APPLIED PROJECT HOW FAST DOES A TANK DRAIN?

If water (or other liquid) drains from a tank, we expect that the flow will be greatest at first (when the water depth is greatest) and will gradually decrease as the water level decreases. But we need a more precise mathematical description of how the flow decreases in order to answer the kinds of questions that engineers ask: How long does it take for a tank to drain completely? How much water should a tank hold in order to guarantee a certain minimum water pressure for a sprinkler system?

Let $h(t)$ and $V(t)$ be the height and volume of water in a tank at time t. If water drains through a hole with area a at the bottom of the tank, then Torricelli's Law says that

$$\boxed{1} \qquad \frac{dV}{dt} = -a\sqrt{2gh}$$

where g is the acceleration due to gravity. So the rate at which water flows from the tank is proportional to the square root of the water height.

1. (a) Suppose the tank is cylindrical with height 6 ft and radius 2 ft and the hole is circular with radius 1 inch. If we take $g = 32 \text{ ft/s}^2$, show that h satisfies the differential equation

$$\frac{dh}{dt} = -\frac{1}{72}\sqrt{h}$$

(b) Solve this equation to find the height of the water at time t, assuming the tank is full at time $t = 0$.

(c) How long will it take for the water to drain completely?

2. Because of the rotation and viscosity of the liquid, the theoretical model given by Equation 1 isn't quite accurate. Instead, the model

$$\boxed{2} \qquad \frac{dh}{dt} = k\sqrt{h}$$

is often used and the constant k (which depends on the physical properties of the liquid) is determined from data concerning the draining of the tank.

(a) Suppose that a hole is drilled in the side of a cylindrical bottle and the height h of the water (above the hole) decreases from 10 cm to 3 cm in 68 seconds. Use Equation 2 to find an expression for $h(t)$. Evaluate $h(t)$ for $t = 10, 20, 30, 40, 50, 60$.

(b) Drill a 4-mm hole near the bottom of the cylindrical part of a two-liter plastic soft-drink bottle. Attach a strip of masking tape marked in centimeters from 0 to 10, with 0 corresponding to the top of the hole. With one finger over the hole, fill the bottle with water to the 10-cm mark. Then take your finger off the hole and record the values of $h(t)$ for $t = 10, 20, 30, 40, 50, 60$ seconds. (You will probably find that it takes 68 seconds for the level to decrease to $h = 3$ cm.) Compare your data with the values of $h(t)$ from part (a). How well did the model predict the actual values?

Problem 2(b) is best done as a classroom demonstration or as a group project with three students in each group: a timekeeper to call out seconds, a bottle keeper to estimate the height every 10 seconds, and a record keeper to record these values.

3. In many parts of the world, the water for sprinkler systems in large hotels and hospitals is supplied by gravity from cylindrical tanks on or near the roofs of the buildings. Suppose such a tank has radius 10 ft and the diameter of the outlet is 2.5 inches. An engineer has to guarantee that the water pressure will be at least 2160 lb/ft^2 for a period of 10 minutes. (When a fire happens, the electrical system might fail and it could take up to 10 minutes for the emergency generator and fire pump to be activated.) What height should the engineer specify for the tank in order to make such a guarantee? (Use the fact that the water pressure at a depth of d feet is $P = 62.5d$. See Section 8.3.)

4. Not all water tanks are shaped like cylinders. Suppose a tank has cross-sectional area $A(h)$ at height h. Then the volume of water up to height h is $V = \int_0^h A(u)\,du$ and so the Fundamental Theorem of Calculus gives $dV/dh = A(h)$. It follows that

$$\frac{dV}{dt} = \frac{dV}{dh}\,\frac{dh}{dt} = A(h)\,\frac{dh}{dt}$$

and so Torricelli's Law becomes

$$A(h)\,\frac{dh}{dt} = -a\sqrt{2gh}$$

(a) Suppose the tank has the shape of a sphere with radius 2 m and is initially half full of water. If the radius of the circular hole is 1 cm and we take $g = 10 \text{ m/s}^2$, show that h satisfies the differential equation

$$(4h - h^2)\,\frac{dh}{dt} = -0.0001\sqrt{20h}$$

(b) How long will it take for the water to drain completely?

APPLIED PROJECT WHICH IS FASTER, GOING UP OR COMING DOWN?

Suppose you throw a ball into the air. Do you think it takes longer to reach its maximum height or to fall back to earth from its maximum height? We will solve the problem in this project but, before getting started, think about that situation and make a guess based on your physical intuition.

In modeling force due to air resistance, various functions have been used, depending on the physical characteristics and speed of the ball. Here we use a linear model, $-pv$, but a quadratic model ($-pv^2$ on the way up and pv^2 on the way down) is another possibility for higher speeds (see Exercise 50 in Section 9.3). For a golf ball, experiments have shown that a good model is $-pv^{1.3}$ going up and $p|v|^{1.3}$ coming down. But no matter which force function $-f(v)$ is used [where $f(v) > 0$ for $v > 0$ and $f(v) < 0$ for $v < 0$], the answer to the question remains the same. See F. Brauer, "What Goes Up Must Come Down, Eventually," *Amer. Math. Monthly* 108 (2001), pp. 437–440.

1. A ball with mass m is projected vertically upward from the earth's surface with a positive initial velocity v_0. We assume the forces acting on the ball are the force of gravity and a retarding force of air resistance with direction opposite to the direction of motion and with magnitude $p|v(t)|$, where p is a positive constant and $v(t)$ is the velocity of the ball at time t. In both the ascent and the descent, the total force acting on the ball is $-pv - mg$. [During ascent, $v(t)$ is positive and the resistance acts downward; during descent, $v(t)$ is negative and the resistance acts upward.] So, by Newton's Second Law, the equation of motion is

$$mv' = -pv - mg$$

Solve this differential equation to show that the velocity is

$$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$$

2. Show that the height of the ball, until it hits the ground, is

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

Graphing calculator or computer required

3. Let t_1 be the time that the ball takes to reach its maximum height. Show that

$$t_1 = \frac{m}{p} \ln\left(\frac{mg + pv_0}{mg}\right)$$

Find this time for a ball with mass 1 kg and initial velocity 20 m/s. Assume the air resistance is $\frac{1}{10}$ of the speed.

4. Let t_2 be the time at which the ball falls back to earth. For the particular ball in Problem 3, estimate t_2 by using a graph of the height function $y(t)$. Which is faster, going up or coming down?

5. In general, it's not easy to find t_2 because it's impossible to solve the equation $y(t) = 0$ explicitly. We can, however, use an indirect method to determine whether ascent or descent is faster: we determine whether $y(2t_1)$ is positive or negative. Show that

$$y(2t_1) = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2 \ln x\right)$$

where $x = e^{pt_1/m}$. Then show that $x > 1$ and the function

$$f(x) = x - \frac{1}{x} - 2 \ln x$$

is increasing for $x > 1$. Use this result to decide whether $y(2t_1)$ is positive or negative. What can you conclude? Is ascent or descent faster?

9.4 Models for Population Growth

In this section we investigate differential equations that are used to model population growth: the law of natural growth, the logistic equation, and several others.

The Law of Natural Growth

One of the models for population growth that we considered in Section 9.1 was based on the assumption that the population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kP$$

Is that a reasonable assumption? Suppose we have a population (of bacteria, for instance) with size $P = 1000$ and at a certain time it is growing at a rate of $P' = 300$ bacteria per hour. Now let's take another 1000 bacteria of the same type and put them with the first population. Each half of the combined population was previously growing at a rate of 300 bacteria per hour. We would expect the total population of 2000 to increase at a rate of 600 bacteria per hour initially (provided there's enough room and nutrition). So if we double the size, we double the growth rate. It seems reasonable that the growth rate should be proportional to the size.

In general, if $P(t)$ is the value of a quantity y at time t and if the rate of change of P with respect to t is proportional to its size $P(t)$ at any time, then

1
$$\boxed{\frac{dP}{dt} = kP}$$

where k is a constant. Equation 1 is sometimes called the **law of natural growth**. If k is positive, then the population increases; if k is negative, it decreases.

Because Equation 1 is a separable differential equation, we can solve it by the methods of Section 9.3:

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C$$

$$|P| = e^{kt+C} = e^C e^{kt}$$

$$P = Ae^{kt}$$

where A $(= \pm e^C$ or $0)$ is an arbitrary constant. To see the significance of the constant A, we observe that

$$P(0) = Ae^{k\cdot 0} = A$$

Therefore A is the initial value of the function.

Examples and exercises on the use of $\boxed{2}$ are given in Section 6.5.

2 The solution of the initial-value problem

$$\frac{dP}{dt} = kP \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

Another way of writing Equation 1 is

$$\frac{1}{P}\frac{dP}{dt} = k$$

which says that the **relative growth rate** (the growth rate divided by the population size) is constant. Then $\boxed{2}$ says that a population with constant relative growth rate must grow exponentially.

We can account for emigration (or "harvesting") from a population by modifying Equation 1: If the rate of emigration is a constant m, then the rate of change of the population is modeled by the differential equation

3
$$\frac{dP}{dt} = kP - m$$

See Exercise 15 for the solution and consequences of Equation 3.

The Logistic Model

As we discussed in Section 9.1, a population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time t, we assume that

$$\frac{dP}{dt} \approx kP \qquad \text{if } P \text{ is small}$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P, we obtain the model for population growth known as the **logistic differential equation**:

$$\boxed{\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)} \tag{4}$$

Notice from Equation 4 that if P is small compared with M, then P/M is close to 0 and so $dP/dt \approx kP$. However, if $P \to M$ (the population approaches its carrying capacity), then $P/M \to 1$, so $dP/dt \to 0$. We can deduce information about whether solutions increase or decrease directly from Equation 4. If the population P lies between 0 and M, then the right side of the equation is positive, so $dP/dt > 0$ and the population increases. But if the population exceeds the carrying capacity ($P > M$), then $1 - P/M$ is negative, so $dP/dt < 0$ and the population decreases.

Let's start our more detailed analysis of the logistic differential equation by looking at a direction field.

V EXAMPLE 1 Draw a direction field for the logistic equation with $k = 0.08$ and carrying capacity $M = 1000$. What can you deduce about the solutions?

SOLUTION In this case the logistic differential equation is

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)$$

A direction field for this equation is shown in Figure 1. We show only the first quadrant because negative populations aren't meaningful and we are interested only in what happens after $t = 0$.

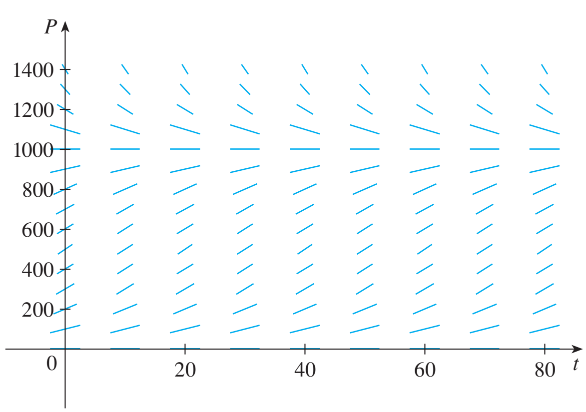


FIGURE 1
Direction field for the logistic equation in Example 1

The logistic equation is autonomous (dP/dt depends only on P, not on t), so the slopes are the same along any horizontal line. As expected, the slopes are positive for $0 < P < 1000$ and negative for $P > 1000$.

The slopes are small when P is close to 0 or 1000 (the carrying capacity). Notice that the solutions move away from the equilibrium solution $P = 0$ and move toward the equilibrium solution $P = 1000$.

In Figure 2 we use the direction field to sketch solution curves with initial populations $P(0) = 100$, $P(0) = 400$, and $P(0) = 1300$. Notice that solution curves that start below $P = 1000$ are increasing and those that start above $P = 1000$ are decreasing. The slopes are greatest when $P \approx 500$ and therefore the solution curves that start below $P = 1000$ have inflection points when $P \approx 500$. In fact we can prove that all solution curves that start below $P = 500$ have an inflection point when P is exactly 500. (See Exercise 11.)

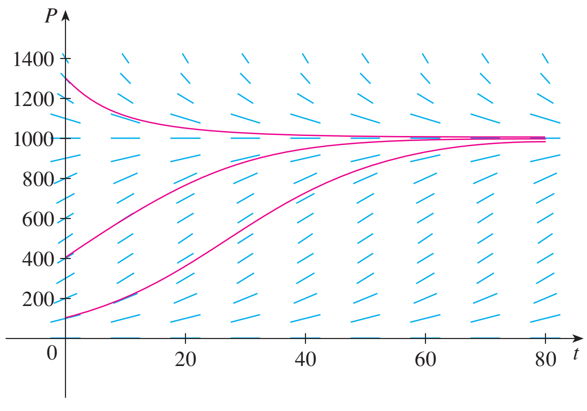


FIGURE 2
Solution curves for the logistic equation in Example 1

The logistic equation $\boxed{4}$ is separable and so we can solve it explicitly using the method of Section 9.3. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\boxed{5} \qquad \int \frac{dP}{P(1 - P/M)} = \int k\,dt$$

To evaluate the integral on the left side, we write

$$\frac{1}{P(1 - P/M)} = \frac{M}{P(M - P)}$$

Using partial fractions (see Section 7.4), we get

$$\frac{M}{P(M - P)} = \frac{1}{P} + \frac{1}{M - P}$$

This enables us to rewrite Equation 5:

$$\int \left(\frac{1}{P} + \frac{1}{M - P} \right) dP = \int k\, dt$$

$$\ln |P| - \ln |M - P| = kt + C$$

$$\ln \left| \frac{M - P}{P} \right| = -kt - C$$

$$\left| \frac{M - P}{P} \right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\boxed{6} \qquad \frac{M - P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$. Solving Equation 6 for P, we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1 + Ae^{-kt}}$$

so

$$P = \frac{M}{1 + Ae^{-kt}}$$

We find the value of A by putting $t = 0$ in Equation 6. If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M - P_0}{P_0} = Ae^0 = A$$

Thus the solution to the logistic equation is

$$\boxed{7} \qquad \boxed{P(t) = \frac{M}{1 + Ae^{-kt}} \qquad \text{where } A = \frac{M - P_0}{P_0}}$$

Using the expression for $P(t)$ in Equation 7, we see that

$$\lim_{t \to \infty} P(t) = M$$

which is to be expected.

EXAMPLE 2 Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) \qquad P(0) = 100$$

and use it to find the population sizes $P(40)$ and $P(80)$. At what time does the population reach 900?

SOLUTION The differential equation is a logistic equation with $k = 0.08$, carrying capacity $M = 1000$, and initial population $P_0 = 100$. So Equation 7 gives the population at time t as

$$P(t) = \frac{1000}{1 + Ae^{-0.08t}} \qquad \text{where } A = \frac{1000 - 100}{100} = 9$$

Thus
$$P(t) = \frac{1000}{1 + 9e^{-0.08t}}$$

So the population sizes when $t = 40$ and 80 are

$$P(40) = \frac{1000}{1 + 9e^{-3.2}} \approx 731.6 \qquad P(80) = \frac{1000}{1 + 9e^{-6.4}} \approx 985.3$$

The population reaches 900 when

$$\frac{1000}{1 + 9e^{-0.08t}} = 900$$

Solving this equation for t, we get

$$1 + 9e^{-0.08t} = \tfrac{10}{9}$$

$$e^{-0.08t} = \tfrac{1}{81}$$

$$-0.08t = \ln \tfrac{1}{81} = -\ln 81$$

$$t = \frac{\ln 81}{0.08} \approx 54.9$$

Compare the solution curve in Figure 3 with the lowest solution curve we drew from the direction field in Figure 2.

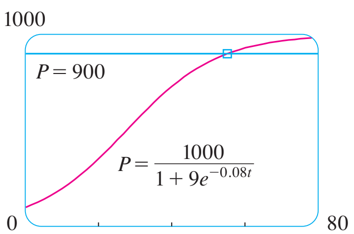


FIGURE 3

So the population reaches 900 when t is approximately 55. As a check on our work, we graph the population curve in Figure 3 and observe where it intersects the line $P = 900$. The cursor indicates that $t \approx 55$.

Comparison of the Natural Growth and Logistic Models

In the 1930s the biologist G. F. Gause conducted an experiment with the protozoan *Paramecium* and used a logistic equation to model his data. The table gives his daily count of the population of protozoa. He estimated the initial relative growth rate to be 0.7944 and the carrying capacity to be 64.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57

V EXAMPLE 3 Find the exponential and logistic models for Gause's data. Compare the predicted values with the observed values and comment on the fit.

SOLUTION Given the relative growth rate $k = 0.7944$ and the initial population $P_0 = 2$, the exponential model is

$$P(t) = P_0 e^{kt} = 2e^{0.7944t}$$

Gause used the same value of k for his logistic model. [This is reasonable because $P_0 = 2$ is small compared with the carrying capacity ($M = 64$). The equation

$$\frac{1}{P_0}\frac{dP}{dt}\bigg|_{t=0} = k\left(1 - \frac{2}{64}\right) \approx k$$

shows that the value of k for the logistic model is very close to the value for the exponential model.]

Then the solution of the logistic equation in Equation 7 gives

$$P(t) = \frac{M}{1 + Ae^{-kt}} = \frac{64}{1 + Ae^{-0.7944t}}$$

where $$A = \frac{M - P_0}{P_0} = \frac{64 - 2}{2} = 31$$

So $$P(t) = \frac{64}{1 + 31e^{-0.7944t}}$$

We use these equations to calculate the predicted values (rounded to the nearest integer) and compare them in the following table.

t (days)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P (observed)	2	3	22	16	39	52	54	47	50	76	69	51	57	70	53	59	57
P (logistic model)	2	4	9	17	28	40	51	57	61	62	63	64	64	64	64	64	64
P (exponential model)	2	4	10	22	48	106	. . .										

We notice from the table and from the graph in Figure 4 that for the first three or four days the exponential model gives results comparable to those of the more sophisticated logistic model. For $t \geqslant 5$, however, the exponential model is hopelessly inaccurate, but the logistic model fits the observations reasonably well.

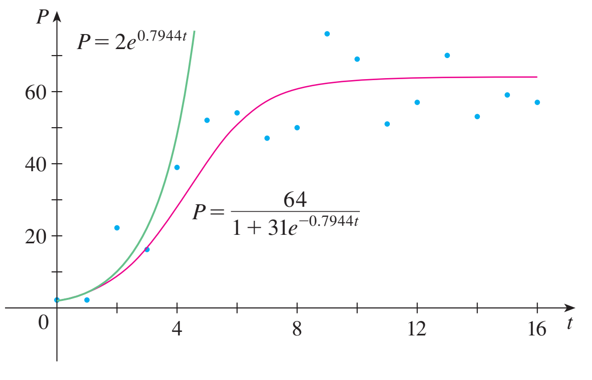


FIGURE 4
The exponential and logistic models for the *Paramecium* data

Many countries that formerly experienced exponential growth are now finding that their rates of population growth are declining and the logistic model provides a better model.

t	$B(t)$	t	$B(t)$
1980	9,847	1992	10,036
1982	9,856	1994	10,109
1984	9,855	1996	10,152
1986	9,862	1998	10,175
1988	9,884	2000	10,186
1990	9,962		

The table in the margin shows midyear values of $B(t)$, the population of Belgium, in thousands, at time t, from 1980 to 2000. Figure 5 shows these data points together with a shifted logistic function obtained from a calculator with the ability to fit a logistic function to these points by regression. We see that the logistic model provides a very good fit.

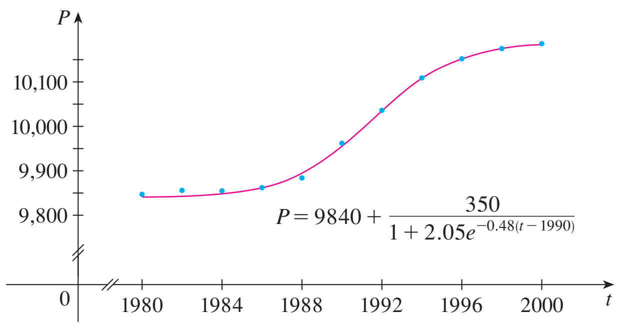


FIGURE 5
Logistic model for the population of Belgium

Other Models for Population Growth

The Law of Natural Growth and the logistic differential equation are not the only equations that have been proposed to model population growth. In Exercise 20 we look at the Gompertz growth function and in Exercises 21 and 22 we investigate seasonal-growth models.

Two of the other models are modifications of the logistic model. The differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) - c$$

has been used to model populations that are subject to harvesting of one sort or another. (Think of a population of fish being caught at a constant rate.) This equation is explored in Exercises 17 and 18.

For some species there is a minimum population level m below which the species tends to become extinct. (Adults may not be able to find suitable mates.) Such populations have been modeled by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

where the extra factor, $1 - m/P$, takes into account the consequences of a sparse population (see Exercise 19).

