plane is a negative z plane.

It also is necessary to adopt a sign convention for the stress. A stress component is considered positive when the direction of the stress component and the plane on which it acts are both positive or both negative. Thus $\tau_{yx} = 5\ \text{lbf/in.}^2$ represents a shear stress on a positive y plane in the positive x direction or a shear stress on a negative y plane in the negative x direction. In Fig. 2.7 all stresses have been drawn as positive stresses. Stress components are negative when the direction of the stress component and the plane on which it acts are of opposite sign.

2-4 VISCOSITY

We have defined a fluid as a substance that deforms continuously under the action of a shear stress. In the absence of a shear stress, there will be no deformation. Fluids may be broadly classified according to the relation between the applied shear stress and the rate of deformation.

Consider the behavior of a fluid element between the two infinite plates shown in Fig. 2.8. The upper plate moves at constant velocity, δu, under the influence of a constant applied force, δF_x. The shear stress, τ_{yx}, applied to the fluid element is given by

$$\tau_{yx} = \lim_{\delta A_y \to 0} \frac{\delta F_x}{\delta A_y} = \frac{dF_x}{dA_y}$$

where δA_y is the area of the fluid element in contact with the plate. During time interval δt, the fluid element is deformed from position $MNOP$ to position $M'NOP'$. The rate of deformation of the fluid is then given by

$$\textit{deformation rate} = \lim_{\delta t \to 0} \frac{\delta \alpha}{\delta t} = \frac{d\alpha}{dt}$$

To calculate the shear stress, τ_{yx}, it is desirable to express $d\alpha/dt$ in terms of readily measurable quantities. This can be done easily. The distance, δl, between the points M and M' is given by

$$\delta l = \delta u\, \delta t$$

or alternatively, for small angles,

$$\delta l = \delta y\, \delta \alpha$$

Equating these two expressions for δl gives

$$\frac{\delta \alpha}{\delta t} = \frac{\delta u}{\delta y}$$

Taking the limit of both sides of the equality, we obtain

$$\frac{d\alpha}{dt} = \frac{du}{dy}$$

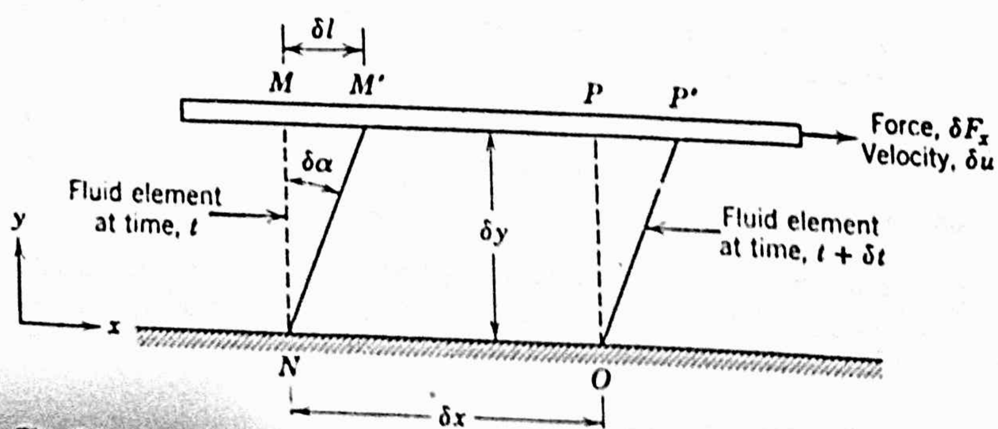


Fig. 2.8 Deformation of a fluid element.

Thus, the fluid element of Fig. 2.8, when subjected to shear stress, τ_{yx}, experiences a rate of deformation (or *shear rate*) given by du/dy. Fluids in which shear stress is directly proportional to rate of deformation are termed *Newtonian fluids*. The term *non-Newtonian* is used to classify all fluids in which shear stress is not directly proportional to shear rate.

