Chapter 2

FUNDAMENTAL CONCEPTS

In Chapter 1 we indicated that our study of fluid mechanics will build on earlier studies in mechanics and thermodynamics. To develop a unified approach, we review some familiar topics and introduce some new concepts and definitions. The purpose of this chapter is to develop these fundamental concepts.

2-1 FLUID AS A CONTINUUM

In our definition of a fluid, no mention was made of the molecular structure of fluids. All fluids are composed of molecules in constant motion. However, in most engineering applications we are interested in the average or macroscopic effects of many molecules. It is these macroscopic effects that we can perceive and measure. We thus treat a fluid as an infinitely divisible substance, a *continuum*, and do not concern ourselves with the behavior of individual molecules.

The concept of a continuum is the basis of classical fluid mechanics. The continuum assumption is valid in treating the behavior of fluids under normal conditions. However, it breaks down whenever the mean free path of the molecules (approximately 10^{-7} mm for gas molecules that show ideal behavior at STP)[1] becomes the same order of magnitude as the smallest significant characteristic dimension of the problem. In problems such as rarefied gas flow (e.g. as encountered in flights into the upper reaches of the atmosphere), we must abandon the concept of a continuum in favor of the microscopic and statistical points of view.

As a consequence of the continuum assumption, each fluid property is assumed to have a definite value at each point in space. Thus fluid properties such as density, temperature, velocity, and so on, are considered to be continuous functions of position and time.

To illustrate the concept of a property at a point, consider the manner in which we determine the density at a point. A region of fluid is shown in Fig. 2.1. We are interested

[1] STP (Standard Temperature and Pressure) for air are 15 C (59 F) and 101.3 kPa absolute (14.696 psia), respectively.

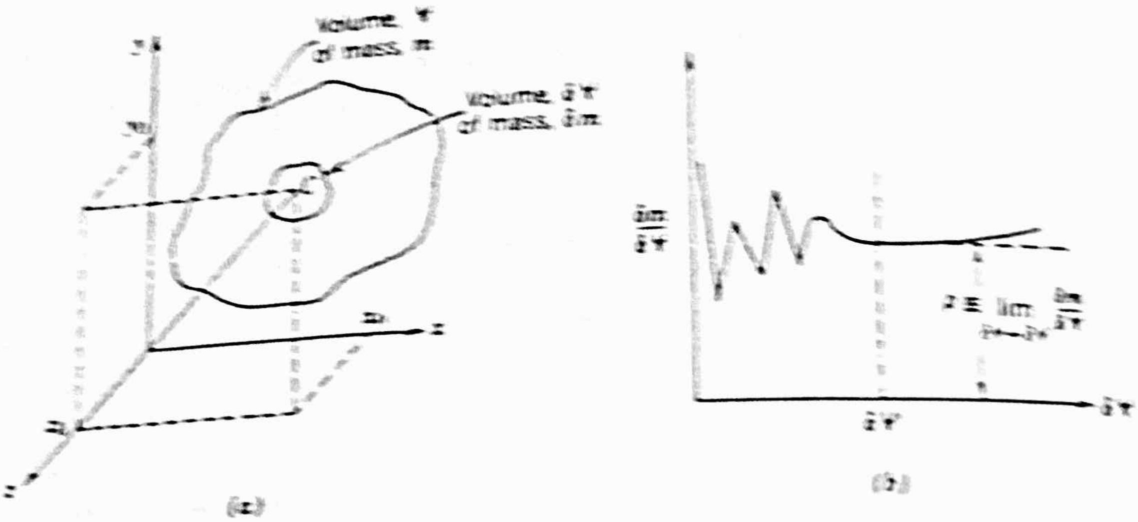

Fig. 2.1 Definition of density at a point.