9.4 Exercises

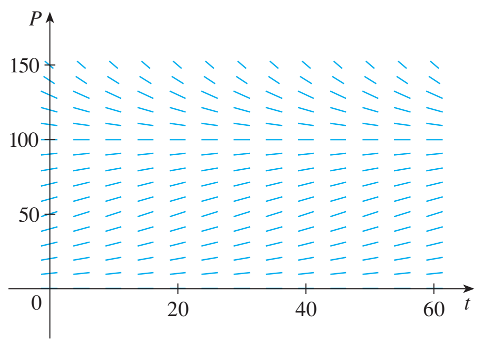
1. Suppose that a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

(b) A direction field for this equation is shown. Where are the slopes close to 0? Where are they largest? Which solutions are increasing? Which solutions are decreasing?



(c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?

(d) What are the equilibrium solutions? How are the other solutions related to these solutions?

2. Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.

(a) Write the logistic differential equation for these data.

(b) Draw a direction field (either by hand or with a computer algebra system). What does it tell you about the solution curves?

(c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?

(d) Program a calculator or computer to use Euler's method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.

(e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).

(f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).

3. The Pacific halibut fishery has been modeled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{M}\right)$$

where $y(t)$ is the biomass (the total mass of the members of the population) in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $k = 0.71$ per year.

(a) If $y(0) = 2 \times 10^7$ kg, find the biomass a year later.

(b) How long will it take for the biomass to reach 4×10^7 kg?

4. Suppose a population $P(t)$ satisfies

$$\frac{dP}{dt} = 0.4P - 0.001P^2 \qquad P(0) = 50$$

where t is measured in years.

(a) What is the carrying capacity?

(b) What is $P'(0)$?

(c) When will the population reach 50% of the carrying capacity?

5. Suppose a population grows according to a logistic model with initial population 1000 and carrying capacity 10,000. If the population grows to 2500 after one year, what will the population be after another three years?

6. The table gives the number of yeast cells in a new laboratory culture.

Time (hours)	Yeast cells	Time (hours)	Yeast cells
0	18	10	509
2	39	12	597
4	80	14	640
6	171	16	664
8	336	18	672

(a) Plot the data and use the plot to estimate the carrying capacity for the yeast population.

(b) Use the data to estimate the initial relative growth rate.

(c) Find both an exponential model and a logistic model for these data.

(d) Compare the predicted values with the observed values, both in a table and with graphs. Comment on how well your models fit the data.

(e) Use your logistic model to estimate the number of yeast cells after 7 hours.

7. The population of the world was about 5.3 billion in 1990. Birth rates in the 1990s ranged from 35 to 40 million per year and death rates ranged from 15 to 20 million per year. Let's assume that the carrying capacity for world population is 100 billion.

(a) Write the logistic differential equation for these data. (Because the initial population is small compared to the carrying capacity, you can take k to be an estimate of the initial relative growth rate.)

(b) Use the logistic model to estimate the world population in the year 2000 and compare with the actual population of 6.1 billion.

(c) Use the logistic model to predict the world population in the years 2100 and 2500.

(d) What are your predictions if the carrying capacity is 50 billion?

8. (a) Make a guess as to the carrying capacity for the US population. Use it and the fact that the population was 250 million in 1990 to formulate a logistic model for the US population.

(b) Determine the value of k in your model by using the fact that the population in 2000 was 275 million.

(c) Use your model to predict the US population in the years 2100 and 2200.

(d) Use your model to predict the year in which the US population will exceed 350 million.

9. One model for the spread of a rumor is that the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

(a) Write a differential equation that is satisfied by y.

(b) Solve the differential equation.

(c) A small town has 1000 inhabitants. At 8 AM, 80 people have heard a rumor. By noon half the town has heard it. At what time will 90% of the population have heard the rumor?

10. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximal population for the fish of that species in that lake) to be 10,000. The number of fish tripled in the first year.

(a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

(b) How long will it take for the population to increase to 5000?

11. (a) Show that if P satisfies the logistic equation $\boxed{4}$, then

$$\frac{d^2P}{dt^2} = k^2P\left(1 - \frac{P}{M}\right)\left(1 - \frac{2P}{M}\right)$$

(b) Deduce that a population grows fastest when it reaches half its carrying capacity.

12. For a fixed value of M (say $M = 10$), the family of logistic functions given by Equation 7 depends on the initial value P_0 and the proportionality constant k. Graph several members of this family. How does the graph change when P_0 varies? How does it change when k varies?

13. The table gives the midyear population of Japan, in thousands, from 1960 to 2005.

Year	Population	Year	Population
1960	94,092	1985	120,754
1965	98,883	1990	123,537
1970	104,345	1995	125,341
1975	111,573	2000	126,700
1980	116,807	2005	127,417

Use a graphing calculator to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models. [*Hint:* Subtract 94,000 from each of the population figures. Then, after obtaining a model from your calculator, add 94,000 to get your final model. It might be helpful to choose $t = 0$ to correspond to 1960 or 1980.]

14. The table gives the midyear population of Spain, in thousands, from 1955 to 2000.

Year	Population	Year	Population
1955	29,319	1980	37,488
1960	30,641	1985	38,535
1965	32,085	1990	39,351
1970	33,876	1995	39,750
1975	35,564	2000	40,016

Use a graphing calculator to fit both an exponential function and a logistic function to these data. Graph the data points and both functions, and comment on the accuracy of the models. [*Hint:* Subtract 29,000 from each of the population figures. Then, after obtaining a model from your calculator, add 29,000 to get your final model. It might be helpful to choose $t = 0$ to correspond to 1955 or 1975.]

15. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad \text{where } k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

(b) What condition on m will lead to an exponential expansion of the population?

(c) What condition on m will result in a constant population? A population decline?

(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time?

16. Let c be a positive number. A differential equation of the form

$$\frac{dy}{dt} = ky^{1+c}$$

where k is a positive constant, is called a *doomsday equation* because the exponent in the expression ky^{1+c} is larger than the exponent 1 for natural growth.

(a) Determine the solution that satisfies the initial condition $y(0) = y_0$.

(b) Show that there is a finite time $t = T$ (doomsday) such that $\lim_{t \to T^-} y(t) = \infty$.

(c) An especially prolific breed of rabbits has the growth term $My^{1.01}$. If 2 such rabbits breed initially and the warren has 16 rabbits after three months, then when is doomsday?

17. Let's modify the logistic differential equation of Example 1 as follows:

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

(a) Suppose $P(t)$ represents a fish population at time t, where t is measured in weeks. Explain the meaning of the final term in the equation (-15).

(b) Draw a direction field for this differential equation.

(c) What are the equilibrium solutions?

(d) Use the direction field to sketch several solution curves. Describe what happens to the fish population for various initial populations.

CAS (e) Solve this differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial populations 200 and 300. Graph the solutions and compare with your sketches in part (d).

CAS **18.** Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population, where t is measured in weeks and c is a constant.

(a) Use a CAS to draw direction fields for various values of c.

(b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?

(c) Use the differential equation to prove what you discovered graphically in part (b).

(d) What would you recommend for a limit to the weekly catch of this fish population?

19. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)\left(1 - \frac{m}{P}\right)$$

(a) Use the differential equation to show that any solution is increasing if $m < P < M$ and decreasing if $0 < P < m$.

(b) For the case where $k = 0.08$, $M = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?

(c) Solve the differential equation explicitly, either by using partial fractions or with a computer algebra system. Use the initial population P_0.

(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. [*Hint:* Show that the numerator in your expression for $P(t)$ is 0 for some value of t.]

20. Another model for a growth function for a limited population is given by the **Gompertz function**, which is a solution of the differential equation

$$\frac{dP}{dt} = c \ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.

(a) Solve this differential equation.

(b) Compute $\lim_{t \to \infty} P(t)$.

(c) Graph the Gompertz growth function for $M = 1000$, $P_0 = 100$, and $c = 0.05$, and compare it with the logistic function in Example 2. What are the similarities? What are the differences?

(d) We know from Exercise 11 that the logistic function grows fastest when $P = M/2$. Use the Gompertz differential equation to show that the Gompertz function grows fastest when $P = M/e$.

21. In a **seasonal-growth model**, a periodic function of time is introduced to account for seasonal variations in the rate of growth. Such variations could, for example, be caused by seasonal changes in the availability of food.

(a) Find the solution of the seasonal-growth model

$$\frac{dP}{dt} = kP\cos(rt - \phi) \qquad P(0) = P_0$$

where k, r, and ϕ are positive constants.

(b) By graphing the solution for several values of k, r, and ϕ, explain how the values of k, r, and ϕ affect the solution. What can you say about $\lim_{t \to \infty} P(t)$?

22. Suppose we alter the differential equation in Exercise 21 as follows:

$$\frac{dP}{dt} = kP\cos^2(rt - \phi) \qquad P(0) = P_0$$

(a) Solve this differential equation with the help of a table of integrals or a CAS.

(b) Graph the solution for several values of k, r, and ϕ. How do the values of k, r, and ϕ affect the solution? What can you say about $\lim_{t\to\infty} P(t)$ in this case?

23. Graphs of logistic functions (Figures 2 and 3) look suspiciously similar to the graph of the hyperbolic tangent function (Figure 3 in Section 6.7). Explain the similarity by showing that the logistic function given by Equation 7 can be written as

$$P(t) = \tfrac{1}{2}M\left[1 + \tanh\left(\tfrac{1}{2}k(t - c)\right)\right]$$

where $c = (\ln A)/k$. Thus the logistic function is really just a shifted hyperbolic tangent.

9.5 Linear Equations

A first-order **linear** differential equation is one that can be put into the form

$$\boxed{1} \qquad \frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions on a given interval. This type of equation occurs frequently in various sciences, as we will see.

An example of a linear equation is $xy' + y = 2x$ because, for $x \neq 0$, it can be written in the form

$$\boxed{2} \qquad y' + \frac{1}{x}y = 2$$

Notice that this differential equation is not separable because it's impossible to factor the expression for y' as a function of x times a function of y. But we can still solve the equation by noticing, by the Product Rule, that

$$xy' + y = (xy)'$$

and so we can rewrite the equation as

$$(xy)' = 2x$$

If we now integrate both sides of this equation, we get

$$xy = x^2 + C \qquad \text{or} \qquad y = x + \frac{C}{x}$$

If we had been given the differential equation in the form of Equation 2, we would have had to take the preliminary step of multiplying each side of the equation by x.

It turns out that every first-order linear differential equation can be solved in a similar fashion by multiplying both sides of Equation 1 by a suitable function $I(x)$ called an *integrating factor*. We try to find I so that the left side of Equation 1, when multiplied by $I(x)$, becomes the derivative of the product $I(x)y$:

$$\boxed{3} \qquad I(x)\big(y' + P(x)y\big) = \big(I(x)y\big)'$$

If we can find such a function I, then Equation 1 becomes

$$\big(I(x)y\big)' = I(x)\,Q(x)$$

Integrating both sides, we would have

$$I(x)y = \int I(x)\,Q(x)\,dx + C$$

so the solution would be

$$\boxed{4} \qquad y(x) = \frac{1}{I(x)}\left[\int I(x)\,Q(x)\,dx + C\right]$$

To find such an I, we expand Equation 3 and cancel terms:

$$I(x)y' + I(x)\,P(x)y = \big(I(x)y\big)' = I'(x)y + I(x)y'$$

$$I(x)\,P(x) = I'(x)$$

This is a separable differential equation for I, which we solve as follows:

$$\int \frac{dI}{I} = \int P(x)\,dx$$

$$\ln|I| = \int P(x)\,dx$$

$$I = Ae^{\int P(x)\,dx}$$

where $A = \pm e^C$. We are looking for a particular integrating factor, not the most general one, so we take $A = 1$ and use

$$\boxed{5} \qquad I(x) = e^{\int P(x)\,dx}$$

Thus a formula for the general solution to Equation 1 is provided by Equation 4, where I is given by Equation 5. Instead of memorizing this formula, however, we just remember the form of the integrating factor.

> To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and integrate both sides.

V EXAMPLE 1 Solve the differential equation $\dfrac{dy}{dx} + 3x^2y = 6x^2$.

SOLUTION The given equation is linear since it has the form of Equation 1 with $P(x) = 3x^2$ and $Q(x) = 6x^2$. An integrating factor is

$$I(x) = e^{\int 3x^2\,dx} = e^{x^3}$$

Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2e^{x^3}y = 6x^2e^{x^3}$$

or

$$\frac{d}{dx}\left(e^{x^3}y\right) = 6x^2e^{x^3}$$

Figure 1 shows the graphs of several members of the family of solutions in Example 1. Notice that they all approach 2 as $x \to \infty$.

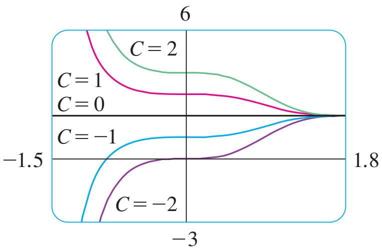


FIGURE 1

Integrating both sides, we have

$$e^{x^3}y = \int 6x^2e^{x^3}\,dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}$$

V EXAMPLE 2 Find the solution of the initial-value problem

$$x^2y' + xy = 1 \qquad x > 0 \qquad y(1) = 2$$

SOLUTION We must first divide both sides by the coefficient of y' to put the differential equation into standard form:

$$\boxed{6} \qquad y' + \frac{1}{x}y = \frac{1}{x^2} \qquad x > 0$$

The integrating factor is

$$I(x) = e^{\int (1/x)\,dx} = e^{\ln x} = x$$

Multiplication of Equation 6 by x gives

$$xy' + y = \frac{1}{x} \qquad \text{or} \qquad (xy)' = \frac{1}{x}$$

Then $$xy = \int \frac{1}{x}\,dx = \ln x + C$$

and so $$y = \frac{\ln x + C}{x}$$

Since $y(1) = 2$, we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}$$

The solution of the initial-value problem in Example 2 is shown in Figure 2.

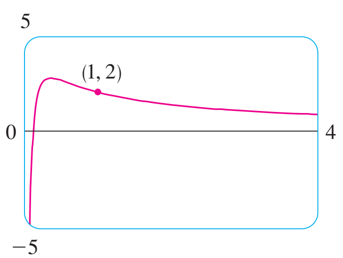


FIGURE 2

EXAMPLE 3 Solve $y' + 2xy = 1$.

SOLUTION The given equation is in the standard form for a linear equation. Multiplying by the integrating factor

$$e^{\int 2x\,dx} = e^{x^2}$$

we get $$e^{x^2}y' + 2xe^{x^2}y = e^{x^2}$$

or $$\left(e^{x^2}y\right)' = e^{x^2}$$

Therefore $$e^{x^2}y = \int e^{x^2}\,dx + C$$

Even though the solutions of the differential equation in Example 3 are expressed in terms of an integral, they can still be graphed by a computer algebra system (Figure 3).

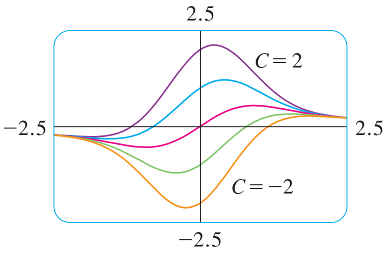


FIGURE 3

Recall from Section 7.5 that $\int e^{x^2}\,dx$ can't be expressed in terms of elementary functions. Nonetheless, it's a perfectly good function and we can leave the answer as

$$y = e^{-x^2}\int e^{x^2}\,dx + Ce^{-x^2}$$

Another way of writing the solution is

$$y = e^{-x^2}\int_0^x e^{t^2}\,dt + Ce^{-x^2}$$

(Any number can be chosen for the lower limit of integration.)

Application to Electric Circuits

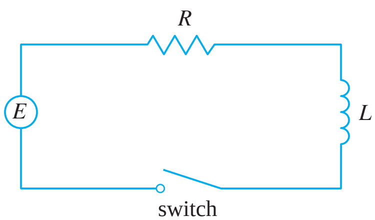


FIGURE 4

In Section 9.2 we considered the simple electric circuit shown in Figure 4: An electromotive force (usually a battery or generator) produces a voltage of $E(t)$ volts (V) and a current of $I(t)$ amperes (A) at time t. The circuit also contains a resistor with a resistance of R ohms (Ω) and an inductor with an inductance of L henries (H).

Ohm's Law gives the drop in voltage due to the resistor as RI. The voltage drop due to the inductor is $L(dI/dt)$. One of Kirchhoff's laws says that the sum of the voltage drops is equal to the supplied voltage $E(t)$. Thus we have

$$\boxed{7} \qquad L\frac{dI}{dt} + RI = E(t)$$

which is a first-order linear differential equation. The solution gives the current I at time t.

V EXAMPLE 4 Suppose that in the simple circuit of Figure 4 the resistance is 12 Ω and the inductance is 4 H. If a battery gives a constant voltage of 60 V and the switch is closed when $t = 0$ so the current starts with $I(0) = 0$, find (a) $I(t)$, (b) the current after 1 s, and (c) the limiting value of the current.

The differential equation in Example 4 is both linear and separable, so an alternative method is to solve it as a separable equation (Example 4 in Section 9.3). If we replace the battery by a generator, however, we get an equation that is linear but not separable (Example 5).

SOLUTION

(a) If we put $L = 4$, $R = 12$, and $E(t) = 60$ in Equation 7, we obtain the initial-value problem

$$4\frac{dI}{dt} + 12I = 60 \qquad I(0) = 0$$

or

$$\frac{dI}{dt} + 3I = 15 \qquad I(0) = 0$$

Multiplying by the integrating factor $e^{\int 3\,dt} = e^{3t}$, we get

$$e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t}$$

$$\frac{d}{dt}(e^{3t}I) = 15e^{3t}$$

$$e^{3t}I = \int 15e^{3t}\,dt = 5e^{3t} + C$$

$$I(t) = 5 + Ce^{-3t}$$

Figure 5 shows how the current in Example 4 approaches its limiting value.

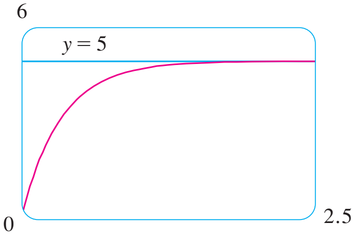


FIGURE 5

Since $I(0) = 0$, we have $5 + C = 0$, so $C = -5$ and

$$I(t) = 5(1 - e^{-3t})$$

(b) After 1 second the current is

$$I(1) = 5(1 - e^{-3}) \approx 4.75 \text{ A}$$

(c) The limiting value of the current is given by

$$\lim_{t\to\infty} I(t) = \lim_{t\to\infty} 5(1 - e^{-3t}) = 5 - 5 \lim_{t\to\infty} e^{-3t} = 5 - 0 = 5$$

EXAMPLE 5 Suppose that the resistance and inductance remain as in Example 4 but, instead of the battery, we use a generator that produces a variable voltage of $E(t) = 60 \sin 30t$ volts. Find $I(t)$.

SOLUTION This time the differential equation becomes

$$4\frac{dI}{dt} + 12I = 60 \sin 30t \qquad \text{or} \qquad \frac{dI}{dt} + 3I = 15 \sin 30t$$

The same integrating factor e^{3t} gives

$$\frac{d}{dt}(e^{3t}I) = e^{3t}\frac{dI}{dt} + 3e^{3t}I = 15e^{3t} \sin 30t$$

Figure 6 shows the graph of the current when the battery is replaced by a generator.

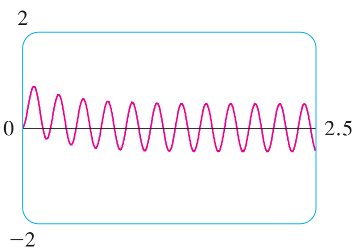


FIGURE 6

Using Formula 98 in the Table of Integrals, we have

$$e^{3t}I = \int 15e^{3t} \sin 30t\, dt = 15\frac{e^{3t}}{909}(3 \sin 30t - 30 \cos 30t) + C$$

$$I = \tfrac{5}{101}(\sin 30t - 10 \cos 30t) + Ce^{-3t}$$

Since $I(0) = 0$, we get

$$-\tfrac{50}{101} + C = 0$$

so
$$I(t) = \tfrac{5}{101}(\sin 30t - 10 \cos 30t) + \tfrac{50}{101}e^{-3t}$$

9.5 Exercises

1–4 Determine whether the differential equation is linear.

1. $x - y' = xy$

2. $y' + xy^2 = \sqrt{x}$

3. $y' = \dfrac{1}{x} + \dfrac{1}{y}$

4. $y \sin x = x^2y' - x$

5–14 Solve the differential equation.

5. $y' + y = 1$

6. $y' - y = e^x$

7. $y' = x - y$

8. $4x^3y + x^4y' = \sin^3 x$

9. $xy' + y = \sqrt{x}$

10. $y' + y = \sin(e^x)$

11. $\sin x \dfrac{dy}{dx} + (\cos x)y = \sin(x^2)$

12. $x\dfrac{dy}{dx} - 4y = x^4e^x$

13. $(1 + t)\dfrac{du}{dt} + u = 1 + t, \quad t > 0$

14. $t \ln t \dfrac{dr}{dt} + r = te^t$

15–20 Solve the initial-value problem.