2-4.1 Newtonian Fluid

Most common fluids such as water, air, and gasoline are Newtonian under normal conditions. If the fluid of Fig. 2.8 is Newtonian, then

$$\tau_{yx} \propto \frac{du}{dy} \tag{2.9}$$

The shear stress acts on a plane normal to the y axis. If one considers the deformation of two different Newtonian fluids, say glycerin and water, one recognizes that they will deform at different rates under the action of the same applied shear stress. Glycerin exhibits a much larger resistance to deformation than water. Thus we say it is much more viscous. The constant of proportionality in Eq. 2.9 is the *absolute* (or *dynamic*) *viscosity*, μ. Thus in terms of the coordinates of Fig. 2.8, Newton's law of viscosity is given for one-dimensional flow by

$$\tau_{yx} = \mu \frac{du}{dy} \tag{2.10}$$

Note that since the dimensions of τ are $[F/L^2]$ and the dimensions of du/dy are $[1/t]$, then μ has dimensions $[Ft/L^2]$. Since the dimensions of force, F, mass, M, length, L, and time, t, are related by Newton's second law of motion, the dimensions of μ can also be expressed as $[M/Lt]$. In the British Gravitational system, the units of viscosity are $\text{lbf} \cdot \text{sec/ft}^2$ or $\text{slug/ft} \cdot \text{sec}$. In the Absolute Metric system, the basic unit of viscosity is called a poise (poise $\equiv$ g/cm · sec); in the SI system the units of viscosity are kg/m · sec or Pa · sec ($= \text{N} \cdot \text{sec/m}^2$). The calculation of viscous shear stress is illustrated in Example Problem 2.2.

In fluid mechanics the ratio of absolute viscosity, μ, to density, ρ, often arises. This ratio is given the name *kinematic viscosity* and is represented by the symbol, ν. Since density has dimensions $[M/L^3]$, the dimensions of ν are $[L^2/t]$. In the Absolute Metric system of units, the unit for ν is a stoke (stoke $\equiv$ cm^2/sec).

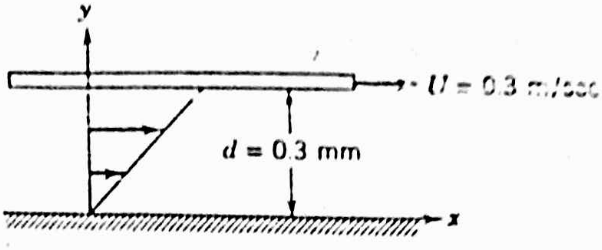
Viscosity data for a number of common Newtonian fluids are given in Appendix A. Note that for gases viscosity increases with temperature while for liquids, viscosity decreases with increasing temperature.

In gases the resistance to deformation is primarily due to the transfer of molecular momentum. Molecules from regions of high bulk velocity collide with molecules moving with lower bulk velocity, and vice versa. These collisions transport momentum from one region of fluid to another. Since the random molecular motions increase with increasing temperature, viscosity also increases with temperature.

For liquids, where molecules are much more closely packed, resistance to deformation is primarily controlled by cohesive forces among molecules. These cohesive forces decrease with increasing temperature and hence the viscosity of liquids decreases with temperature.

Example 2.2

An infinite plate is moved over a second plate on a layer of liquid as shown. For small gap width, d, we assume a linear velocity distribution in the liquid. The liquid viscosity is 0.65 centipoise and its specific gravity is 0.88. Calculate:

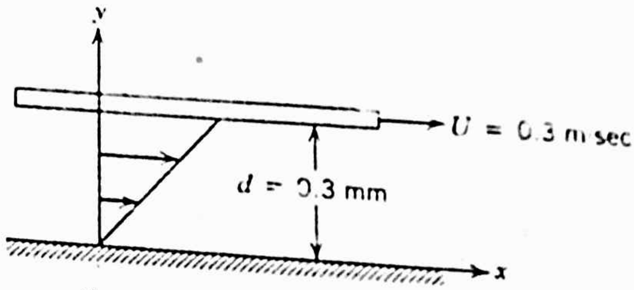


(a) The absolute viscosity of the liquid, in $\text{lbf} \cdot \text{sec/ft}^2$
(b) The kinematic viscosity of the liquid, in m^2/sec
(c) The shear stress on the upper plate, in lbf/ft^2
(d) The shear stress on the lower plate, in Pa
(e) Indicate the direction of each shear stress calculated in parts (c) and (d)

EXAMPLE PROBLEM 2.2

GIVEN: Linear velocity profile in the liquid between infinite parallel plates as shown.

$\mu = 0.65$ cp (1 poise = 1 g/cm · sec)
SG = 0.88



FIND: (a) μ in units of $\text{lbf} \cdot \text{sec/ft}^2$.
(b) ν in units of m^2/sec.
(c) τ on upper plate in units of lbf/ft^2.
(d) τ on lower plate in units of Pa.
(e) Direction of stress in parts (c) and (d).