in determining the density at the point C, whose coordinates are x_0, y_0, and z_0. The density is defined as mass per unit volume. Thus the mean density within the volume, $\mathcal{V}$, would be given by $\rho = m/\mathcal{V}$. In general, this will not be equal to the value of the density at point C. To determine the density at point C, we must select a small volume, $\delta\mathcal{V}$, surrounding point C and determine the ratio, $\delta m/\delta\mathcal{V}$. The question is how small can we make the volume, $\delta\mathcal{V}$? Let us answer this question by plotting the ratio, $\delta m/\delta\mathcal{V}$, and allowing the volume to shrink continuously in size. Assuming that the volume, $\delta\mathcal{V}$, is initially relatively large (but still small compared to the volume, $\mathcal{V}$) a typical plot of $\delta m/\delta\mathcal{V}$ might appear as in Fig. 2.1*b*. The average density tends to approach an asymptotic value as the volume is shrunk to enclose only homogeneous fluid in the immediate neighborhood of point C. When $\delta\mathcal{V}$ becomes so small that it contains only a small number of molecules, it becomes impossible to fix a definite value for $\delta m/\delta\mathcal{V}$; the value will vary erratically as molecules cross into and out of the volume. Thus there is a lower limiting value of $\delta\mathcal{V}$, designated $\delta\mathcal{V}'$ in Fig. 2.1*b*, allowable for use in defining fluid density at a point.[2] The density at a point is then defined as

$$\rho \equiv \lim_{\delta\mathcal{V}\to\delta\mathcal{V}'} \frac{\delta m}{\delta\mathcal{V}} \tag{2.1}$$

Since point C was arbitrary, the density at any point in the fluid could be determined in a like manner. If density determinations were made simultaneously at an infinite number of points in the fluid, we would obtain an expression for the density distribution as a function of the space coordinates, $\rho = \rho(x, y, z)$, at the given instant in time. Clearly, the density at a point may vary with time as a result of work done on or

[2] The size of $\delta\mathcal{V}'$ is extremely small. For example 1 m^3 of air at STP contains approximately 2.5×10^{25} molecules. Thus the number of molecules in a volume of 10^{-12} m^3 (about the size of a grain of sand) would be 2.5×10^{13}. This number is certainly large enough to insure that the average mass within $\delta\mathcal{V}'$ will be constant.

by the fluid and/or heat transfer to the fluid. Thus the complete representation of density (the *field* representation) is given by

$$\rho = \rho(x, y, z, t) \tag{2.2}$$

Since density is a scalar quantity, requiring only the specification of a magnitude for a complete description, the field represented by Eq. 2.2 is a scalar field.

2-2 VELOCITY FIELD

In the previous section we saw that the continuum assumption led directly to the notion of the density field. Other fluid properties are described by fields.

To deal with fluids in motion, we shall necessarily be concerned with the description of a velocity field. Refer again to Fig. 2.1*a*. Define the fluid velocity at point *C* as the instantaneous velocity of the center of gravity of the volume, $\delta \forall'$, instantaneously surrounding point *C*. Define a *fluid particle* as the small mass of fluid of fixed identity of volume, $\delta \forall'$. Thus we define the velocity at point *C* as the instantaneous velocity of the fluid particle which, at a given instant, is passing through point *C*. The velocity at any point in the flow field is defined similarly. At a given instant the velocity field, $\vec{V}$, is a function of the space coordinates x, y, z. The velocity at any point in the flow field might vary from one instant to another. Thus the complete representation of velocity (the velocity field) is given by

$$\vec{V} = \vec{V}(x, y, z, t) \tag{2.3}$$

The velocity vector, $\vec{V}$, can be written in terms of its three scalar components. Denoting the components in the x, y, and z directions by u, v, and w, then

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} \tag{2.4}$$

In general, each of the components u, v, and w will be a function of x, y, z, and t.

If properties at each point in a flow field do not change with time, the flow is termed *steady*. Stated mathematically, the definition of steady flow is

$$\frac{\partial \eta}{\partial t} = 0$$

where η represents any fluid property. For steady flow,

$$\frac{\partial \rho}{\partial t} = 0, \qquad \text{or} \qquad \rho = \rho(x, y, z)$$

and

$$\frac{\partial \vec{V}}{\partial t} = 0 \qquad \text{or} \qquad \vec{V} = \vec{V}(x, y, z)$$

Thus, in steady flow, any property may vary from point to point in the field, but all properties remain constant with time at each point.

2-2.1 One-, Two-, and Three-Dimensional Flows

A flow is classified as one-, two-, or three-dimensional depending on the number of space coordinates required to specify the velocity field.[3] Equation 2.3 indicates that the velocity field may be a function of three space coordinates and time. Such a flow field is termed *three-dimensional* (it is also *unsteady*) because the velocity at any point in the flow field depends on the three coordinates required to locate the point in space.