15. $x^2y' + 2xy = \ln x, \quad y(1) = 2$

16. $t^3 \dfrac{dy}{dt} + 3t^2 y = \cos t, \quad y(\pi) = 0$

17. $t \dfrac{du}{dt} = t^2 + 3u, \quad t > 0, \quad u(2) = 4$

18. $2xy' + y = 6x, \quad x > 0, \quad y(4) = 20$

19. $xy' = y + x^2 \sin x, \quad y(\pi) = 0$

20. $(x^2 + 1) \dfrac{dy}{dx} + 3x(y - 1) = 0, \quad y(0) = 2$

21–22 Solve the differential equation and use a graphing calculator or computer to graph several members of the family of solutions. How does the solution curve change as C varies?

21. $xy' + 2y = e^x$

22. $xy' = x^2 + 2y$

23. A **Bernoulli differential equation** (named after James Bernoulli) is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, show that the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x)$$

24–25 Use the method of Exercise 23 to solve the differential equation.

24. $xy' + y = -xy^2$

25. $y' + \dfrac{2}{x}y = \dfrac{y^3}{x^2}$

26. Solve the second-order equation $xy'' + 2y' = 12x^2$ by making the substitution $u = y'$.

27. In the circuit shown in Figure 4, a battery supplies a constant voltage of 40 V, the inductance is 2 H, the resistance is 10 Ω, and $I(0) = 0$.
(a) Find $I(t)$.
(b) Find the current after 0.1 s.

28. In the circuit shown in Figure 4, a generator supplies a voltage of $E(t) = 40 \sin 60t$ volts, the inductance is 1 H, the resistance is 20 Ω, and $I(0) = 1$ A.
(a) Find $I(t)$.
(b) Find the current after 0.1 s.
(c) Use a graphing device to draw the graph of the current function.

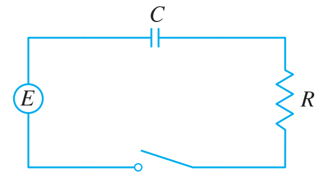
29. The figure shows a circuit containing an electromotive force, a capacitor with a capacitance of C farads (F), and a resistor with a resistance of R ohms (Ω). The voltage drop across the capacitor is Q/C, where Q is the charge (in coulombs), so in this case Kirchhoff's Law gives

$$RI + \frac{Q}{C} = E(t)$$

But $I = dQ/dt$ (see Example 3 in Section 2.7), so we have

$$R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

Suppose the resistance is 5 Ω, the capacitance is 0.05 F, a battery gives a constant voltage of 60 V, and the initial charge is $Q(0) = 0$ C. Find the charge and the current at time t.



30. In the circuit of Exercise 29, $R = 2\ \Omega$, $C = 0.01$ F, $Q(0) = 0$, and $E(t) = 10 \sin 60t$. Find the charge and the current at time t.

31. Let $P(t)$ be the performance level of someone learning a skill as a function of the training time t. The graph of P is called a *learning curve*. In Exercise 15 in Section 9.1 we proposed the differential equation

$$\frac{dP}{dt} = k[M - P(t)]$$

as a reasonable model for learning, where k is a positive constant. Solve it as a linear differential equation and use your solution to graph the learning curve.

32. Two new workers were hired for an assembly line. Jim processed 25 units during the first hour and 45 units during the second hour. Mark processed 35 units during the first hour and 50 units the second hour. Using the model of Exercise 31 and assuming that $P(0) = 0$, estimate the maximum number of units per hour that each worker is capable of processing.

33. In Section 9.3 we looked at mixing problems in which the volume of fluid remained constant and saw that such problems give rise to separable equations. (See Example 6 in that section.) If the rates of flow into and out of the system are different, then the volume is not constant and the resulting differential equation is linear but not separable.

A tank contains 100 L of water. A solution with a salt concentration of 0.4 kg/L is added at a rate of 5 L/min. The solution is kept mixed and is drained from the tank at a rate of 3 L/min. If $y(t)$ is the amount of salt (in kilograms) after t minutes, show that y satisfies the differential equation

$$\frac{dy}{dt} = 2 - \frac{3y}{100 + 2t}$$

Solve this equation and find the concentration after 20 minutes.

34. A tank with a capacity of 400 L is full of a mixture of water and chlorine with a concentration of 0.05 g of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at a rate of 4 L/s. The mixture is kept stirred and is pumped out at a rate of 10 L/s. Find the amount of chlorine in the tank as a function of time.

35. An object with mass m is dropped from rest and we assume that the air resistance is proportional to the speed of the object. If $s(t)$ is the distance dropped after t seconds, then the speed is $v = s'(t)$ and the acceleration is $a = v'(t)$. If g is the acceleration due to gravity, then the downward force on the object is $mg - cv$, where c is a positive constant, and Newton's Second Law gives

$$m\frac{dv}{dt} = mg - cv$$

(a) Solve this as a linear equation to show that

$$v = \frac{mg}{c}(1 - e^{-ct/m})$$

(b) What is the limiting velocity?
(c) Find the distance the object has fallen after t seconds.

36. If we ignore air resistance, we can conclude that heavier objects fall no faster than lighter objects. But if we take air resistance into account, our conclusion changes. Use the expression for the velocity of a falling object in Exercise 35(a) to find dv/dm and show that heavier objects *do* fall faster than lighter ones.

37. (a) Show that the substitution $z = 1/P$ transforms the logistic differential equation $P' = kP(1 - P/M)$ into the linear differential equation

$$z' + kz = \frac{k}{M}$$

(b) Solve the linear differential equation in part (a) and thus obtain an expression for $P(t)$. Compare with Equation 9.4.7.

38. To account for seasonal variation in the logistic differential equation we could allow k and M to be functions of t:

$$\frac{dP}{dt} = k(t)P\left(1 - \frac{P}{M(t)}\right)$$

(a) Verify that the substitution $z = 1/P$ transforms this equation into the linear equation

$$\frac{dz}{dt} + k(t)z = \frac{k(t)}{M(t)}$$

(b) Write an expression for the solution of the linear equation in part (a) and use it to show that if the carrying capacity M is constant, then

$$P(t) = \frac{M}{1 + CMe^{-\int k(t)\,dt}}$$

Deduce that if $\int_0^\infty k(t)\,dt = \infty$, then $\lim_{t\to\infty} P(t) = M$. [This will be true if $k(t) = k_0 + a\cos bt$ with $k_0 > 0$, which describes a positive intrinsic growth rate with a periodic seasonal variation.]

(c) If k is constant but M varies, show that

$$z(t) = e^{-kt}\int_0^t \frac{ke^{ks}}{M(s)}\,ds + Ce^{-kt}$$

and use l'Hospital's Rule to deduce that if $M(t)$ has a limit as $t\to\infty$, then $P(t)$ has the same limit.

9.6 Predator-Prey Systems

We have looked at a variety of models for the growth of a single species that lives alone in an environment. In this section we consider more realistic models that take into account the interaction of two species in the same habitat. We will see that these models take the form of a pair of linked differential equations.

We first consider the situation in which one species, called the *prey*, has an ample food supply and the second species, called the *predators*, feeds on the prey. Examples of prey and predators include rabbits and wolves in an isolated forest, food fish and sharks, aphids and ladybugs, and bacteria and amoebas. Our model will have two dependent variables and both are functions of time. We let $R(t)$ be the number of prey (using R for rabbits) and $W(t)$ be the number of predators (with W for wolves) at time t.

In the absence of predators, the ample food supply would support exponential growth of the prey, that is,

$$\frac{dR}{dt} = kR \qquad \text{where } k \text{ is a positive constant}$$

In the absence of prey, we assume that the predator population would decline at a rate proportional to itself, that is,

$$\frac{dW}{dt} = -rW \qquad \text{where } r \text{ is a positive constant}$$

With both species present, however, we assume that the principal cause of death among the prey is being eaten by a predator, and the birth and survival rates of the predators depend on their available food supply, namely, the prey. We also assume that the two species encounter each other at a rate that is proportional to both populations and is therefore proportional to the product RW. (The more there are of either population, the more encounters there are likely to be.) A system of two differential equations that incorporates these assumptions is as follows:

W represents the predator.
R represents the prey.

$$\boxed{1} \qquad \frac{dR}{dt} = kR - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

where k, r, a, and b are positive constants. Notice that the term $-aRW$ decreases the natural growth rate of the prey and the term bRW increases the natural growth rate of the predators.

The Lotka-Volterra equations were proposed as a model to explain the variations in the shark and food-fish populations in the Adriatic Sea by the Italian mathematician Vito Volterra (1860–1940).

The equations in $\boxed{1}$ are known as the **predator-prey equations**, or the **Lotka-Volterra equations**. A **solution** of this system of equations is a pair of functions $R(t)$ and $W(t)$ that describe the populations of prey and predator as functions of time. Because the system is coupled (R and W occur in both equations), we can't solve one equation and then the other; we have to solve them simultaneously. Unfortunately, it is usually impossible to find explicit formulas for R and W as functions of t. We can, however, use graphical methods to analyze the equations.

V EXAMPLE 1 Suppose that populations of rabbits and wolves are described by the Lotka-Volterra equations $\boxed{1}$ with $k = 0.08$, $a = 0.001$, $r = 0.02$, and $b = 0.00002$. The time t is measured in months.
(a) Find the constant solutions (called the **equilibrium solutions**) and interpret the answer.
(b) Use the system of differential equations to find an expression for dW/dR.
(c) Draw a direction field for the resulting differential equation in the RW-plane. Then use that direction field to sketch some solution curves.
(d) Suppose that, at some point in time, there are 1000 rabbits and 40 wolves. Draw the corresponding solution curve and use it to describe the changes in both population levels.
(e) Use part (d) to make sketches of R and W as functions of t.

SOLUTION
(a) With the given values of k, a, r, and b, the Lotka-Volterra equations become

$$\frac{dR}{dt} = 0.08R - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

Both R and W will be constant if both derivatives are 0, that is,

$$R' = R(0.08 - 0.001W) = 0$$
$$W' = W(-0.02 + 0.00002R) = 0$$

One solution is given by $R = 0$ and $W = 0$. (This makes sense: If there are no rabbits or wolves, the populations are certainly not going to increase.) The other constant solution is

$$W = \frac{0.08}{0.001} = 80 \qquad R = \frac{0.02}{0.00002} = 1000$$

So the equilibrium populations consist of 80 wolves and 1000 rabbits. This means that 1000 rabbits are just enough to support a constant wolf population of 80. There are neither too many wolves (which would result in fewer rabbits) nor too few wolves (which would result in more rabbits).

(b) We use the Chain Rule to eliminate t:

$$\frac{dW}{dt} = \frac{dW}{dR}\frac{dR}{dt}$$

so

$$\frac{dW}{dR} = \frac{\dfrac{dW}{dt}}{\dfrac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

(c) If we think of W as a function of R, we have the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

We draw the direction field for this differential equation in Figure 1 and we use it to sketch several solution curves in Figure 2. If we move along a solution curve, we observe how the relationship between R and W changes as time passes. Notice that the curves appear to be closed in the sense that if we travel along a curve, we always return to the same point. Notice also that the point (1000, 80) is inside all the solution curves. That point is called an *equilibrium point* because it corresponds to the equilibrium solution $R = 1000$, $W = 80$.

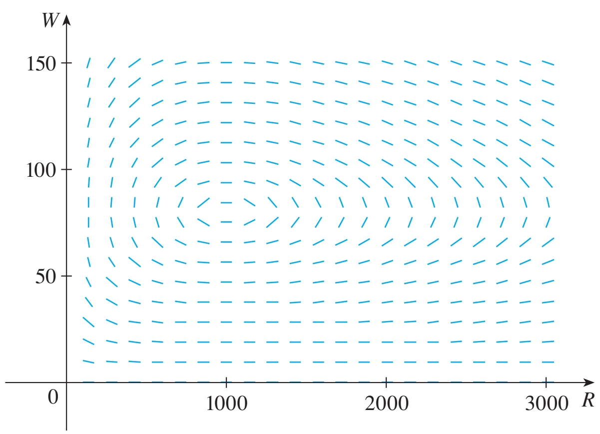


FIGURE 1 Direction field for the predator-prey system

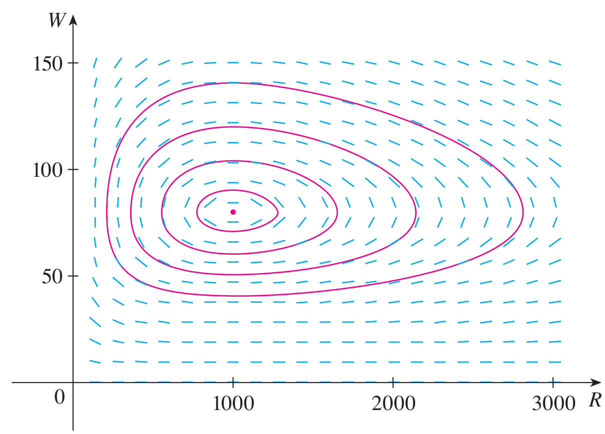


FIGURE 2 Phase portrait of the system

When we represent solutions of a system of differential equations as in Figure 2, we refer to the RW-plane as the **phase plane**, and we call the solution curves **phase trajectories**. So a phase trajectory is a path traced out by solutions (R, W) as time goes by. A **phase portrait** consists of equilibrium points and typical phase trajectories, as shown in Figure 2.

(d) Starting with 1000 rabbits and 40 wolves corresponds to drawing the solution curve through the point $P_0(1000, 40)$. Figure 3 shows this phase trajectory with the direction field removed. Starting at the point P_0 at time $t = 0$ and letting t increase, do we move clockwise or counterclockwise around the phase trajectory? If we put $R = 1000$ and $W = 40$ in the first differential equation, we get

$$\frac{dR}{dt} = 0.08(1000) - 0.001(1000)(40) = 80 - 40 = 40$$

Since $dR/dt > 0$, we conclude that R is increasing at P_0 and so we move counterclockwise around the phase trajectory.

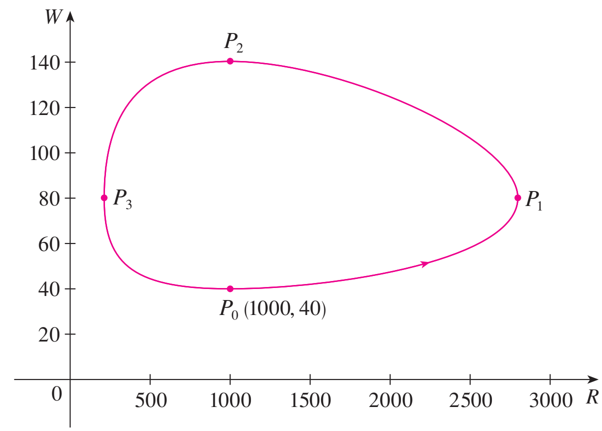


FIGURE 3
Phase trajectory through (1000, 40)

We see that at P_0 there aren't enough wolves to maintain a balance between the populations, so the rabbit population increases. That results in more wolves and eventually there are so many wolves that the rabbits have a hard time avoiding them. So the number of rabbits begins to decline (at P_1, where we estimate that R reaches its maximum population of about 2800). This means that at some later time the wolf population starts to fall (at P_2, where $R = 1000$ and $W \approx 140$). But this benefits the rabbits, so their population later starts to increase (at P_3, where $W = 80$ and $R \approx 210$). As a consequence, the wolf population eventually starts to increase as well. This happens when the populations return to their initial values of $R = 1000$ and $W = 40$, and the entire cycle begins again.

(e) From the description in part (d) of how the rabbit and wolf populations rise and fall, we can sketch the graphs of $R(t)$ and $W(t)$. Suppose the points P_1, P_2, and P_3 in Figure 3 are reached at times t_1, t_2, and t_3. Then we can sketch graphs of R and W as in Figure 4.

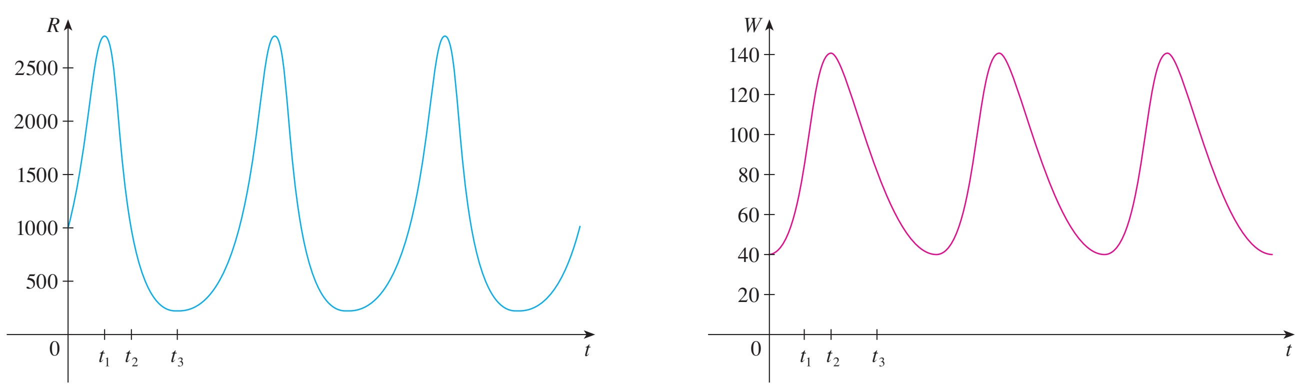


FIGURE 4 Graphs of the rabbit and wolf populations as functions of time

TEC In Module 9.6 you can change the coefficients in the Lotka-Volterra equations and observe the resulting changes in the phase trajectory and graphs of the rabbit and wolf populations.

To make the graphs easier to compare, we draw the graphs on the same axes but with different scales for R and W, as in Figure 5. Notice that the rabbits reach their maximum populations about a quarter of a cycle before the wolves.

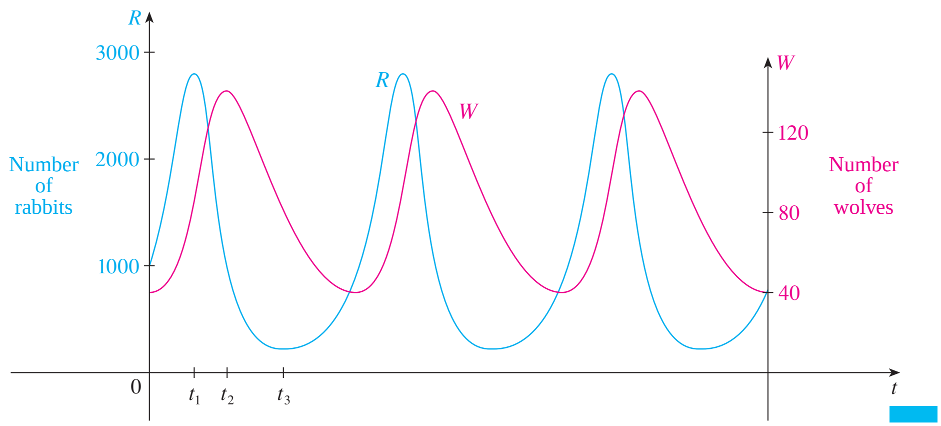


FIGURE 5
Comparison of the rabbit and wolf populations

An important part of the modeling process, as we discussed in Section 1.2, is to interpret our mathematical conclusions as real-world predictions and to test the predictions against real data. The Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s. Figure 6 shows graphs of the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded by the company over a 90-year period. You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur and the period of these cycles is roughly 10 years.

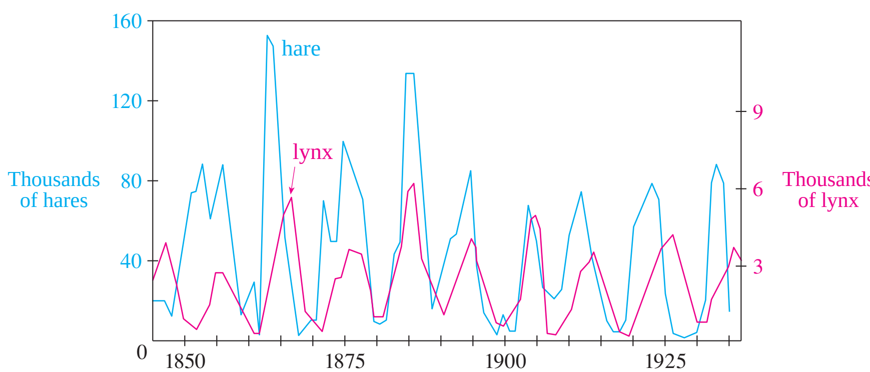


FIGURE 6
Relative abundance of hare and lynx from Hudson's Bay Company records

Although the relatively simple Lotka-Volterra model has had some success in explaining and predicting coupled populations, more sophisticated models have also been proposed. One way to modify the Lotka-Volterra equations is to assume that, in the absence of predators, the prey grow according to a logistic model with carrying capacity M. Then the Lotka-Volterra equations $\boxed{1}$ are replaced by the system of differential equations

$$\frac{dR}{dt} = kR\left(1 - \frac{R}{M}\right) - aRW \qquad \frac{dW}{dt} = -rW + bRW$$

This model is investigated in Exercises 11 and 12.