SOLUTION:

Basic equation: $\tau_{yx} = \mu \dfrac{du}{dy}$ Definition: $\nu = \dfrac{\mu}{\rho}$

(a) $$\mu = 0.65\ \text{cp} \times \frac{\text{poise}}{100\ \text{cp}} \times \frac{\text{g}}{\text{cm} \cdot \text{sec} \cdot \text{poise}} \times \frac{\text{lbm}}{453.6\ \text{g}} \times \frac{\text{slug}}{32.2\ \text{lbm}} \times 30.48\ \frac{\text{cm}}{\text{ft}} \times \frac{\text{lbf} \cdot \text{sec}^2}{\text{slug} \cdot \text{ft}}$$

$\mu = 1.36 \times 10^{-5}\ \text{lbf} \cdot \text{sec/ft}^2$ μ

(b) $$\nu = \frac{\mu}{\rho} = \frac{\mu}{SG\rho_{H_2O}}$$

$$= 1.36 \times 10^{-5}\ \frac{\text{lbf} \cdot \text{sec}}{\text{ft}^2} \times \frac{\text{ft}^3}{(0.88)1.94\ \text{slug}} \times \frac{\text{slug} \cdot \text{ft}}{\text{lbf} \cdot \text{sec}^2} \times (0.3048)^2\ \frac{\text{m}^2}{\text{ft}^2}$$

$\nu = 7.40 \times 10^{-7}\ \text{m}^2/\text{sec}$ ν

(c) $\tau_{upper} = \tau_{yx,upper} = \mu \left.\frac{du}{dy}\right)_{y=d}$. Since u varies linearly with y,

$$\frac{du}{dy} = \frac{\Delta u}{\Delta y} = \frac{U-0}{d-0} = \frac{U}{d} = 0.3\,\frac{\text{m}}{\text{sec}} \times \frac{1}{0.3\text{ mm}} \times 1000\,\frac{\text{mm}}{\text{m}} = 1000\text{ sec}^{-1}$$

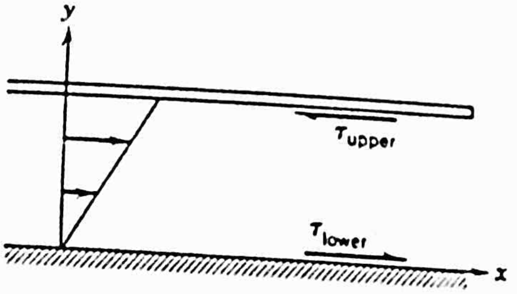
$$\tau_{upper} = \mu\frac{U}{d} = 1.36 \times 10^{-5}\,\frac{\text{lbf}\cdot\text{sec}}{\text{ft}^2} \times \frac{1000}{\text{sec}} = 0.0136\text{ lbf/ft}^2$$

τ_{upper}

(d) $$\tau_{lower} = \mu\frac{U}{d} = 0.0136\,\frac{\text{lbf}}{\text{ft}^2} \times 4.448\,\frac{\text{N}}{\text{lbf}} \times \frac{\text{ft}^2}{(0.3048)^2\text{ m}^2} \times \frac{\text{Pa}\cdot\text{m}^2}{\text{N}} = 0.651\text{ Pa}$$

τ_{lower}

(c) Direction of shear stress on upper and lower plates.



{The upper plate is a negative y surface, so positive τ_{yx} acts in the negative x direction.}

{The lower plate is a positive y surface, so positive τ_{yx} acts in the positive x direction.}

(e)

2-4.2 Non-Newtonian Fluids

Many common fluids exhibit non-Newtonian behavior. Two familiar examples are toothpaste and Lucite[6] paint. The latter is very "thick" when in the can, but becomes "thin" when sheared by brushing. Toothpaste behaves as a "fluid" when squeezed from the tube. However, it does not run out by itself when the cap is removed. There is a threshold or yield stress below which toothpaste behaves as a solid. Strictly speaking, our definition of a fluid is valid only for materials that have zero yield stress. The term non-Newtonian is used to classify all fluids in which shear stress is not directly proportional to deformation rate. Such fluids commonly are classified as having time-independent, time-dependent, or viscoelastic behavior. Four examples of time-independent behavior are shown in the rheological diagram of Fig. 2.9.