Not all flow fields are three-dimensional. Consider, for example, the steady flow through a long straight pipe of constant cross section. Far from the entrance to the pipe the velocity distribution may be described by

$$u = u_{max}\left[1 - \left(\frac{r}{R}\right)^2\right] \tag{2.5}$$

This profile is shown in Fig. 2.2, where cylindrical coordinates r, θ, and x are used to locate any point in the flow field. The velocity field is a function of r only; it is independent of the coordinates x and θ. Thus this is a one-dimensional flow.

An example of a two-dimensional flow is illustrated in Fig. 2.3; the velocity distribution is depicted for a flow between diverging straight walls that are imagined to be infinite in extent (in the z direction). Since the channel is considered to be infinite in the z direction, the velocity field will be identical in all planes perpendicular to the z axis. Consequently, the velocity field is a function only of the space coordinates x and y; the flow field is classified as two-dimensional.

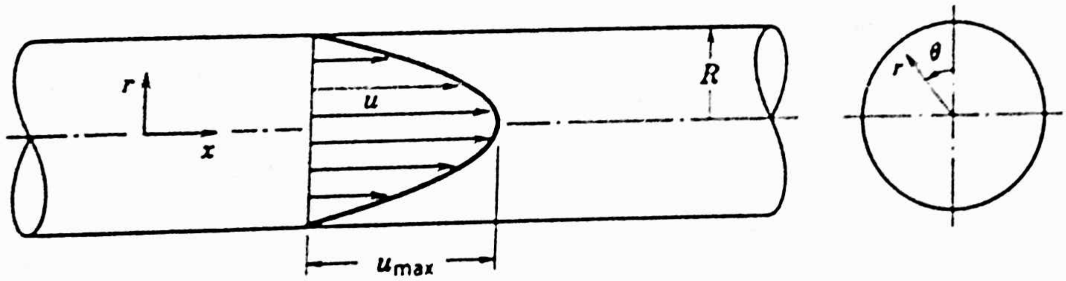

Fig. 2.2 Example of one-dimensional flow.

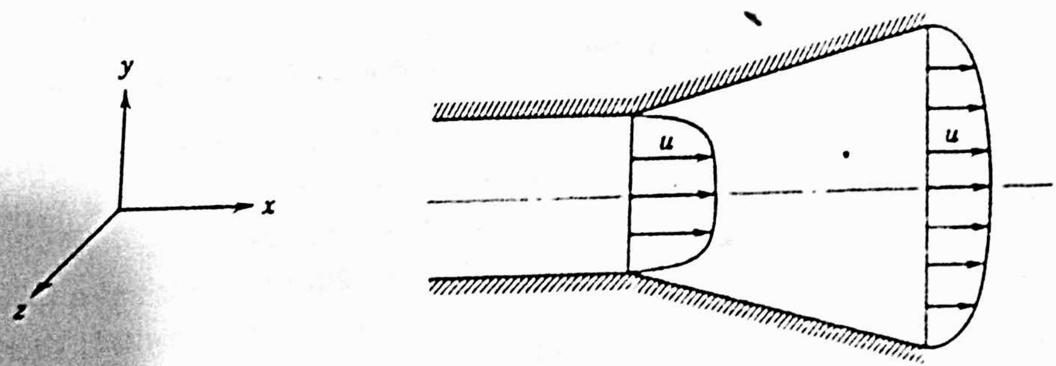

Fig. 2.3 Example of two-dimensional flow.

[3] Some authors choose to classify a flow as one-, two-, or three-dimensional on the basis of the number of space coordinates required to specify all fluid properties. In this text, classification of flow fields will be based on the number of space coordinates required to specify the velocity field only.

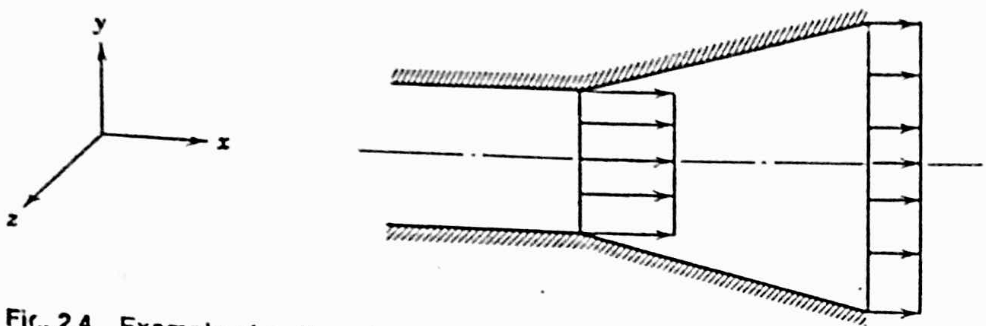

Fig. 2.4 Example of uniform flow at a section.