Models have also been proposed to describe and predict population levels of two or more species that compete for the same resources or cooperate for mutual benefit. Such models are explored in Exercises 2–4.

9.6 Exercises

1. For each predator-prey system, determine which of the variables, x or y, represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Do the predators feed only on the prey or do they have additional food sources? Explain.

(a) $\dfrac{dx}{dt} = -0.05x + 0.0001xy$

$\dfrac{dy}{dt} = 0.1y - 0.005xy$

(b) $\dfrac{dx}{dt} = 0.2x - 0.0002x^2 - 0.006xy$

$\dfrac{dy}{dt} = -0.015y + 0.00008xy$

2. Each system of differential equations is a model for two species that either compete for the same resources or cooperate for mutual benefit (flowering plants and insect pollinators, for instance). Decide whether each system describes competition or cooperation and explain why it is a reasonable model. (Ask yourself what effect an increase in one species has on the growth rate of the other.)

(a) $\dfrac{dx}{dt} = 0.12x - 0.0006x^2 + 0.00001xy$

$\dfrac{dy}{dt} = 0.08x + 0.00004xy$

(b) $\dfrac{dx}{dt} = 0.15x - 0.0002x^2 - 0.0006xy$

$\dfrac{dy}{dt} = 0.2y - 0.00008y^2 - 0.0002xy$

3. The system of differential equations

$$\frac{dx}{dt} = 0.5x - 0.004x^2 - 0.001xy$$

$$\frac{dy}{dt} = 0.4y - 0.001y^2 - 0.002xy$$

is a model for the populations of two species.

(a) Does the model describe cooperation, or competition, or a predator-prey relationship?

(b) Find the equilibrium solutions and explain their significance.

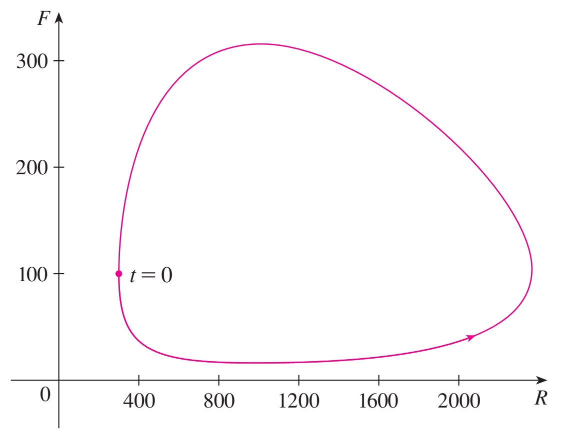
4. Flies, frogs, and crocodiles coexist in an environment. To survive, frogs need to eat flies and crocodiles need to eat frogs. In the absence of frogs, the fly population will grow exponentially and the crocodile population will decay exponentially. In the absence of crocodiles and flies, the frog population will decay exponentially. If $P(t)$, $Q(t)$, and $R(t)$ represent the populations of these three species at time t, write a system of differential equations as a model for their evolution. If the constants in your equation are all positive, explain why you have used plus or minus signs.

5–6 A phase trajectory is shown for populations of rabbits (R) and foxes (F).

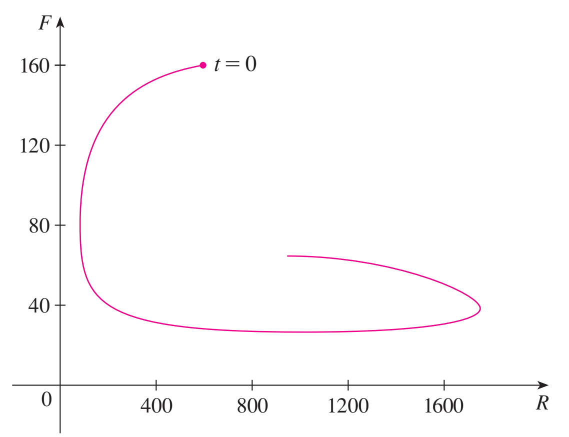
(a) Describe how each population changes as time goes by.

(b) Use your description to make a rough sketch of the graphs of R and F as functions of time.

5.

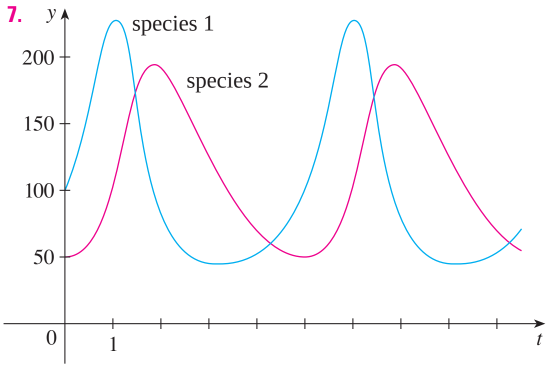


6.

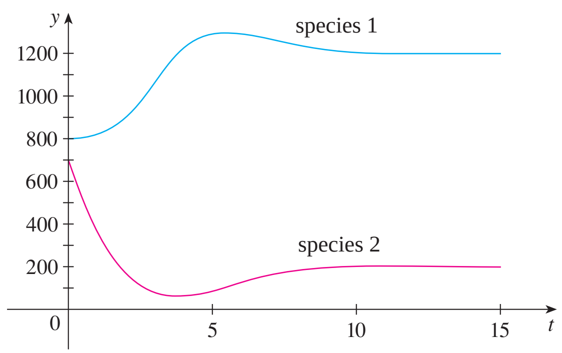


7–8 Graphs of populations of two species are shown. Use them to sketch the corresponding phase trajectory.

7.



8.



9. In Example 1(b) we showed that the rabbit and wolf populations satisfy the differential equation

$$\frac{dW}{dR} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

By solving this separable differential equation, show that

$$\frac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C$$

where C is a constant.

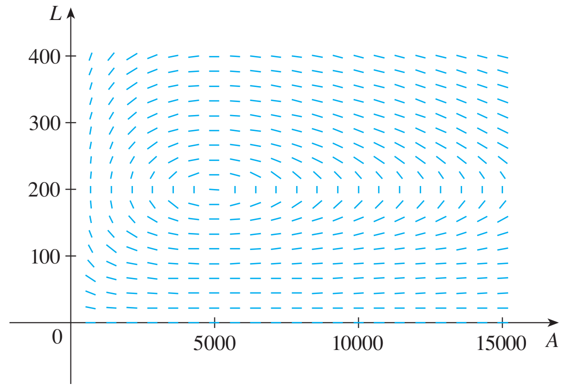
It is impossible to solve this equation for W as an explicit function of R (or vice versa). If you have a computer algebra system that graphs implicitly defined curves, use this equation and your CAS to draw the solution curve that passes through the point (1000, 40) and compare with Figure 3.

10. Populations of aphids and ladybugs are modeled by the equations

$$\frac{dA}{dt} = 2A - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) Find the equilibrium solutions and explain their significance.
(b) Find an expression for dL/dA.
(c) The direction field for the differential equation in part (b) is shown. Use it to sketch a phase portrait. What do the phase trajectories have in common?



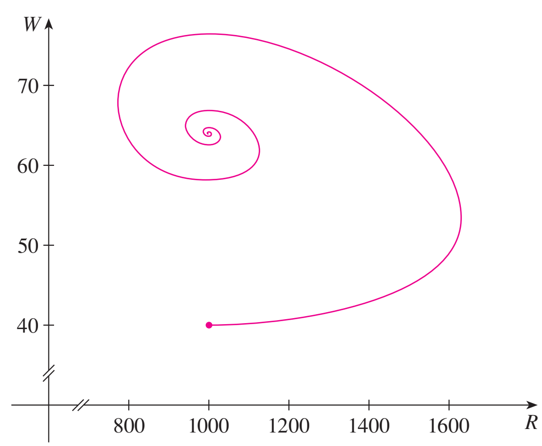
(d) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
(e) Use part (d) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

11. In Example 1 we used Lotka-Volterra equations to model populations of rabbits and wolves. Let's modify those equations as follows:

$$\frac{dR}{dt} = 0.08R(1 - 0.0002R) - 0.001RW$$

$$\frac{dW}{dt} = -0.02W + 0.00002RW$$

(a) According to these equations, what happens to the rabbit population in the absence of wolves?
(b) Find all the equilibrium solutions and explain their significance.
(c) The figure shows the phase trajectory that starts at the point (1000, 40). Describe what eventually happens to the rabbit and wolf populations.



(d) Sketch graphs of the rabbit and wolf populations as functions of time.

CAS **12.** In Exercise 10 we modeled populations of aphids and ladybugs with a Lotka-Volterra system. Suppose we modify those equations as follows:

$$\frac{dA}{dt} = 2A(1 - 0.0001A) - 0.01AL$$

$$\frac{dL}{dt} = -0.5L + 0.0001AL$$

(a) In the absence of ladybugs, what does the model predict about the aphids?
(b) Find the equilibrium solutions.
(c) Find an expression for dL/dA.
(d) Use a computer algebra system to draw a direction field for the differential equation in part (c). Then use the direction field to sketch a phase portrait. What do the phase trajectories have in common?
(e) Suppose that at time $t = 0$ there are 1000 aphids and 200 ladybugs. Draw the corresponding phase trajectory and use it to describe how both populations change.
(f) Use part (e) to make rough sketches of the aphid and ladybug populations as functions of t. How are the graphs related to each other?

9 Review

Concept Check

1. (a) What is a differential equation?
(b) What is the order of a differential equation?
(c) What is an initial condition?

2. What can you say about the solutions of the equation $y' = x^2 + y^2$ just by looking at the differential equation?

3. What is a direction field for the differential equation $y' = F(x, y)$?

4. Explain how Euler's method works.

5. What is a separable differential equation? How do you solve it?

6. What is a first-order linear differential equation? How do you solve it?

7. (a) Write a differential equation that expresses the law of natural growth. What does it say in terms of relative growth rate?
(b) Under what circumstances is this an appropriate model for population growth?
(c) What are the solutions of this equation?

8. (a) Write the logistic equation.
(b) Under what circumstances is this an appropriate model for population growth?

9. (a) Write Lotka-Volterra equations to model populations of food fish (F) and sharks (S).
(b) What do these equations say about each population in the absence of the other?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. All solutions of the differential equation $y' = -1 - y^4$ are decreasing functions.

2. The function $f(x) = (\ln x)/x$ is a solution of the differential equation $x^2y' + xy = 1$.

3. The equation $y' = x + y$ is separable.

4. The equation $y' = 3y - 2x + 6xy - 1$ is separable.

5. The equation $e^x y' = y$ is linear.

6. The equation $y' + xy = e^y$ is linear.

7. If y is the solution of the initial-value problem

$$\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right) \qquad y(0) = 1$$

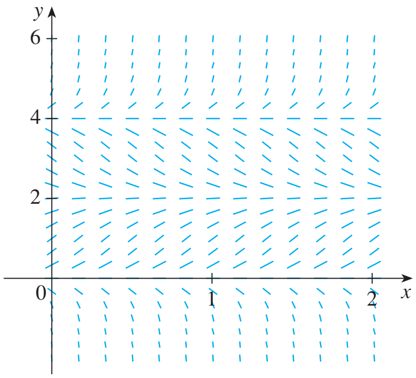
then $\lim_{t\to\infty} y = 5$.

Exercises

1. (a) A direction field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) $y(0) = -0.3$ (ii) $y(0) = 1$
(iii) $y(0) = 3$ (iv) $y(0) = 4.3$

(b) If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite? What are the equilibrium solutions?



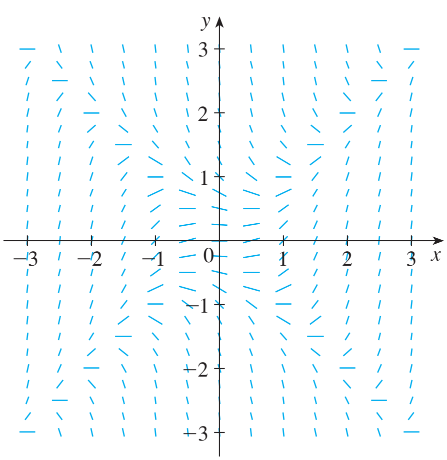
2. (a) Sketch a direction field for the differential equation $y' = x/y$. Then use it to sketch the four solutions that satisfy the initial conditions $y(0) = 1$, $y(0) = -1$, $y(2) = 1$, and $y(-2) = 1$.

(b) Check your work in part (a) by solving the differential equation explicitly. What type of curve is each solution curve?

3. (a) A direction field for the differential equation $y' = x^2 - y^2$ is shown. Sketch the solution of the initial-value problem

$$y' = x^2 - y^2 \qquad y(0) = 1$$

Use your graph to estimate the value of $y(0.3)$.



(b) Use Euler's method with step size 0.1 to estimate $y(0.3)$, where $y(x)$ is the solution of the initial-value problem in part (a). Compare with your estimate from part (a).

(c) On what lines are the centers of the horizontal line segments of the direction field in part (a) located? What happens when a solution curve crosses these lines?

4. (a) Use Euler's method with step size 0.2 to estimate $y(0.4)$, where $y(x)$ is the solution of the initial-value problem

$$y' = 2xy^2 \qquad y(0) = 1$$

(b) Repeat part (a) with step size 0.1.

(c) Find the exact solution of the differential equation and compare the value at 0.4 with the approximations in parts (a) and (b).

5–8 Solve the differential equation.

5. $y' = xe^{-\sin x} - y\cos x$

6. $\dfrac{dx}{dt} = 1 - t + x - tx$

7. $2ye^{y^2}y' = 2x + 3\sqrt{x}$

8. $x^2y' - y = 2x^3e^{-1/x}$

9–11 Solve the initial-value problem.

9. $\dfrac{dr}{dt} + 2tr = r, \quad r(0) = 5$

10. $(1 + \cos x)y' = (1 + e^{-y})\sin x, \quad y(0) = 0$

11. $xy' - y = x\ln x, \quad y(1) = 2$

12. Solve the initial-value problem $y' = 3x^2e^y$, $y(0) = 1$, and graph the solution.

13–14 Find the orthogonal trajectories of the family of curves.

13. $y = ke^x$

14. $y = e^{kx}$

15. (a) Write the solution of the initial-value problem

$$\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{2000}\right) \qquad P(0) = 100$$

and use it to find the population when $t = 20$.

(b) When does the population reach 1200?

16. (a) The population of the world was 5.28 billion in 1990 and 6.07 billion in 2000. Find an exponential model for these data and use the model to predict the world population in the year 2020.

(b) According to the model in part (a), when will the world population exceed 10 billion?

(c) Use the data in part (a) to find a logistic model for the population. Assume a carrying capacity of 100 billion. Then use the logistic model to predict the population in 2020. Compare with your prediction from the exponential model.

(d) According to the logistic model, when will the world population exceed 10 billion? Compare with your prediction in part (b).

17. The von Bertalanffy growth model is used to predict the length $L(t)$ of a fish over a period of time. If L_∞ is the largest length for a species, then the hypothesis is that the rate of growth in length is proportional to $L_\infty - L$, the length yet to be achieved.

(a) Formulate and solve a differential equation to find an expression for $L(t)$.

(b) For the North Sea haddock it has been determined that $L_\infty = 53$ cm, $L(0) = 10$ cm, and the constant of proportionality is 0.2. What does the expression for $L(t)$ become with these data?

18. A tank contains 100 L of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate. How much salt is in the tank after 6 minutes?

19. One model for the spread of an epidemic is that the rate of spread is jointly proportional to the number of infected people and the number of uninfected people. In an isolated town of 5000 inhabitants, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week. How long does it take for 80% of the population to become infected?

20. The Brentano-Stevens Law in psychology models the way that a subject reacts to a stimulus. It states that if R represents the reaction to an amount S of stimulus, then the relative rates of increase are proportional:

$$\frac{1}{R}\frac{dR}{dt} = \frac{k}{S}\frac{dS}{dt}$$

where k is a positive constant. Find R as a function of S.

21. The transport of a substance across a capillary wall in lung physiology has been modeled by the differential equation

$$\frac{dh}{dt} = -\frac{R}{V}\left(\frac{h}{k+h}\right)$$

where h is the hormone concentration in the bloodstream, t is time, R is the maximum transport rate, V is the volume of the capillary, and k is a positive constant that measures the affinity between the hormones and the enzymes that assist the process. Solve this differential equation to find a relationship between h and t.

22. Populations of birds and insects are modeled by the equations

$$\frac{dx}{dt} = 0.4x - 0.002xy$$

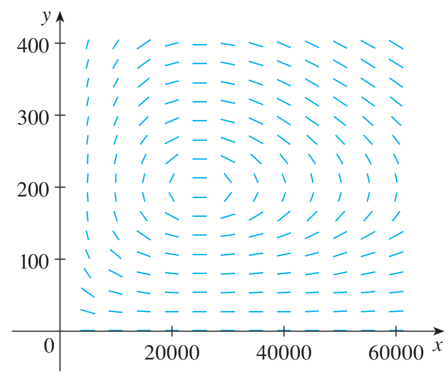
$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) Which of the variables, x or y, represents the bird population and which represents the insect population? Explain.

(b) Find the equilibrium solutions and explain their significance.

(c) Find an expression for dy/dx.

(d) The direction field for the differential equation in part (c) is shown. Use it to sketch the phase trajectory corresponding to initial populations of 100 birds and 40,000 insects. Then use the phase trajectory to describe how both populations change.



(e) Use part (d) to make rough sketches of the bird and insect populations as functions of time. How are these graphs related to each other?

23. Suppose the model of Exercise 22 is replaced by the equations

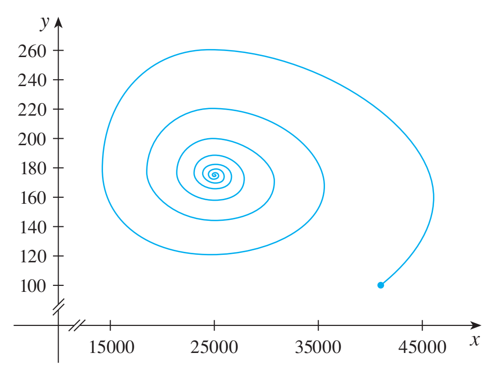
$$\frac{dx}{dt} = 0.4x(1 - 0.000005x) - 0.002xy$$

$$\frac{dy}{dt} = -0.2y + 0.000008xy$$

(a) According to these equations, what happens to the insect population in the absence of birds?

(b) Find the equilibrium solutions and explain their significance.

(c) The figure shows the phase trajectory that starts with 100 birds and 40,000 insects. Describe what eventually happens to the bird and insect populations.



(d) Sketch graphs of the bird and insect populations as functions of time.

24. Barbara weighs 60 kg and is on a diet of 1600 calories per day, of which 850 are used automatically by basal metabolism. She spends about 15 cal/kg/day times her weight doing exercise. If 1 kg of fat contains 10,000 cal and we assume that the storage of calories in the form of fat is 100% efficient, formulate a differential equation and solve it to find her weight as a function of time. Does her weight ultimately approach an equilibrium weight?

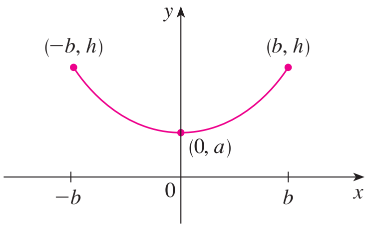
25. When a flexible cable of uniform density is suspended between two fixed points and hangs of its own weight, the shape $y = f(x)$ of the cable must satisfy a differential equation of the form

$$\frac{d^2y}{dx^2} = k\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where k is a positive constant. Consider the cable shown in the figure.

(a) Let $z = dy/dx$ in the differential equation. Solve the resulting first-order differential equation (in z), and then integrate to find y.

(b) Determine the length of the cable.



Problems Plus

1. Find all functions f such that f' is continuous and

$$[f(x)]^2 = 100 + \int_0^x \{[f(t)]^2 + [f'(t)]^2\}\,dt \qquad \text{for all real } x$$

2. A student forgot the Product Rule for differentiation and made the mistake of thinking that $(fg)' = f'g'$. However, he was lucky and got the correct answer. The function f that he used was $f(x) = e^{x^2}$ and the domain of his problem was the interval $\left(\frac{1}{2}, \infty\right)$. What was the function g?

3. Let f be a function with the property that $f(0) = 1$, $f'(0) = 1$, and $f(a + b) = f(a)f(b)$ for all real numbers a and b. Show that $f'(x) = f(x)$ for all x and deduce that $f(x) = e^x$.

4. Find all functions f that satisfy the equation

$$\left(\int f(x)\,dx\right)\left(\int \frac{1}{f(x)}\,dx\right) = -1$$

5. Find the curve $y = f(x)$ such that $f(x) \geqslant 0$, $f(0) = 0$, $f(1) = 1$, and the area under the graph of f from 0 to x is proportional to the $(n + 1)$st power of $f(x)$.

6. A *subtangent* is a portion of the x-axis that lies directly beneath the segment of a tangent line from the point of contact to the x-axis. Find the curves that pass through the point $(c, 1)$ and whose subtangents all have length c.