Numerous empirical equations have been proposed to model the observed relations between τ_{yx} and du/dy for time-independent fluids. They may be adequately represented for many engineering applications by the power law model, which for one-dimensional flow becomes

$$\tau_{yx} = k\left(\frac{du}{dy}\right)^n \tag{2.11}$$

where the exponent, n, is called the flow behavior index and k, the consistency index. This equation reduces to Newton's law of viscosity for $n = 1$ with $k = \mu$.

[6] Trademark, E. I. du Pont de Nemours & Company.

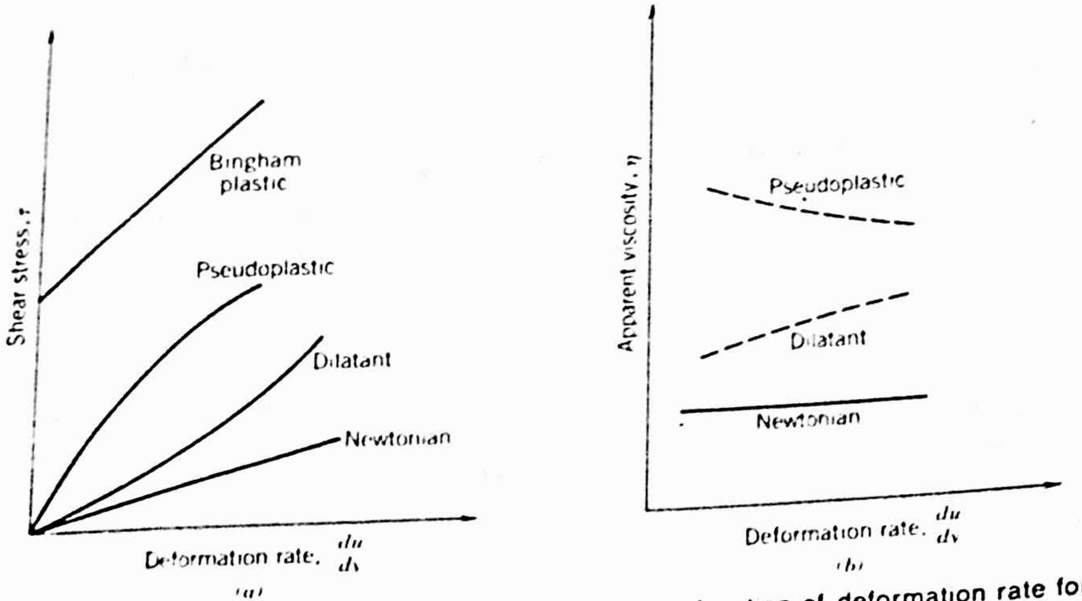


Fig. 2.9 (a) Shear stress, τ, and (b) apparent viscosity, η, as a function of deformation rate for one-dimensional flow of various non-Newtonian fluids.

If Eq. 2.11 is rewritten in the form,

$$\tau_{yx} = k\left|\frac{du}{dy}\right|^{n-1}\frac{du}{dy} = \eta\frac{du}{dy} \tag{2.12}$$

then $\eta = k|du/dy|^{n-1}$ is referred to as the *apparent viscosity*.

Fluids in which the apparent viscosity decreases with increasing deformation rate ($n < 1$) are called *pseudoplastic* (or shear thinning) fluids. Most non-Newtonian fluids fall into this group; examples include polymer solutions, colloidal suspensions, and paper pulp in water. If the apparent viscosity increases with increasing deformation rate ($n > 1$) the fluid is termed *dilatant* (or shear thickening). Suspensions of starch and of sand are examples of dilatant fluids.

A "fluid" that behaves as a solid until a minimum yield stress, τ_y, is exceeded and subsequently exhibits a linear relation between stress and rate of deformation is referred to as an ideal or *Bingham plastic*. The shear stress model is then

$$\tau_{yx} = \tau_y + \mu_p\frac{du}{dy} \tag{2.13}$$

Clay suspensions, drilling muds, and toothpaste are examples of substances exhibiting this behavior.

Most non-Newtonian fluids have apparent viscosities that are relatively high compared to the viscosity of water.

The study of non-Newtonian fluids is further complicated by the fact that the apparent viscosity may be time-dependent.[7] Fluids that show a decrease in η with time

[7] Examples of time-dependent fluids are illustrated in the film, *Rheological Behavior of Fluids*, H. Markovitz, principal.

under a constant applied shear stress are called *thixotropic*; many paints are thixotropic. Fluids that show an increase in η with time are termed *rheopectic*. In addition, some fluids after deformation partially return to their original shape when the applied stress is released; such fluids are called *viscoelastic*.