As you might suspect, the complexity of analysis increases considerably with the number of dimensions of the flow field. The simplest to analyze is one-dimensional flow. For many problems encountered in engineering, a one-dimensional analysis is adequate to provide approximate solutions of engineering accuracy.

Since all fluids satisfying the continuum assumption must have a zero relative velocity at a solid surface (to satisfy the no-slip condition), most flows are inherently two- or three-dimensional. For purposes of analysis it often is convenient to introduce the notion of *uniform flow* at a given cross section. In a flow that is uniform at a given cross section, the velocity is constant across any section normal to the flow. Under this assumption,[4] the two-dimensional flow of Fig. 2.3 is modeled as the flow shown in Fig. 2.4. In the flow of Fig. 2.4, the velocity field is a function of x alone, and thus the flow model is one-dimensional. (Other properties, such as density or pressure, also may be assumed uniform at a section, if appropriate.)

The term *uniform flow field* (as opposed to uniform flow at a cross section) is used to describe a flow in which the magnitude and direction of the velocity vector are constant, i.e. independent of all space coordinates, throughout the entire flow field.

2-2.2 Timelines, Pathlines, Streaklines, and Streamlines

In the analysis of problems in fluid mechanics, frequently it is advantageous to obtain a visual representation of a flow field. Such a representation is provided by timelines, pathlines, streaklines, and streamlines.[5]

If a number of adjacent fluid particles in a flow field are marked at a given instant, they form a line in the fluid at that instant; this line is called a *timeline*. Subsequent observations of the line may provide information about the flow field. For example, in discussing the behavior of a fluid under the action of a constant shear force

[4] Convenience alone does not justify this assumption; often results of acceptable accuracy are obtained. Sweeping assumptions such as uniform flow at a cross section should always be reviewed carefully to be sure they provide a reasonable analytical model of the real flow.

[5] Timelines, pathlines, streaklines, and streamlines are demonstrated in the film loops: S-FM047, *Pathlines, Streaklines, Streamlines, and Timelines in Steady Flow*, and S-FM048, *Pathlines, Streaklines, and Streamlines in Unsteady Flow*. These two loops are taken from the film *Flow Visualization*, S. J. Kline, principal.

(Section 1-2) timelines were introduced to demonstrate the deformation of a fluid at successive instants.

A *pathline* is the path or trajectory traced out by a moving fluid particle. To make a pathline visible, we might identify a fluid particle at a given instant, e.g. by the use of dye, and then take a long exposure photograph of its subsequent motion. The line traced out by the particle is a pathline.

On the other hand, we might choose to focus our attention on a fixed location in space and identify, again by the use of dye, all fluid particles passing through this point. After a short period of time we would have a number of identifiable fluid particles in the flow, all of which had, at some time, passed through one fixed location in space. The line joining these fluid particles is defined as a *streakline*.

Streamlines are lines drawn in the flow field so that at a given instant they are tangent to the direction of flow at every point in the flow field. Since the streamlines are tangent to the velocity vector at every point in the flow field, there can be no flow across a streamline.

In steady flow, the velocity at each point in the flow field remains constant with time and, consequently, the streamlines do not vary from one instant to the next. This implies that a particle located on a given streamline will remain on the same streamline. Furthermore, consecutive particles passing through a fixed point in space will be on the same streamline and, subsequently, will remain on this streamline. Thus in a steady flow, pathlines, streaklines, and streamlines are identical lines in the flow field.

The shape of the streamlines may vary from instant to instant if the flow is unsteady. In the case of unsteady flow, pathlines, streaklines, and streamlines do not coincide.

Example 2.1

A velocity field is given by $\vec{V} = ax\hat{i} - ay\hat{j}$; the units of velocity are m/sec; x and y are given in meters; $a = 0.1\ \text{sec}^{-1}$.