7. A peach pie is taken out of the oven at 5:00 PM. At that time it is piping hot, 100°C. At 5:10 PM its temperature is 80°C; at 5:20 PM it is 65°C. What is the temperature of the room?

8. Snow began to fall during the morning of February 2 and continued steadily into the afternoon. At noon a snowplow began removing snow from a road at a constant rate. The plow traveled 6 km from noon to 1 PM but only 3 km from 1 PM to 2 PM. When did the snow begin to fall? [*Hints:* To get started, let t be the time measured in hours after noon; let $x(t)$ be the distance traveled by the plow at time t; then the speed of the plow is dx/dt. Let b be the number of hours before noon that it began to snow. Find an expression for the height of the snow at time t. Then use the given information that the rate of removal R (in m^3/h) is constant.]

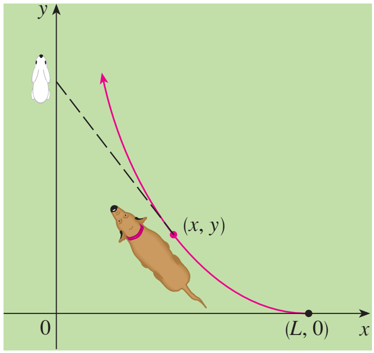


FIGURE FOR PROBLEM 9

9. A dog sees a rabbit running in a straight line across an open field and gives chase. In a rectangular coordinate system (as shown in the figure), assume:

(i) The rabbit is at the origin and the dog is at the point $(L, 0)$ at the instant the dog first sees the rabbit.

(ii) The rabbit runs up the y-axis and the dog always runs straight for the rabbit.

(iii) The dog runs at the same speed as the rabbit.

(a) Show that the dog's path is the graph of the function $y = f(x)$, where y satisfies the differential equation

$$x\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

(b) Determine the solution of the equation in part (a) that satisfies the initial conditions $y = y' = 0$ when $x = L$. [*Hint:* Let $z = dy/dx$ in the differential equation and solve the resulting first-order equation to find z; then integrate z to find y.]

(c) Does the dog ever catch the rabbit?

Graphing calculator or computer required

10. (a) Suppose that the dog in Problem 9 runs twice as fast as the rabbit. Find a differential equation for the path of the dog. Then solve it to find the point where the dog catches the rabbit.

(b) Suppose the dog runs half as fast as the rabbit. How close does the dog get to the rabbit? What are their positions when they are closest?

11. A planning engineer for a new alum plant must present some estimates to his company regarding the capacity of a silo designed to contain bauxite ore until it is processed into alum. The ore resembles pink talcum powder and is poured from a conveyor at the top of the silo. The silo is a cylinder 100 ft high with a radius of 200 ft. The conveyor carries ore at a rate of $60{,}000\pi\ \text{ft}^3/\text{h}$ and the ore maintains a conical shape whose radius is 1.5 times its height.

(a) If, at a certain time t, the pile is 60 ft high, how long will it take for the pile to reach the top of the silo?

(b) Management wants to know how much room will be left in the floor area of the silo when the pile is 60 ft high. How fast is the floor area of the pile growing at that height?

(c) Suppose a loader starts removing the ore at the rate of $20{,}000\pi\ \text{ft}^3/\text{h}$ when the height of the pile reaches 90 ft. Suppose, also, that the pile continues to maintain its shape. How long will it take for the pile to reach the top of the silo under these conditions?

12. Find the curve that passes through the point $(3, 2)$ and has the property that if the tangent line is drawn at any point P on the curve, then the part of the tangent line that lies in the first quadrant is bisected at P.

13. Recall that the normal line to a curve at a point P on the curve is the line that passes through P and is perpendicular to the tangent line at P. Find the curve that passes through the point $(3, 2)$ and has the property that if the normal line is drawn at any point on the curve, then the y-intercept of the normal line is always 6.

14. Find all curves with the property that if the normal line is drawn at any point P on the curve, then the part of the normal line between P and the x-axis is bisected by the y-axis.

15. Find all curves with the property that if a line is drawn from the origin to any point (x, y) on the curve, and then a tangent is drawn to the curve at that point and extended to meet the x-axis, the result is an isosceles triangle with equal sides meeting at (x, y).

16. (a) An outfielder fields a baseball 280 ft away from home plate and throws it directly to the catcher with an initial velocity of 100 ft/s. Assume that the velocity $v(t)$ of the ball after t seconds satisfies the differential equation $dv/dt = -\frac{1}{10}v$ because of air resistance. How long does it take for the ball to reach home plate? (Ignore any vertical motion of the ball.)

(b) The manager of the team wonders whether the ball will reach home plate sooner if it is relayed by an infielder. The shortstop can position himself directly between the outfielder and home plate, catch the ball thrown by the outfielder, turn, and throw the ball to the catcher with an initial velocity of 105 ft/s. The manager clocks the relay time of the shortstop (catching, turning, throwing) at half a second. How far from home plate should the shortstop position himself to minimize the total time for the ball to reach home plate? Should the manager encourage a direct throw or a relayed throw? What if the shortstop can throw at 115 ft/s?

(c) For what throwing velocity of the shortstop does a relayed throw take the same time as a direct throw?

10 Parametric Equations and Polar Coordinates

The Hale-Bopp comet, with its blue ion tail and white dust tail, appeared in the sky in March 1997. In Section 10.6 you will see how polar coordinates provide a convenient equation for the path of this comet.

So far we have described plane curves by giving y as a function of x $[y = f(x)]$ or x as a function of y $[x = g(y)]$ or by giving a relation between x and y that defines y implicitly as a function of x $[f(x, y) = 0]$. In this chapter we discuss two new methods for describing curves.

Some curves, such as the cycloid, are best handled when both x and y are given in terms of a third variable t called a parameter $[x = f(t), y = g(t)]$. Other curves, such as the cardioid, have their most convenient description when we use a new coordinate system, called the polar coordinate system.

10.1 Curves Defined by Parametric Equations

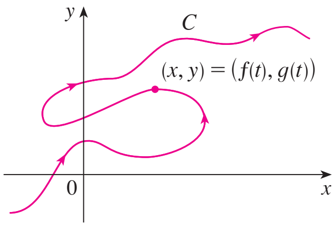


FIGURE 1

Imagine that a particle moves along the curve C shown in Figure 1. It is impossible to describe C by an equation of the form $y = f(x)$ because C fails the Vertical Line Test. But the x- and y-coordinates of the particle are functions of time and so we can write $x = f(t)$ and $y = g(t)$. Such a pair of equations is often a convenient way of describing a curve and gives rise to the following definition.

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t) \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**. The parameter t does not necessarily represent time and, in fact, we could use a letter other than t for the parameter. But in many applications of parametric curves, t does denote time and therefore we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

EXAMPLE 1 Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t \qquad y = t + 1$$

SOLUTION Each value of t gives a point on the curve, as shown in the table. For instance, if $t = 0$, then $x = 0$, $y = 1$ and so the corresponding point is $(0, 1)$. In Figure 2 we plot the points (x, y) determined by several values of the parameter and we join them to produce a curve.

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3
3	3	4
4	8	5

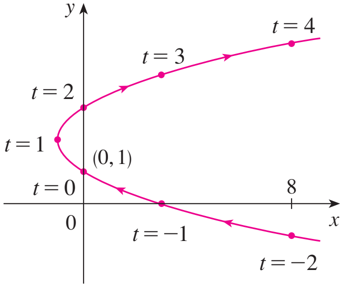


FIGURE 2

A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t increases. Notice that the consecutive points marked on the curve appear at equal time intervals but not at equal distances. That is because the particle slows down and then speeds up as t increases.

It appears from Figure 2 that the curve traced out by the particle may be a parabola. This can be confirmed by eliminating the parameter t as follows. We obtain $t = y - 1$ from the second equation and substitute into the first equation. This gives

$$x = t^2 - 2t = (y - 1)^2 - 2(y - 1) = y^2 - 4y + 3$$

This equation in x and y describes *where* the particle has been, but it doesn't tell us *when* the particle was at a particular point. The parametric equations have an advantage—they tell us *when* the particle was at a point. They also indicate the *direction* of the motion.

and so the curve represented by the given parametric equations is the parabola $x = y^2 - 4y + 3$.

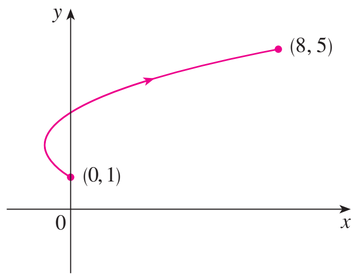


FIGURE 3

No restriction was placed on the parameter t in Example 1, so we assumed that t could be any real number. But sometimes we restrict t to lie in a finite interval. For instance, the parametric curve

$$x = t^2 - 2t \qquad y = t + 1 \qquad 0 \leqslant t \leqslant 4$$

shown in Figure 3 is the part of the parabola in Example 1 that starts at the point (0, 1) and ends at the point (8, 5). The arrowhead indicates the direction in which the curve is traced as t increases from 0 to 4.

In general, the curve with parametric equations

$$x = f(t) \qquad y = g(t) \qquad a \leqslant t \leqslant b$$

has **initial point** $(f(a), g(a))$ and **terminal point** $(f(b), g(b))$.

V EXAMPLE 2 What curve is represented by the following parametric equations?

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

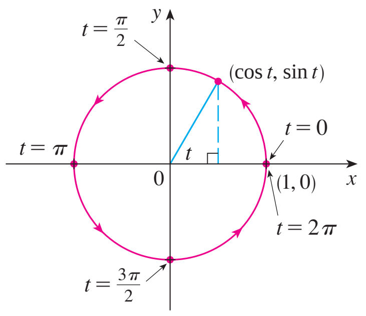


FIGURE 4

SOLUTION If we plot points, it appears that the curve is a circle. We can confirm this impression by eliminating t. Observe that

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus the point (x, y) moves on the unit circle $x^2 + y^2 = 1$. Notice that in this example the parameter t can be interpreted as the angle (in radians) shown in Figure 4. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ moves once around the circle in the counterclockwise direction starting from the point (1, 0).

EXAMPLE 3 What curve is represented by the given parametric equations?

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

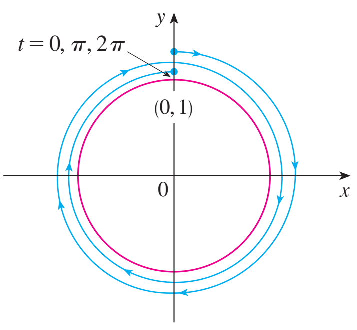


FIGURE 5

SOLUTION Again we have

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

so the parametric equations again represent the unit circle $x^2 + y^2 = 1$. But as t increases from 0 to 2π, the point $(x, y) = (\sin 2t, \cos 2t)$ starts at (0, 1) and moves *twice* around the circle in the clockwise direction as indicated in Figure 5.

Examples 2 and 3 show that different sets of parametric equations can represent the same curve. Thus we distinguish between a *curve*, which is a set of points, and a *parametric curve*, in which the points are traced in a particular way.

EXAMPLE 4 Find parametric equations for the circle with center (h, k) and radius r.

SOLUTION If we take the equations of the unit circle in Example 2 and multiply the expressions for x and y by r, we get $x = r\cos t$, $y = r\sin t$. You can verify that these equations represent a circle with radius r and center the origin traced counterclockwise. We now shift h units in the x-direction and k units in the y-direction and obtain parametric equations of the circle (Figure 6) with center (h, k) and radius r:

$$x = h + r\cos t \qquad y = k + r\sin t \qquad 0 \leqslant t \leqslant 2\pi$$

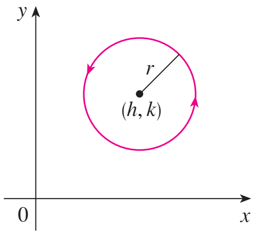


FIGURE 6
$x = h + r\cos t,\ y = k + r\sin t$

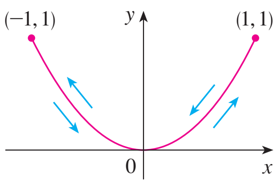


FIGURE 7

V EXAMPLE 5 Sketch the curve with parametric equations $x = \sin t$, $y = \sin^2 t$.

SOLUTION Observe that $y = (\sin t)^2 = x^2$ and so the point (x, y) moves on the parabola $y = x^2$. But note also that, since $-1 \leqslant \sin t \leqslant 1$, we have $-1 \leqslant x \leqslant 1$, so the parametric equations represent only the part of the parabola for which $-1 \leqslant x \leqslant 1$. Since $\sin t$ is periodic, the point $(x, y) = (\sin t, \sin^2 t)$ moves back and forth infinitely often along the parabola from $(-1, 1)$ to $(1, 1)$. (See Figure 7.)

TEC Module 10.1A gives an animation of the relationship between motion along a parametric curve $x = f(t)$, $y = g(t)$ and motion along the graphs of f and g as functions of t. Clicking on TRIG gives you the family of parametric curves

$$x = a\cos bt \qquad y = c\sin dt$$

If you choose $a = b = c = d = 1$ and click on **animate**, you will see how the graphs of $x = \cos t$ and $y = \sin t$ relate to the circle in Example 2. If you choose $a = b = c = 1$, $d = 2$, you will see graphs as in Figure 8. By clicking on **animate** or moving the t-slider to the right, you can see from the color coding how motion along the graphs of $x = \cos t$ and $y = \sin 2t$ corresponds to motion along the parametric curve, which is called a **Lissajous figure**.

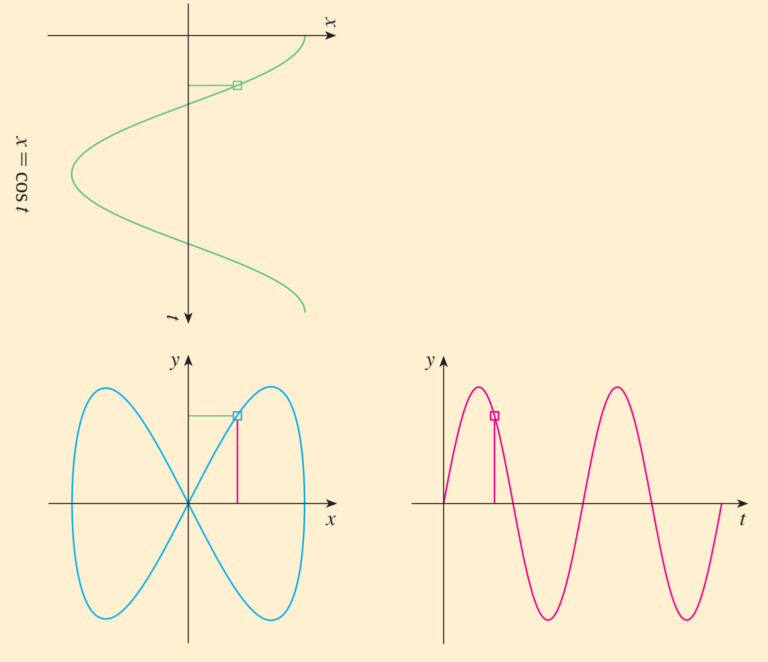


FIGURE 8 $x = \cos t \quad y = \sin 2t$ $\qquad y = \sin 2t$

Graphing Devices

Most graphing calculators and computer graphing programs can be used to graph curves defined by parametric equations. In fact, it's instructive to watch a parametric curve being drawn by a graphing calculator because the points are plotted in order as the corresponding parameter values increase.

EXAMPLE 6 Use a graphing device to graph the curve $x = y^4 - 3y^2$.

SOLUTION If we let the parameter be $t = y$, then we have the equations

$$x = t^4 - 3t^2 \qquad y = t$$

Using these parametric equations to graph the curve, we obtain Figure 9. It would be possible to solve the given equation ($x = y^4 - 3y^2$) for y as four functions of x and graph them individually, but the parametric equations provide a much easier method.

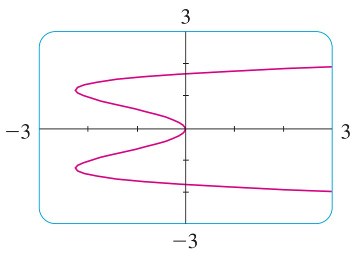


FIGURE 9

In general, if we need to graph an equation of the form $x = g(y)$, we can use the parametric equations

$$x = g(t) \qquad y = t$$

Notice also that curves with equations $y = f(x)$ (the ones we are most familiar with—graphs of functions) can also be regarded as curves with parametric equations

$$x = t \qquad y = f(t)$$

Graphing devices are particularly useful for sketching complicated curves. For instance, the curves shown in Figures 10, 11, and 12 would be virtually impossible to produce by hand.

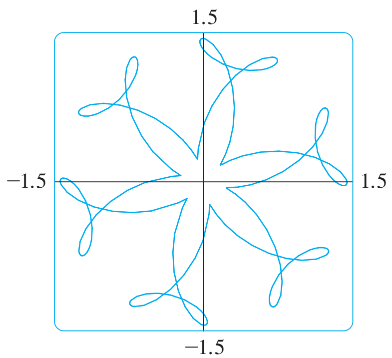


FIGURE 10

$x = \sin t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 13t$

$y = \cos t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 13t$

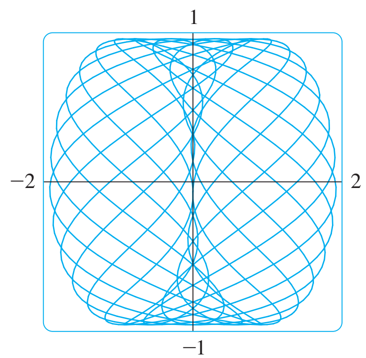


FIGURE 11

$x = \sin t - \sin 2.3t$

$y = \cos t$

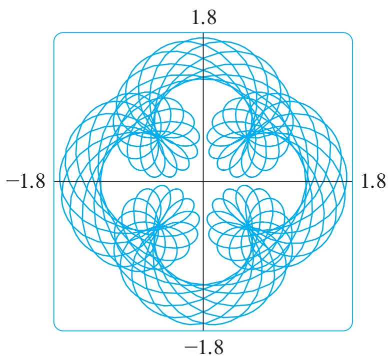


FIGURE 12

$x = \sin t + \frac{1}{2}\sin 5t + \frac{1}{4}\cos 2.3t$

$y = \cos t + \frac{1}{2}\cos 5t + \frac{1}{4}\sin 2.3t$

One of the most important uses of parametric curves is in computer-aided design (CAD). In the Laboratory Project after Section 10.2 we will investigate special parametric curves, called **Bézier curves**, that are used extensively in manufacturing, especially in the automotive industry. These curves are also employed in specifying the shapes of letters and other symbols in laser printers.

The Cycloid

TEC An animation in Module 10.1B shows how the cycloid is formed as the circle moves.

EXAMPLE 7 The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see Figure 13). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.

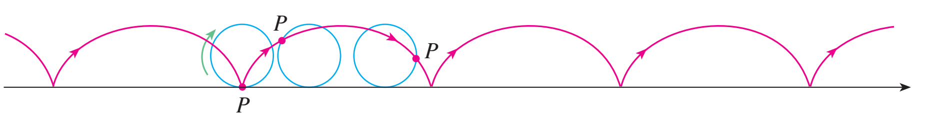


FIGURE 13

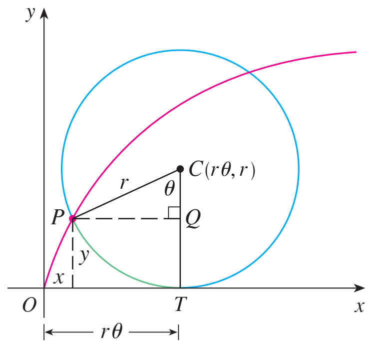


FIGURE 14

SOLUTION We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from Figure 14 that the distance it has rolled from the origin is

$$|OT| = \text{arc } PT = r\theta$$

Therefore the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from Figure 14 we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore parametric equations of the cycloid are

$$\boxed{1} \qquad x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta) \qquad \theta \in \mathbb{R}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \leqslant \theta \leqslant 2\pi$. Although Equations 1 were derived from Figure 14, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ (see Exercise 39).

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated and not as convenient to work with as the parametric equations.

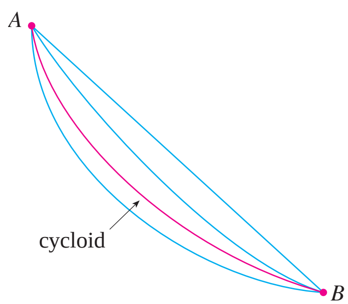


FIGURE 15

One of the first people to study the cycloid was Galileo, who proposed that bridges be built in the shape of cycloids and who tried to find the area under one arch of a cycloid. Later this curve arose in connection with the **brachistochrone problem**: Find the curve along which a particle will slide in the shortest time (under the influence of gravity) from a point A to a lower point B not directly beneath A. The Swiss mathematician John Bernoulli, who posed this problem in 1696, showed that among all possible curves that join A to B, as in Figure 15, the particle will take the least time sliding from A to B if the curve is part of an inverted arch of a cycloid.

The Dutch physicist Huygens had already shown that the cycloid is also the solution to the **tautochrone problem**; that is, no matter where a particle P is placed on an inverted cycloid, it takes the same time to slide to the bottom (see Figure 16). Huygens proposed that pendulum clocks (which he invented) should swing in cycloidal arcs because then the pendulum would take the same time to make a complete oscillation whether it swings through a wide or a small arc.

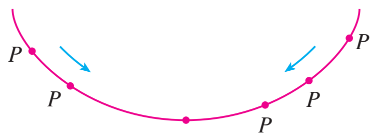


FIGURE 16

Families of Parametric Curves

V EXAMPLE 8 Investigate the family of curves with parametric equations

$$x = a + \cos t \qquad y = a\tan t + \sin t$$

What do these curves have in common? How does the shape change as a increases?

SOLUTION We use a graphing device to produce the graphs for the cases $a = -2, -1, -0.5, -0.2, 0, 0.5, 1$, and 2 shown in Figure 17. Notice that all of these curves (except the case $a = 0$) have two branches, and both branches approach the vertical asymptote $x = a$ as x approaches a from the left or right.

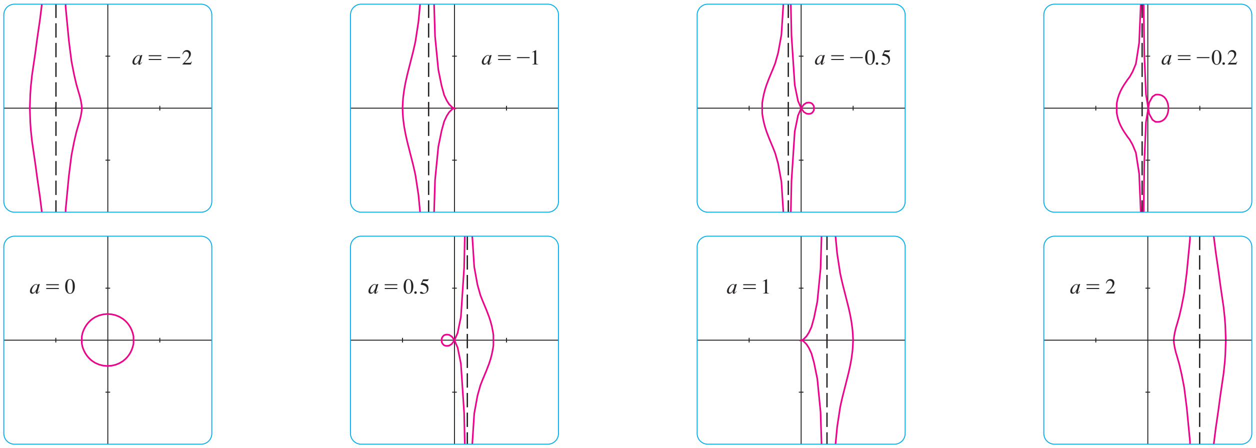


FIGURE 17 Members of the family $x = a + \cos t$, $y = a \tan t + \sin t$, all graphed in the viewing rectangle $[-4, 4]$ by $[-4, 4]$

When $a < -1$, both branches are smooth; but when a reaches -1, the right branch acquires a sharp point, called a *cusp*. For a between -1 and 0 the cusp turns into a loop, which becomes larger as a approaches 0. When $a = 0$, both branches come together and form a circle (see Example 2). For a between 0 and 1, the left branch has a loop, which shrinks to become a cusp when $a = 1$. For $a > 1$, the branches become smooth again, and as a increases further, they become less curved. Notice that the curves with a positive are reflections about the y-axis of the corresponding curves with a negative.

These curves are called **conchoids of Nicomedes** after the ancient Greek scholar Nicomedes. He called them conchoids because the shape of their outer branches resembles that of a conch shell or mussel shell.

10.1 Exercises

1–4 Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.

1. $x = t^2 + t, \quad y = t^2 - t, \quad -2 \leqslant t \leqslant 2$

2. $x = t^2, \quad y = t^3 - 4t, \quad -3 \leqslant t \leqslant 3$

3. $x = \cos^2 t, \quad y = 1 - \sin t, \quad 0 \leqslant t \leqslant \pi/2$

4. $x = e^{-t} + t, \quad y = e^t - t, \quad -2 \leqslant t \leqslant 2$

5–10
(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as t increases.
(b) Eliminate the parameter to find a Cartesian equation of the curve.

5. $x = 3 - 4t, \quad y = 2 - 3t$

6. $x = 1 - 2t, \quad y = \frac{1}{2}t - 1, \quad -2 \leqslant t \leqslant 4$

7. $x = 1 - t^2, \quad y = t - 2, \quad -2 \leqslant t \leqslant 2$

8. $x = t - 1, \quad y = t^3 + 1, \quad -2 \leqslant t \leqslant 2$

9. $x = \sqrt{t}, \quad y = 1 - t$

10. $x = t^2, \quad y = t^3$

11–18
(a) Eliminate the parameter to find a Cartesian equation of the curve.
(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

11. $x = \sin \frac{1}{2}\theta, \quad y = \cos \frac{1}{2}\theta, \quad -\pi \leqslant \theta \leqslant \pi$

12. $x = \frac{1}{2}\cos\theta, \quad y = 2\sin\theta, \quad 0 \leqslant \theta \leqslant \pi$

13. $x = \sin t, \quad y = \csc t, \quad 0 < t < \pi/2$

14. $x = e^t - 1, \quad y = e^{2t}$

15. $x = e^{2t}, \quad y = t + 1$

16. $y = \sqrt{t + 1}, \quad y = \sqrt{t - 1}$

17. $x = \sinh t, \quad y = \cosh t$

18. $x = \tan^2\theta, \quad y = \sec\theta, \quad -\pi/2 < \theta < \pi/2$

19–22 Describe the motion of a particle with position (x, y) as t varies in the given interval.

19. $x = 3 + 2\cos t, \quad y = 1 + 2\sin t, \quad \pi/2 \leqslant t \leqslant 3\pi/2$

20. $x = 2\sin t, \quad y = 4 + \cos t, \quad 0 \leqslant t \leqslant 3\pi/2$

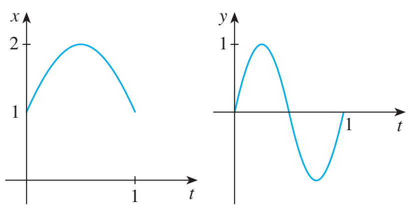
21. $x = 5\sin t, \quad y = 2\cos t, \quad -\pi \leqslant t \leqslant 5\pi$

22. $x = \sin t, \quad y = \cos^2 t, \quad -2\pi \leqslant t \leqslant 2\pi$

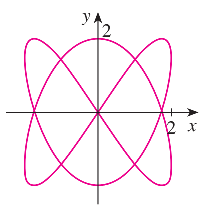
23. Suppose a curve is given by the parametric equations $x = f(t)$, $y = g(t)$, where the range of f is $[1, 4]$ and the range of g is $[2, 3]$. What can you say about the curve?

24. Match the graphs of the parametric equations $x = f(t)$ and $y = g(t)$ in (a)–(d) with the parametric curves labeled I–IV. Give reasons for your choices.

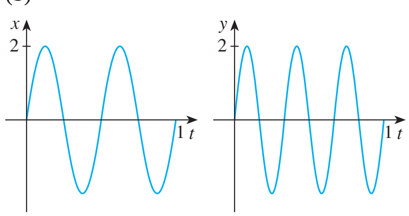
(a)



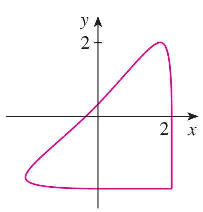
I



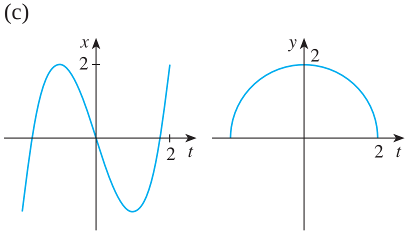
(b)



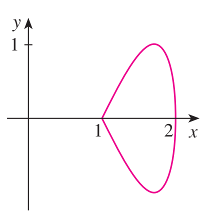
II



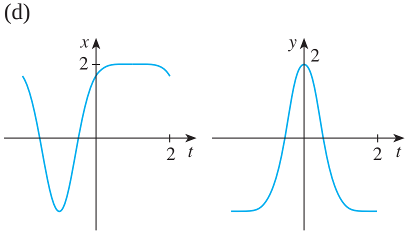
(c)



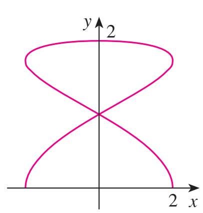
III



(d)

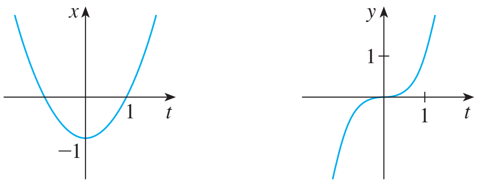


IV



25–27 Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.

25.



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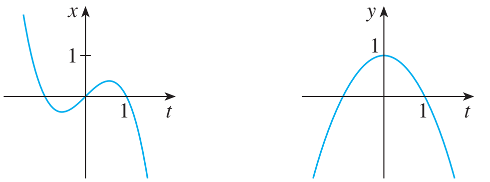
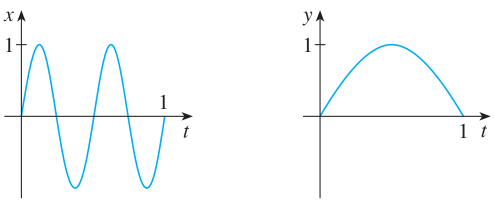


27.



28. Match the parametric equations with the graphs labeled I-VI. Give reasons for your choices. (Do not use a graphing device.)

(a) $x = t^4 - t + 1, \quad y = t^2$

(b) $x = t^2 - 2t, \quad y = \sqrt{t}$

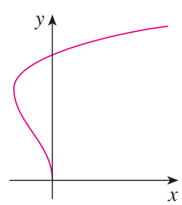
(c) $x = \sin 2t, \quad y = \sin(t + \sin 2t)$

(d) $x = \cos 5t, \quad y = \sin 2t$

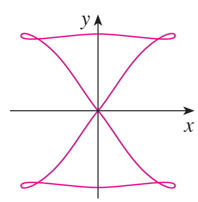
(e) $x = t + \sin 4t, \quad y = t^2 + \cos 3t$

(f) $x = \dfrac{\sin 2t}{4 + t^2}, \quad y = \dfrac{\cos 2t}{4 + t^2}$

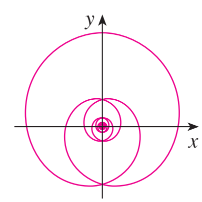
I



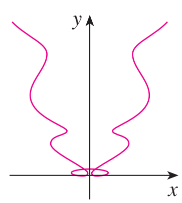
II



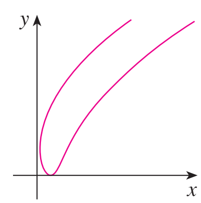
III



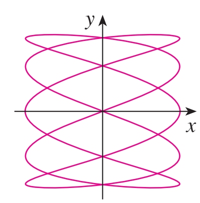
IV



V



VI



29. Graph the curve $x = y - 2 \sin \pi y$.

30. Graph the curves $y = x^3 - 4x$ and $x = y^3 - 4y$ and find their points of intersection correct to one decimal place.

31. (a) Show that the parametric equations

$$x = x_1 + (x_2 - x_1)t \qquad y = y_1 + (y_2 - y_1)t$$

where $0 \leqslant t \leqslant 1$, describe the line segment that joins the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

(b) Find parametric equations to represent the line segment from $(-2, 7)$ to $(3, -1)$.

32. Use a graphing device and the result of Exercise 31(a) to draw the triangle with vertices $A(1, 1)$, $B(4, 2)$, and $C(1, 5)$.

33. Find parametric equations for the path of a particle that moves along the circle $x^2 + (y - 1)^2 = 4$ in the manner described.

(a) Once around clockwise, starting at $(2, 1)$

(b) Three times around counterclockwise, starting at $(2, 1)$

(c) Halfway around counterclockwise, starting at $(0, 3)$

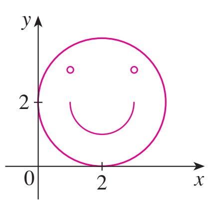
34. (a) Find parametric equations for the ellipse $x^2/a^2 + y^2/b^2 = 1$. [*Hint:* Modify the equations of the circle in Example 2.]

(b) Use these parametric equations to graph the ellipse when $a = 3$ and $b = 1, 2, 4$, and 8.

(c) How does the shape of the ellipse change as b varies?

35–36 Use a graphing calculator or computer to reproduce the picture.

35.



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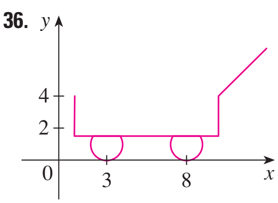


37–38 Compare the curves represented by the parametric equations. How do they differ?

37. (a) $x = t^3, \quad y = t^2$ (b) $x = t^6, \quad y = t^4$
(c) $x = e^{-3t}, \quad y = e^{-2t}$

38. (a) $x = t, \quad y = t^{-2}$ (b) $x = \cos t, \quad y = \sec^2 t$
(c) $x = e^t, \quad y = e^{-2t}$

---

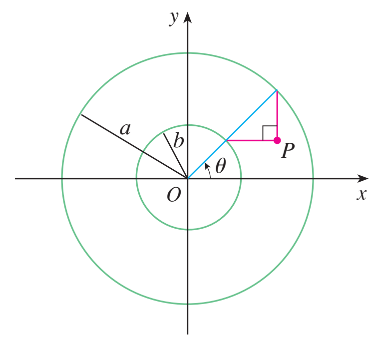
39. Derive Equations 1 for the case $\pi/2 < \theta < \pi$.

40. Let P be a point at a distance d from the center of a circle of radius r. The curve traced out by P as the circle rolls along a straight line is called a **trochoid**. (Think of the motion of a point on a spoke of a bicycle wheel.) The cycloid is the special case of a trochoid with $d = r$. Using the same parameter θ as for the cycloid and, assuming the line is the x-axis and $\theta = 0$ when P is at one of its lowest points, show that parametric equations of the trochoid are

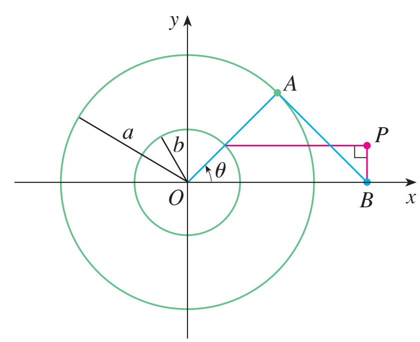
$$x = r\theta - d \sin \theta \qquad y = r - d \cos \theta$$

Sketch the trochoid for the cases $d < r$ and $d > r$.

41. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. Then eliminate the parameter and identify the curve.



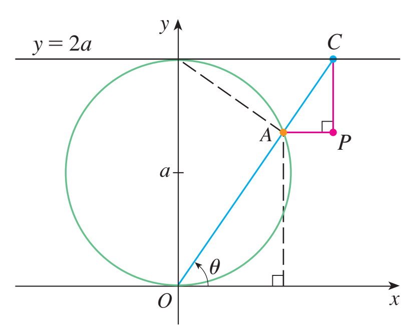
42. If a and b are fixed numbers, find parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.



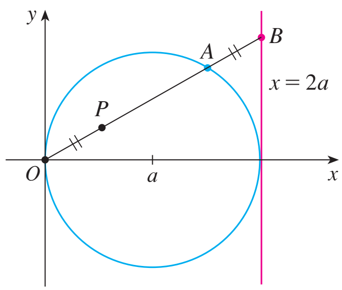
43. A curve, called a **witch of Maria Agnesi**, consists of all possible positions of the point P in the figure. Show that parametric equations for this curve can be written as

$$x = 2a \cot \theta \qquad y = 2a \sin^2\theta$$

Sketch the curve.



44. (a) Find parametric equations for the set of all points P as shown in the figure such that $|OP| = |AB|$. (This curve is called the **cissoid of Diocles** after the Greek scholar Diocles, who introduced the cissoid as a graphical method for constructing the edge of a cube whose volume is twice that of a given cube.)
(b) Use the geometric description of the curve to draw a rough sketch of the curve by hand. Check your work by using the parametric equations to graph the curve.



45. Suppose that the position of one particle at time t is given by

$$x_1 = 3 \sin t \qquad y_1 = 2 \cos t \qquad 0 \leqslant t \leqslant 2\pi$$

and the position of a second particle is given by

$$x_2 = -3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

(a) Graph the paths of both particles. How many points of intersection are there?
(b) Are any of these points of intersection *collision points*? In other words, are the particles ever at the same place at the same time? If so, find the collision points.
(c) Describe what happens if the path of the second particle is given by

$$x_2 = 3 + \cos t \qquad y_2 = 1 + \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

46. If a projectile is fired with an initial velocity of v_0 meters per second at an angle α above the horizontal and air resistance is assumed to be negligible, then its position after t seconds is given by the parametric equations

$$x = (v_0 \cos \alpha)t \qquad y = (v_0 \sin \alpha)t - \tfrac{1}{2}gt^2$$

where g is the acceleration due to gravity (9.8 m/s^2).
(a) If a gun is fired with $\alpha = 30°$ and $v_0 = 500$ m/s, when will the bullet hit the ground? How far from the gun will it hit the ground? What is the maximum height reached by the bullet?
(b) Use a graphing device to check your answers to part (a). Then graph the path of the projectile for several other values of the angle α to see where it hits the ground. Summarize your findings.
(c) Show that the path is parabolic by eliminating the parameter.

47. Investigate the family of curves defined by the parametric equations $x = t^2$, $y = t^3 - ct$. How does the shape change as c increases? Illustrate by graphing several members of the family.

48. The **swallowtail catastrophe curves** are defined by the parametric equations $x = 2ct - 4t^3$, $y = -ct^2 + 3t^4$. Graph several of these curves. What features do the curves have in common? How do they change when c increases?

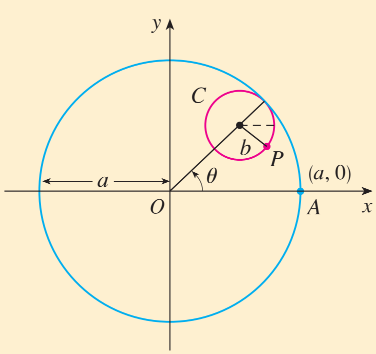
49. Graph several members of the family of curves with parametric equations $x = t + a \cos t$, $y = t + a \sin t$, where $a > 0$. How does the shape change as a increases? For what values of a does the curve have a loop?

50. Graph several members of the family of curves $x = \sin t + \sin nt$, $y = \cos t + \cos nt$ where n is a positive integer. What features do the curves have in common? What happens as n increases?

51. The curves with equations $x = a \sin nt$, $y = b \cos t$ are called **Lissajous figures**. Investigate how these curves vary when a, b, and n vary. (Take n to be a positive integer.)

52. Investigate the family of curves defined by the parametric equations $x = \cos t$, $y = \sin t - \sin ct$, where $c > 0$. Start by letting c be a positive integer and see what happens to the shape as c increases. Then explore some of the possibilities that occur when c is a fraction.

LABORATORY PROJECT RUNNING CIRCLES AROUND CIRCLES



In this project we investigate families of curves, called *hypocycloids* and *epicycloids*, that are generated by the motion of a point on a circle that rolls inside or outside another circle.

1. A **hypocycloid** is a curve traced out by a fixed point P on a circle C of radius b as C rolls on the inside of a circle with center O and radius a. Show that if the initial position of P is $(a, 0)$ and the parameter θ is chosen as in the figure, then parametric equations of the hypocycloid are

$$x = (a - b) \cos \theta + b \cos\left(\frac{a - b}{b}\,\theta\right) \qquad y = (a - b) \sin \theta - b \sin\left(\frac{a - b}{b}\,\theta\right)$$

Graphing calculator or computer required

TEC Look at Module 10.1B to see how hypocycloids and epicycloids are formed by the motion of rolling circles.

2. Use a graphing device (or the interactive graphic in TEC Module 10.1B) to draw the graphs of hypocycloids with a a positive integer and $b = 1$. How does the value of a affect the graph? Show that if we take $a = 4$, then the parametric equations of the hypocycloid reduce to

$$x = 4\cos^3\theta \qquad y = 4\sin^3\theta$$

This curve is called a **hypocycloid of four cusps**, or an **astroid**.

3. Now try $b = 1$ and $a = n/d$, a fraction where n and d have no common factor. First let $n = 1$ and try to determine graphically the effect of the denominator d on the shape of the graph. Then let n vary while keeping d constant. What happens when $n = d + 1$?

4. What happens if $b = 1$ and a is irrational? Experiment with an irrational number like $\sqrt{2}$ or $e - 2$. Take larger and larger values for θ and speculate on what would happen if we were to graph the hypocycloid for all real values of θ.

5. If the circle C rolls on the *outside* of the fixed circle, the curve traced out by P is called an **epicycloid**. Find parametric equations for the epicycloid.