(a) Determine the equation for the streamline passing through the point $(x_0, y_0, 0) = (2, 8, 0)$.
(b) Determine the velocity of a particle at the point (2, 8, 0).
(c) If the particle passing through the point $(x_0, y_0, 0)$ is marked at time $t_0 = 0$, determine the location of the particle at time $t = 20$ sec.
(d) What is the velocity of the particle at $t = 20$ sec?
(e) Show that the equation of the particle path (the pathline) is the same as the equation of the streamline.

EXAMPLE PROBLEM 2.1

GIVEN: Velocity field, $\vec{V} = ax\hat{i} - ay\hat{j}$; x and y in meters, $a = 0.1\ \text{sec}^{-1}$

FIND:
(a) Equation of streamline through point (2, 8, 0).
(b) Velocity of particle at point (2, 8, 0).
(c) Position at $t = 20$ sec of particle located at (2, 8, 0) at $t = 0$.
(d) Velocity of particle at position found in (c).
(e) Equation of pathline of particle located at (2, 8, 0) at $t = 0$.

SOLUTION:

(a) Streamlines are lines drawn in the flow field such that, at a given instant, they are tangent to the direction of flow at every point.

Consequently,

$$\left.\frac{dy}{dx}\right)_{\text{streamline}} = \frac{v}{u} = \frac{-ay}{ax} = \frac{-y}{x}$$

Separating variables and integrating, we obtain

$$\int \frac{dy}{y} = -\int \frac{dx}{x} \quad \text{or} \quad \ln y = -\ln x + c_1$$

This can be written as $xy = c$.

For the streamline passing through the point $(x_0, y_0, 0) = (2, 8, 0)$ the constant, c, has a value of 16 and the equation of the streamline through the point (2, 8, 0) is

$$xy = x_0 y_0 = 16 \text{ m}^2$$

(b) The velocity field is $\vec{V} = ax\hat{i} - ay\hat{j}$. At the point (2, 8, 0)

$$\vec{V} = a(x\hat{i} - y\hat{j}) = 0.1 \text{ sec}^{-1}(2\hat{i} - 8\hat{j})\text{m} = 0.2\hat{i} - 0.8\hat{j} \text{ m/sec}$$

(c) A particle moving in the flow field will have velocity given by

$$\vec{V} = ax\hat{i} - ay\hat{j}$$

Thus

$$u_p = \frac{dx}{dt} = ax \quad \text{and} \quad v_p = \frac{dy}{dt} = -ay$$

Separating variables and integrating (in each equation) gives

$$\int_{x_0}^{x} \frac{dx}{x} = \int_0^t a\,dt \quad \text{and} \quad \int_{y_0}^{y} \frac{dy}{y} = \int_0^t -a\,dt$$

Then

$$\ln \frac{x}{x_0} = at \quad \text{and} \quad \ln \frac{y}{y_0} = -at$$

or

$$x = x_0 e^{at} \quad \text{and} \quad y = y_0 e^{-at}$$

At $t = 20$ sec

$$x = 2 \text{ m } e^{(0.1)(20)} = 14.8 \text{ m} \quad \text{and} \quad y = 8 \text{ m } e^{-(0.1)(20)} = 1.08 \text{ m}$$

At $t = 20$ sec, particle is at (14.8, 1.08, 0)m

(d) At the point (14.8, 1.08, 0)m

$$\vec{V} = a(x\hat{i} - y\hat{j}) = 0.1 \text{ sec}^{-1}(14.8\hat{i} - 1.08\hat{j})\text{m} = 1.48\hat{i} - 0.108\hat{j} \text{ m/sec}$$

(e) To determine the equation of the pathline, we use the parametric equations

$$x = x_0 e^{at} \quad \text{and} \quad y = y_0 e^{-at}$$

and eliminate t. Solving for e^{at} from both equations

$$e^{at} = \frac{y_0}{y} = \frac{x}{x_0} \qquad \therefore\ xy = x_0 y_0 = 16\ \text{m}^2$$

Note: (i) the equation of the streamline through $(x_0, y_0, 0)$ and the equation of pathline traced out by the particle passing through $(x_0, y_0, 0)$ are the same this steady flow.

(ii) in following a particle (Lagrangian method of description), both the coordin of the particle (x, y) and the components of the particle velocity ($u_p = d$ and $v_p = dy/dt$) are functions of time.