6. Investigate the possible shapes for epicycloids. Use methods similar to Problems 2–4.

10.2 Calculus with Parametric Curves

Having seen how to represent curves by parametric equations, we now apply the methods of calculus to these parametric curves. In particular, we solve problems involving tangents, area, arc length, and surface area.

Tangents

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the curve where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dy/dx:

If we think of the curve as being traced out by a moving particle, then dy/dt and dx/dt are the vertical and horizontal velocities of the particle and Formula 1 says that the slope of the tangent is the ratio of these velocities.

$$\boxed{1} \qquad \boxed{\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} \qquad \text{if} \quad \frac{dx}{dt} \neq 0}$$

Equation 1 (which you can remember by thinking of canceling the dt's) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t. We see from $\boxed{1}$ that the curve has a horizontal tangent when $dy/dt = 0$ (provided that $dx/dt \neq 0$) and it has a vertical tangent when $dx/dt = 0$ (provided that $dy/dt \neq 0$). This information is useful for sketching parametric curves.

As we know from Chapter 4, it is also useful to consider d^2y/dx^2. This can be found by replacing y by dy/dx in Equation 1:

Note that $\dfrac{d^2y}{dx^2} \neq \dfrac{\dfrac{d^2y}{dt^2}}{\dfrac{d^2x}{dt^2}}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}}$$

EXAMPLE 1 A curve C is defined by the parametric equations $x = t^2$, $y = t^3 - 3t$.
(a) Show that C has two tangents at the point (3, 0) and find their equations.
(b) Find the points on C where the tangent is horizontal or vertical.
(c) Determine where the curve is concave upward or downward.
(d) Sketch the curve.

SOLUTION
(a) Notice that $y = t^3 - 3t = t(t^2 - 3) = 0$ when $t = 0$ or $t = \pm\sqrt{3}$. Therefore the point (3, 0) on C arises from two values of the parameter, $t = \sqrt{3}$ and $t = -\sqrt{3}$. This indicates that C crosses itself at (3, 0). Since

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

the slope of the tangent when $t = \pm\sqrt{3}$ is $dy/dx = \pm 6/(2\sqrt{3}) = \pm\sqrt{3}$, so the equations of the tangents at (3, 0) are

$$y = \sqrt{3}\,(x - 3) \qquad \text{and} \qquad y = -\sqrt{3}\,(x - 3)$$

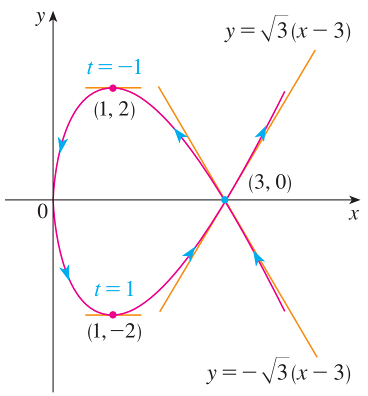


FIGURE 1

(b) C has a horizontal tangent when $dy/dx = 0$, that is, when $dy/dt = 0$ and $dx/dt \neq 0$. Since $dy/dt = 3t^2 - 3$, this happens when $t^2 = 1$, that is, $t = \pm 1$. The corresponding points on C are $(1, -2)$ and $(1, 2)$. C has a vertical tangent when $dx/dt = 2t = 0$, that is, $t = 0$. (Note that $dy/dt \neq 0$ there.) The corresponding point on C is (0, 0).

(c) To determine concavity we calculate the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{\dfrac{dx}{dt}} = \frac{\dfrac{3}{2}\left(1 + \dfrac{1}{t^2}\right)}{2t} = \frac{3(t^2 + 1)}{4t^3}$$

Thus the curve is concave upward when $t > 0$ and concave downward when $t < 0$.

(d) Using the information from parts (b) and (c), we sketch C in Figure 1.

V EXAMPLE 2
(a) Find the tangent to the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$ at the point where $\theta = \pi/3$. (See Example 7 in Section 10.1.)
(b) At what points is the tangent horizontal? When is it vertical?

SOLUTION
(a) The slope of the tangent line is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\sin\theta}{r(1 - \cos\theta)} = \frac{\sin\theta}{1 - \cos\theta}$$

When $\theta = \pi/3$, we have

$$x = r\left(\frac{\pi}{3} - \sin\frac{\pi}{3}\right) = r\left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) \qquad y = r\left(1 - \cos\frac{\pi}{3}\right) = \frac{r}{2}$$

and

$$\frac{dy}{dx} = \frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1 - \frac{1}{2}} = \sqrt{3}$$

Therefore the slope of the tangent is $\sqrt{3}$ and its equation is

$$y - \frac{r}{2} = \sqrt{3}\left(x - \frac{r\pi}{3} + \frac{r\sqrt{3}}{2}\right) \qquad \text{or} \qquad \sqrt{3}\,x - y = r\left(\frac{\pi}{\sqrt{3}} - 2\right)$$

The tangent is sketched in Figure 2.

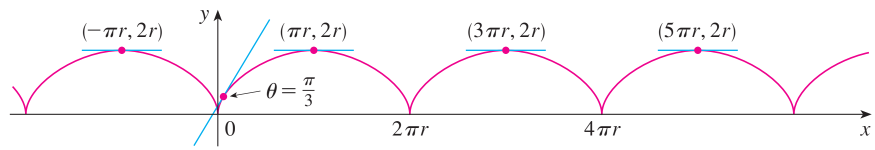


FIGURE 2

(b) The tangent is horizontal when $dy/dx = 0$, which occurs when $\sin\theta = 0$ and $1 - \cos\theta \neq 0$, that is, $\theta = (2n - 1)\pi$, n an integer. The corresponding point on the cycloid is $((2n - 1)\pi r, 2r)$.

When $\theta = 2n\pi$, both $dx/d\theta$ and $dy/d\theta$ are 0. It appears from the graph that there are vertical tangents at those points. We can verify this by using l'Hospital's Rule as follows:

$$\lim_{\theta\to 2n\pi^+} \frac{dy}{dx} = \lim_{\theta\to 2n\pi^+} \frac{\sin\theta}{1 - \cos\theta} = \lim_{\theta\to 2n\pi^+} \frac{\cos\theta}{\sin\theta} = \infty$$

A similar computation shows that $dy/dx \to -\infty$ as $\theta \to 2n\pi^-$, so indeed there are vertical tangents when $\theta = 2n\pi$, that is, when $x = 2n\pi r$.

Areas

We know that the area under a curve $y = F(x)$ from a to b is $A = \int_a^b F(x)\,dx$, where $F(x) \geqslant 0$. If the curve is traced out once by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, then we can calculate an area formula by using the Substitution Rule for Definite Integrals as follows:

The limits of integration for t are found as usual with the Substitution Rule. When $x = a$, t is either α or β. When $x = b$, t is the remaining value.

$$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)\,f'(t)\,dt \qquad \left[\text{or} \quad \int_\beta^\alpha g(t)\,f'(t)\,dt\right]$$

V EXAMPLE 3 Find the area under one arch of the cycloid

$$x = r(\theta - \sin\theta) \qquad y = r(1 - \cos\theta)$$

(See Figure 3.)

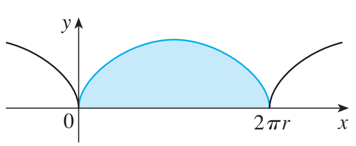


FIGURE 3

SOLUTION One arch of the cycloid is given by $0 \leqslant \theta \leqslant 2\pi$. Using the Substitution Rule with $y = r(1 - \cos\theta)$ and $dx = r(1 - \cos\theta)\,d\theta$, we have

$$\begin{aligned} A &= \int_0^{2\pi r} y\,dx = \int_0^{2\pi} r(1 - \cos\theta)\,r(1 - \cos\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} (1 - \cos\theta)^2\,d\theta = r^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta)\,d\theta \\ &= r^2 \int_0^{2\pi} \left[1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\right] d\theta \\ &= r^2\left[\tfrac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{2\pi} \\ &= r^2\left(\tfrac{3}{2}\cdot 2\pi\right) = 3\pi r^2 \end{aligned}$$

The result of Example 3 says that the area under one arch of the cycloid is three times the area of the rolling circle that generates the cycloid (see Example 7 in Section 10.1). Galileo guessed this result but it was first proved by the French mathematician Roberval and the Italian mathematician Torricelli.

Arc Length

We already know how to find the length L of a curve C given in the form $y = F(x)$, $a \leqslant x \leqslant b$. Formula 8.1.3 says that if F' is continuous, then

2
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that C can also be described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β and $f(\alpha) = a$, $f(\beta) = b$. Putting Formula 1 into Formula 2 and using the Substitution Rule, we obtain

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt}\, dt$$

Since $dx/dt > 0$, we have

3
$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

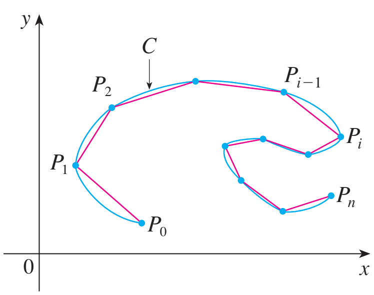


FIGURE 4

Even if C can't be expressed in the form $y = F(x)$, Formula 3 is still valid but we obtain it by polygonal approximations. We divide the parameter interval $[\alpha, \beta]$ into n subintervals of equal width Δt. If $t_0, t_1, t_2, \ldots, t_n$ are the endpoints of these subintervals, then $x_i = f(t_i)$ and $y_i = g(t_i)$ are the coordinates of points $P_i(x_i, y_i)$ that lie on C and the polygon with vertices $P_0, P_1, \ldots, P_n$ approximates C. (See Figure 4.)

As in Section 8.1, we define the length L of C to be the limit of the lengths of these approximating polygons as $n \to \infty$:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

The Mean Value Theorem, when applied to f on the interval $[t_{i-1}, t_i]$, gives a number t_i^* in (t_{i-1}, t_i) such that

$$f(t_i) - f(t_{i-1}) = f'(t_i^*)(t_i - t_{i-1})$$

If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_i = y_i - y_{i-1}$, this equation becomes

$$\Delta x_i = f'(t_i^*)\, \Delta t$$

Similarly, when applied to g, the Mean Value Theorem gives a number t_i^{**} in (t_{i-1}, t_i) such that

$$\Delta y_i = g'(t_i^{**})\, \Delta t$$

Therefore

$$\begin{aligned} |P_{i-1}P_i| &= \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{[f'(t_i^*)\Delta t]^2 + [g'(t_i^{**})\Delta t]^2} \\ &= \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t \end{aligned}$$

and so

4
$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{[f'(t_i^*)]^2 + [g'(t_i^{**})]^2}\, \Delta t$$

The sum in $\boxed{4}$ resembles a Riemann sum for the function $\sqrt{[f'(t)]^2 + [g'(t)]^2}$ but it is not exactly a Riemann sum because $t_i^* \neq t_i^{**}$ in general. Nevertheless, if f' and g' are continuous, it can be shown that the limit in $\boxed{4}$ is the same as if t_i^* and t_i^{**} were equal, namely,

$$L = \int_\alpha^\beta \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

Thus, using Leibniz notation, we have the following result, which has the same form as Formula 3.

$\boxed{5}$ **Theorem** If a curve C is described by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Notice that the formula in Theorem 5 is consistent with the general formulas $L = \int ds$ and $(ds)^2 = (dx)^2 + (dy)^2$ of Section 8.1.

EXAMPLE 4 If we use the representation of the unit circle given in Example 2 in Section 10.1,

$$x = \cos t \qquad y = \sin t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = -\sin t$ and $dy/dt = \cos t$, so Theorem 5 gives

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\, dt = \int_0^{2\pi} dt = 2\pi$$

as expected. If, on the other hand, we use the representation given in Example 3 in Section 10.1,

$$x = \sin 2t \qquad y = \cos 2t \qquad 0 \leqslant t \leqslant 2\pi$$

then $dx/dt = 2\cos 2t$, $dy/dt = -2\sin 2t$, and the integral in Theorem 5 gives

$$\int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^{2\pi} \sqrt{4\cos^2 2t + 4\sin^2 2t}\, dt = \int_0^{2\pi} 2\, dt = 4\pi$$

⊘ Notice that the integral gives twice the arc length of the circle because as t increases from 0 to 2π, the point $(\sin 2t, \cos 2t)$ traverses the circle twice. In general, when finding the length of a curve C from a parametric representation, we have to be careful to ensure that C is traversed only once as t increases from α to β.

V EXAMPLE 5 Find the length of one arch of the cycloid $x = r(\theta - \sin\theta)$, $y = r(1 - \cos\theta)$.

SOLUTION From Example 3 we see that one arch is described by the parameter interval $0 \leqslant \theta \leqslant 2\pi$. Since

$$\frac{dx}{d\theta} = r(1 - \cos\theta) \qquad \text{and} \qquad \frac{dy}{d\theta} = r\sin\theta$$

we have

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - \cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(1 - 2\cos\theta + \cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= r\int_0^{2\pi} \sqrt{2(1 - \cos\theta)}\, d\theta$$

The result of Example 5 says that the length of one arch of a cycloid is eight times the radius of the generating circle (see Figure 5). This was first proved in 1658 by Sir Christopher Wren, who later became the architect of St. Paul's Cathedral in London.

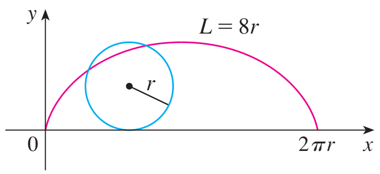


FIGURE 5

To evaluate this integral we use the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ with $\theta = 2x$, which gives $1 - \cos\theta = 2\sin^2(\theta/2)$. Since $0 \leqslant \theta \leqslant 2\pi$, we have $0 \leqslant \theta/2 \leqslant \pi$ and so $\sin(\theta/2) \geqslant 0$. Therefore

$$\sqrt{2(1 - \cos\theta)} = \sqrt{4\sin^2(\theta/2)} = 2\left|\sin(\theta/2)\right| = 2\sin(\theta/2)$$

and so

$$L = 2r\int_0^{2\pi} \sin(\theta/2)\, d\theta = 2r\Big[-2\cos(\theta/2)\Big]_0^{2\pi}$$

$$= 2r[2 + 2] = 8r$$

Surface Area

In the same way as for arc length, we can adapt Formula 8.2.5 to obtain a formula for surface area. If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leqslant t \leqslant \beta$, is rotated about the x-axis, where f', g' are continuous and $g(t) \geqslant 0$, then the area of the resulting surface is given by

6
$$S = \int_\alpha^\beta 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The general symbolic formulas $S = \int 2\pi y\, ds$ and $S = \int 2\pi x\, ds$ (Formulas 8.2.7 and 8.2.8) are still valid, but for parametric curves we use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

EXAMPLE 6 Show that the surface area of a sphere of radius r is $4\pi r^2$.

SOLUTION The sphere is obtained by rotating the semicircle

$$x = r\cos t \qquad y = r\sin t \qquad 0 \leqslant t \leqslant \pi$$

about the x-axis. Therefore, from Formula 6, we get

$$S = \int_0^\pi 2\pi r\sin t\sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt$$

$$= 2\pi\int_0^\pi r\sin t\sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt = 2\pi\int_0^\pi r\sin t \cdot r\, dt$$

$$= 2\pi r^2\int_0^\pi \sin t\, dt = 2\pi r^2(-\cos t)\Big]_0^\pi = 4\pi r^2$$

10.2 Exercises

1–2 Find dy/dx.

1. $x = t\sin t, \quad y = t^2 + t$

2. $x = 1/t, \quad y = \sqrt{t}\,e^{-t}$

3–6 Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

3. $x = 1 + 4t - t^2, \quad y = 2 - t^3; \quad t = 1$

4. $x = t - t^{-1}, \quad y = 1 + t^2; \quad t = 1$

5. $x = t\cos t, \quad y = t\sin t; \quad t = \pi$

6. $x = \sin^3\theta, \quad y = \cos^3\theta; \quad \theta = \pi/6$

7–8 Find an equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

7. $x = 1 + \ln t, \quad y = t^2 + 2; \quad (1, 3)$

8. $x = 1 + \sqrt{t}, \quad y = e^{t^2}; \quad (2, e)$

9–10 Find an equation of the tangent(s) to the curve at the given point. Then graph the curve and the tangent(s).

9. $x = 6\sin t, \quad y = t^2 + t; \quad (0, 0)$

10. $x = \cos t + \cos 2t, \quad y = \sin t + \sin 2t; \quad (-1, 1)$

11–16 Find dy/dx and d^2y/dx^2. For which values of t is the curve concave upward?

11. $x = t^2 + 1, \quad y = t^2 + t$

12. $x = t^3 + 1, \quad y = t^2 - t$

13. $x = e^t, \quad y = te^{-t}$

14. $x = t^2 + 1, \quad y = e^t - 1$

15. $x = 2\sin t, \quad y = 3\cos t, \quad 0 < t < 2\pi$

16. $x = \cos 2t, \quad y = \cos t, \quad 0 < t < \pi$

17–20 Find the points on the curve where the tangent is horizontal or vertical. If you have a graphing device, graph the curve to check your work.

17. $x = t^3 - 3t, \quad y = t^2 - 3$

18. $x = t^3 - 3t, \quad y = t^3 - 3t^2$

19. $x = \cos\theta, \quad y = \cos 3\theta$

20. $x = e^{\sin\theta}, \quad y = e^{\cos\theta}$

21. Use a graph to estimate the coordinates of the rightmost point on the curve $x = t - t^6$, $y = e^t$. Then use calculus to find the exact coordinates.

22. Use a graph to estimate the coordinates of the lowest point and the leftmost point on the curve $x = t^4 - 2t$, $y = t + t^4$. Then find the exact coordinates.

23–24 Graph the curve in a viewing rectangle that displays all the important aspects of the curve.

23. $x = t^4 - 2t^3 - 2t^2, \quad y = t^3 - t$

24. $x = t^4 + 4t^3 - 8t^2, \quad y = 2t^2 - t$

25. Show that the curve $x = \cos t$, $y = \sin t\cos t$ has two tangents at $(0, 0)$ and find their equations. Sketch the curve.

26. Graph the curve $x = \cos t + 2\cos 2t$, $y = \sin t + 2\sin 2t$ to discover where it crosses itself. Then find equations of both tangents at that point.

27. (a) Find the slope of the tangent line to the trochoid $x = r\theta - d\sin\theta$, $y = r - d\cos\theta$ in terms of θ. (See Exercise 40 in Section 10.1.)
(b) Show that if $d < r$, then the trochoid does not have a vertical tangent.

28. (a) Find the slope of the tangent to the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$ in terms of θ. (Astroids are explored in the Laboratory Project on page 668.)
(b) At what points is the tangent horizontal or vertical?
(c) At what points does the tangent have slope 1 or -1?

29. At what points on the curve $x = 2t^3$, $y = 1 + 4t - t^2$ does the tangent line have slope 1?

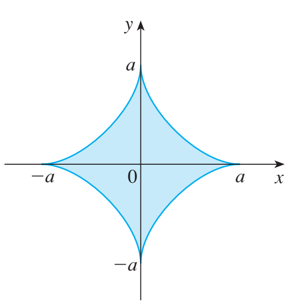
30. Find equations of the tangents to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ that pass through the point $(4, 3)$.

31. Use the parametric equations of an ellipse, $x = a\cos\theta$, $y = b\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$, to find the area that it encloses.

32. Find the area enclosed by the curve $x = t^2 - 2t$, $y = \sqrt{t}$ and the y-axis.

33. Find the area enclosed by the x-axis and the curve $x = 1 + e^t$, $y = t - t^2$.

34. Find the area of the region enclosed by the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$. (Astroids are explored in the Laboratory Project on page 668.)



35. Find the area under one arch of the trochoid of Exercise 40 in Section 10.1 for the case $d < r$.

36. Let $\mathcal{R}$ be the region enclosed by the loop of the curve in Example 1.
(a) Find the area of $\mathcal{R}$.
(b) If $\mathcal{R}$ is rotated about the x-axis, find the volume of the resulting solid.
(c) Find the centroid of $\mathcal{R}$.

37–40 Set up an integral that represents the length of the curve. Then use your calculator to find the length correct to four decimal places.

37. $x = t + e^{-t}, \quad y = t - e^{-t}, \quad 0 \leqslant t \leqslant 2$

38. $x = t^2 - t, \quad y = t^4, \quad 1 \leqslant t \leqslant 4$

39. $x = t - 2\sin t, \quad y = 1 - 2\cos t, \quad 0 \leqslant t \leqslant 4\pi$

40. $x = t + \sqrt{t}, \quad y = t - \sqrt{t}, \quad 0 \leqslant t \leqslant 1$

41–44 Find the exact length of the curve.

41. $x = 1 + 3t^2, \quad y = 4 + 2t^3, \quad 0 \leqslant t \leqslant 1$

42. $x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \leqslant t \leqslant 3$

43. $x = t\sin t, \quad y = t\cos t, \quad 0 \leqslant t \leqslant 1$

44. $x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t, \quad 0 \leqslant t \leqslant \pi$

45–46 Graph the curve and find its length.

45. $x = e^t\cos t, \quad y = e^t\sin t, \quad 0 \leqslant t \leqslant \pi$

46. $x = \cos t + \ln(\tan\frac{1}{2}t), \quad y = \sin t, \quad \pi/4 \leqslant t \leqslant 3\pi/4$

47. Graph the curve $x = \sin t + \sin 1.5t$, $y = \cos t$ and find its length correct to four decimal places.

48. Find the length of the loop of the curve $x = 3t - t^3$, $y = 3t^2$.

49. Use Simpson's Rule with $n = 6$ to estimate the length of the curve $x = t - e^t$, $y = t + e^t$, $-6 \leqslant t \leqslant 6$.

50. In Exercise 43 in Section 10.1 you were asked to derive the parametric equations $x = 2a\cot\theta$, $y = 2a\sin^2\theta$ for the curve called the witch of Maria Agnesi. Use Simpson's Rule with $n = 4$ to estimate the length of the arc of this curve given by $\pi/4 \leqslant \theta \leqslant \pi/2$.

51–52 Find the distance traveled by a particle with position (x, y) as t varies in the given time interval. Compare with the length of the curve.

51. $x = \sin^2 t, \quad y = \cos^2 t, \quad 0 \leqslant t \leqslant 3\pi$

52. $x = \cos^2 t, \quad y = \cos t, \quad 0 \leqslant t \leqslant 4\pi$

53. Show that the total length of the ellipse $x = a\sin\theta$, $y = b\cos\theta$, $a > b > 0$, is

$$L = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta$$

where e is the eccentricity of the ellipse $\left(e = c/a\text{, where } c = \sqrt{a^2 - b^2}\right)$.

54. Find the total length of the astroid $x = a\cos^3\theta$, $y = a\sin^3\theta$, where $a > 0$.

CAS **55.** (a) Graph the **epitrochoid** with equations

$$x = 11\cos t - 4\cos(11t/2)$$
$$y = 11\sin t - 4\sin(11t/2)$$

What parameter interval gives the complete curve?
(b) Use your CAS to find the approximate length of this curve.

CAS **56.** A curve called **Cornu's spiral** is defined by the parametric equations

$$x = C(t) = \int_0^t \cos(\pi u^2/2)\,du$$

$$y = S(t) = \int_0^t \sin(\pi u^2/2)\,du$$

where C and S are the Fresnel functions that were introduced in Chapter 4.
(a) Graph this curve. What happens as $t \to \infty$ and as $t \to -\infty$?
(b) Find the length of Cornu's spiral from the origin to the point with parameter value t.

57–60 Set up an integral that represents the area of the surface obtained by rotating the given curve about the x-axis. Then use your calculator to find the surface area correct to four decimal places.

57. $x = t\sin t, \quad y = t\cos t, \quad 0 \leqslant t \leqslant \pi/2$

58. $x = \sin t, \quad y = \sin 2t, \quad 0 \leqslant t \leqslant \pi/2$

59. $x = 1 + te^t, \quad y = (t^2 + 1)e^t, \quad 0 \leqslant t \leqslant 1$

60. $x = t^2 - t^3, \quad y = t + t^4, \quad 0 \leqslant t \leqslant 1$

61–63 Find the exact area of the surface obtained by rotating the given curve about the x-axis.

61. $x = t^3, \quad y = t^2, \quad 0 \leqslant t \leqslant 1$

62. $x = 3t - t^3, \quad y = 3t^2, \quad 0 \leqslant t \leqslant 1$

63. $x = a\cos^3\theta, \quad y = a\sin^3\theta, \quad 0 \leqslant \theta \leqslant \pi/2$

64. Graph the curve

$$x = 2\cos\theta - \cos 2\theta \qquad y = 2\sin\theta - \sin 2\theta$$

If this curve is rotated about the x-axis, find the area of the resulting surface. (Use your graph to help find the correct parameter interval.)

65–66 Find the surface area generated by rotating the given curve about the y-axis.

65. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 5$

66. $x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \leqslant t \leqslant 1$

---

67. If f' is continuous and $f'(t) \neq 0$ for $a \leqslant t \leqslant b$, show that the parametric curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, can be put in the form $y = F(x)$. [*Hint:* Show that f^{-1} exists.]

68. Use Formula 2 to derive Formula 7 from Formula 8.2.5 for the case in which the curve can be represented in the form $y = F(x)$, $a \leqslant x \leqslant b$.

69. The **curvature** at a point P of a curve is defined as

$$\kappa = \left| \frac{d\phi}{ds} \right|$$

where ϕ is the angle of inclination of the tangent line at P, as shown in the figure. Thus the curvature is the absolute value of the rate of change of ϕ with respect to arc length. It can be regarded as a measure of the rate of change of direction of the curve at P and will be studied in greater detail in Chapter 13.

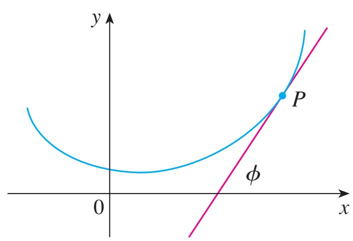
(a) For a parametric curve $x = x(t)$, $y = y(t)$, derive the formula

$$\kappa = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{[\dot{x}^2 + \dot{y}^2]^{3/2}}$$

where the dots indicate derivatives with respect to t, so $\dot{x} = dx/dt$. [*Hint:* Use $\phi = \tan^{-1}(dy/dx)$ and Formula 2 to find $d\phi/dt$. Then use the Chain Rule to find $d\phi/ds$.]

(b) By regarding a curve $y = f(x)$ as the parametric curve $x = x$, $y = f(x)$, with parameter x, show that the formula in part (a) becomes

$$\kappa = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}$$



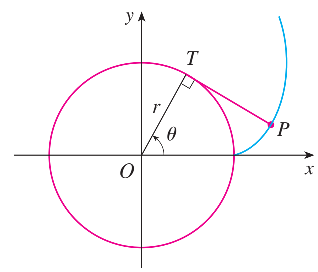
70. (a) Use the formula in Exercise 69(b) to find the curvature of the parabola $y = x^2$ at the point $(1, 1)$.
(b) At what point does this parabola have maximum curvature?

71. Use the formula in Exercise 69(a) to find the curvature of the cycloid $x = \theta - \sin\theta$, $y = 1 - \cos\theta$ at the top of one of its arches.

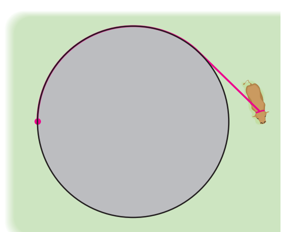
72. (a) Show that the curvature at each point of a straight line is $\kappa = 0$.
(b) Show that the curvature at each point of a circle of radius r is $\kappa = 1/r$.

73. A string is wound around a circle and then unwound while being held taut. The curve traced by the point P at the end of the string is called the **involute** of the circle. If the circle has radius r and center O and the initial position of P is $(r, 0)$, and if the parameter θ is chosen as in the figure, show that parametric equations of the involute are

$$x = r(\cos\theta + \theta\sin\theta) \qquad y = r(\sin\theta - \theta\cos\theta)$$



74. A cow is tied to a silo with radius r by a rope just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.



LABORATORY PROJECT — BÉZIER CURVES

Bézier curves are used in computer-aided design and are named after the French mathematician Pierre Bézier (1910–1999), who worked in the automotive industry. A cubic Bézier curve is determined by four *control points*, $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, and $P_3(x_3, y_3)$, and is defined by the parametric equations

$$x = x_0(1-t)^3 + 3x_1t(1-t)^2 + 3x_2t^2(1-t) + x_3t^3$$

$$y = y_0(1-t)^3 + 3y_1t(1-t)^2 + 3y_2t^2(1-t) + y_3t^3$$

Graphing calculator or computer required

where $0 \leqslant t \leqslant 1$. Notice that when $t = 0$ we have $(x, y) = (x_0, y_0)$ and when $t = 1$ we have $(x, y) = (x_3, y_3)$, so the curve starts at P_0 and ends at P_3.

1. Graph the Bézier curve with control points $P_0(4, 1)$, $P_1(28, 48)$, $P_2(50, 42)$, and $P_3(40, 5)$. Then, on the same screen, graph the line segments P_0P_1, P_1P_2, and P_2P_3. (Exercise 31 in Section 10.1 shows how to do this.) Notice that the middle control points P_1 and P_2 don't lie on the curve; the curve starts at P_0, heads toward P_1 and P_2 without reaching them, and ends at P_3.

2. From the graph in Problem 1, it appears that the tangent at P_0 passes through P_1 and the tangent at P_3 passes through P_2. Prove it.

3. Try to produce a Bézier curve with a loop by changing the second control point in Problem 1.

4. Some laser printers use Bézier curves to represent letters and other symbols. Experiment with control points until you find a Bézier curve that gives a reasonable representation of the letter C.

5. More complicated shapes can be represented by piecing together two or more Bézier curves. Suppose the first Bézier curve has control points P_0, P_1, P_2, P_3 and the second one has control points P_3, P_4, P_5, P_6. If we want these two pieces to join together smoothly, then the tangents at P_3 should match and so the points P_2, P_3, and P_4 all have to lie on this common tangent line. Using this principle, find control points for a pair of Bézier curves that represent the letter S.

10.3 Polar Coordinates

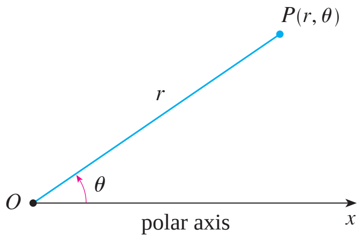


FIGURE 1

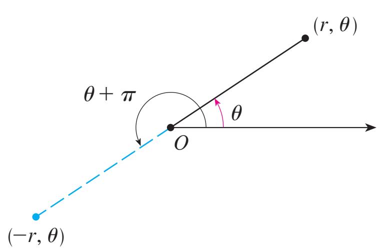


FIGURE 2

A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates. Usually we use Cartesian coordinates, which are directed distances from two perpendicular axes. Here we describe a coordinate system introduced by Newton, called the **polar coordinate system**, which is more convenient for many purposes.

We choose a point in the plane that is called the **pole** (or origin) and is labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drawn horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in Figure 1. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = O$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.

We extend the meaning of polar coordinates (r, θ) to the case in which r is negative by agreeing that, as in Figure 2, the points $(-r, \theta)$ and (r, θ) lie on the same line through O and at the same distance $|r|$ from O, but on opposite sides of O. If $r > 0$, the point (r, θ) lies in the same quadrant as θ; if $r < 0$, it lies in the quadrant on the opposite side of the pole. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.

EXAMPLE 1 Plot the points whose polar coordinates are given.
(a) $(1, 5\pi/4)$ (b) $(2, 3\pi)$ (c) $(2, -2\pi/3)$ (d) $(-3, 3\pi/4)$

SOLUTION The points are plotted in Figure 3. In part (d) the point $(-3, 3\pi/4)$ is located three units from the pole in the fourth quadrant because the angle $3\pi/4$ is in the second quadrant and $r = -3$ is negative.

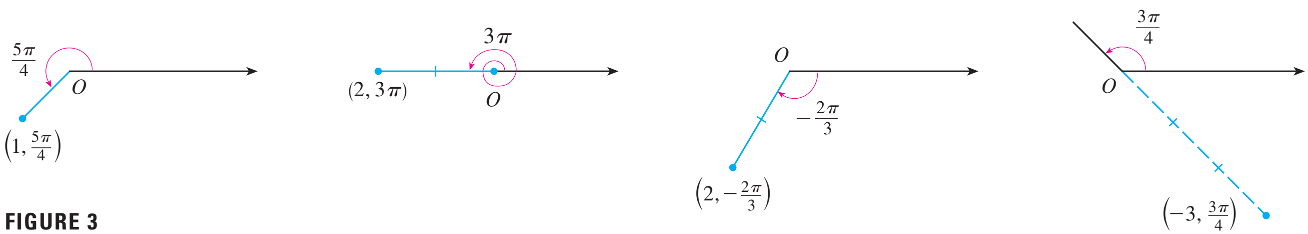


FIGURE 3

In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. For instance, the point $(1, 5\pi/4)$ in Example 1(a) could be written as $(1, -3\pi/4)$ or $(1, 13\pi/4)$ or $(-1, \pi/4)$. (See Figure 4.)

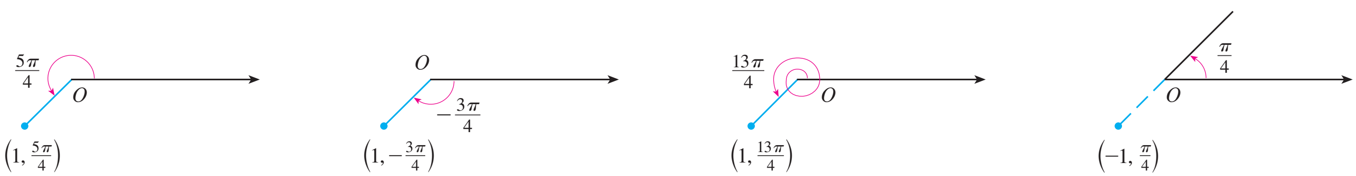


FIGURE 4

In fact, since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2n\pi) \qquad \text{and} \qquad (-r, \theta + (2n + 1)\pi)$$

where n is any integer.

The connection between polar and Cartesian coordinates can be seen from Figure 5, in which the pole corresponds to the origin and the polar axis coincides with the positive x-axis. If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure, we have

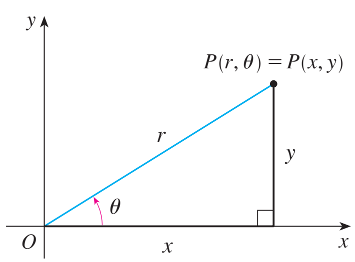


FIGURE 5

$$\cos \theta = \frac{x}{r} \qquad \sin \theta = \frac{y}{r}$$

and so

1 $$\boxed{x = r \cos \theta \qquad y = r \sin \theta}$$

Although Equations 1 were deduced from Figure 5, which illustrates the case where $r > 0$ and $0 < \theta < \pi/2$, these equations are valid for all values of r and θ. (See the general definition of $\sin \theta$ and $\cos \theta$ in Appendix D.)

Equations 1 allow us to find the Cartesian coordinates of a point when the polar coordinates are known. To find r and θ when x and y are known, we use the equations

2
$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}}$$

which can be deduced from Equations 1 or simply read from Figure 5.

EXAMPLE 2 Convert the point $(2, \pi/3)$ from polar to Cartesian coordinates.

SOLUTION Since $r = 2$ and $\theta = \pi/3$, Equations 1 give

$$x = r\cos\theta = 2\cos\frac{\pi}{3} = 2 \cdot \frac{1}{2} = 1$$

$$y = r\sin\theta = 2\sin\frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore the point is $\left(1, \sqrt{3}\right)$ in Cartesian coordinates.

EXAMPLE 3 Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

SOLUTION If we choose r to be positive, then Equations 2 give

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\pi/4$ or $\theta = 7\pi/4$. Thus one possible answer is $\left(\sqrt{2}, -\pi/4\right)$; another is $\left(\sqrt{2}, 7\pi/4\right)$.

NOTE Equations 2 do not uniquely determine θ when x and y are given because, as θ increases through the interval $0 \leqslant \theta < 2\pi$, each value of $\tan\theta$ occurs twice. Therefore, in converting from Cartesian to polar coordinates, it's not good enough just to find r and θ that satisfy Equations 2. As in Example 3, we must choose θ so that the point (r, θ) lies in the correct quadrant.

Polar Curves

The **graph of a polar equation** $r = f(\theta)$, or more generally $F(r, \theta) = 0$, consists of all points P that have at least one polar representation (r, θ) whose coordinates satisfy the equation.

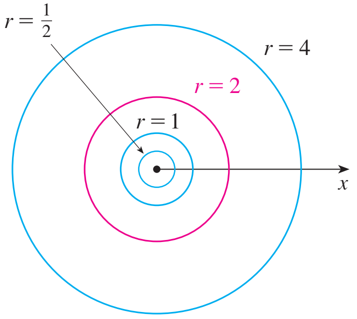


FIGURE 6

V EXAMPLE 4 What curve is represented by the polar equation $r = 2$?

SOLUTION The curve consists of all points (r, θ) with $r = 2$. Since r represents the distance from the point to the pole, the curve $r = 2$ represents the circle with center O and radius 2. In general, the equation $r = a$ represents a circle with center O and radius $|a|$. (See Figure 6.)

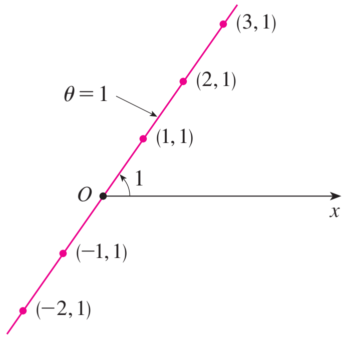


FIGURE 7

EXAMPLE 5 Sketch the polar curve $\theta = 1$.

SOLUTION This curve consists of all points (r, θ) such that the polar angle θ is 1 radian. It is the straight line that passes through O and makes an angle of 1 radian with the polar axis (see Figure 7). Notice that the points $(r, 1)$ on the line with $r > 0$ are in the first quadrant, whereas those with $r < 0$ are in the third quadrant.

EXAMPLE 6

(a) Sketch the curve with polar equation $r = 2\cos\theta$.
(b) Find a Cartesian equation for this curve.

SOLUTION

(a) In Figure 8 we find the values of r for some convenient values of θ and plot the corresponding points (r, θ). Then we join these points to sketch the curve, which appears to be a circle. We have used only values of θ between 0 and π, since if we let θ increase beyond π, we obtain the same points again.

θ	$r = 2\cos\theta$
0	2
$\pi/6$	$\sqrt{3}$
$\pi/4$	$\sqrt{2}$
$\pi/3$	1
$\pi/2$	0
$2\pi/3$	-1
$3\pi/4$	$-\sqrt{2}$
$5\pi/6$	$-\sqrt{3}$
π	-2

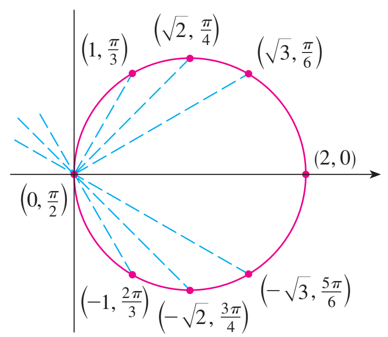


FIGURE 8
Table of values and graph of $r = 2\cos\theta$

(b) To convert the given equation to a Cartesian equation we use Equations 1 and 2. From $x = r\cos\theta$ we have $\cos\theta = x/r$, so the equation $r = 2\cos\theta$ becomes $r = 2x/r$, which gives

$$2x = r^2 = x^2 + y^2 \qquad \text{or} \qquad x^2 + y^2 - 2x = 0$$

Completing the square, we obtain

$$(x - 1)^2 + y^2 = 1$$

which is an equation of a circle with center $(1, 0)$ and radius 1.

Figure 9 shows a geometrical illustration that the circle in Example 6 has the equation $r = 2\cos\theta$. The angle OPQ is a right angle (Why?) and so $r/2 = \cos\theta$.

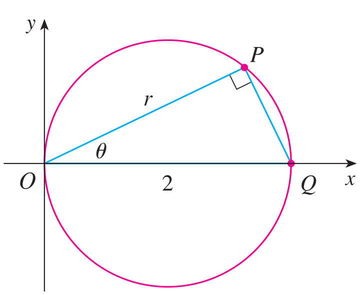


FIGURE 9

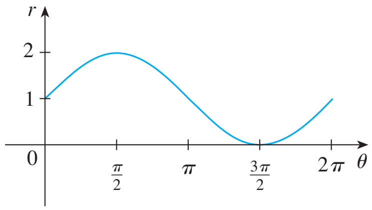


FIGURE 10
$r = 1 + \sin\theta$ in Cartesian coordinates, $0 \leqslant \theta \leqslant 2\pi$

V EXAMPLE 7 Sketch the curve $r = 1 + \sin\theta$.

SOLUTION Instead of plotting points as in Example 6, we first sketch the graph of $r = 1 + \sin\theta$ in *Cartesian* coordinates in Figure 10 by shifting the sine curve up one unit. This enables us to read at a glance the values of r that correspond to increasing values of θ. For instance, we see that as θ increases from 0 to $\pi/2$, r (the distance from O) increases from 1 to 2, so we sketch the corresponding part of the polar curve in Figure 11(a). As θ increases from $\pi/2$ to π, Figure 10 shows that r decreases from 2 to 1, so we sketch the next part of the curve as in Figure 11(b). As θ increases from π to $3\pi/2$, r decreases from 1 to 0 as shown in part (c). Finally, as θ increases from $3\pi/2$ to 2π, r increases from 0 to 1 as shown in part (d). If we let θ increase beyond 2π or decrease beyond 0, we would simply retrace our path. Putting together the parts of the curve from Figure 11(a)–(d), we sketch the complete curve in part (e). It is called a **cardioid** because it's shaped like a heart.

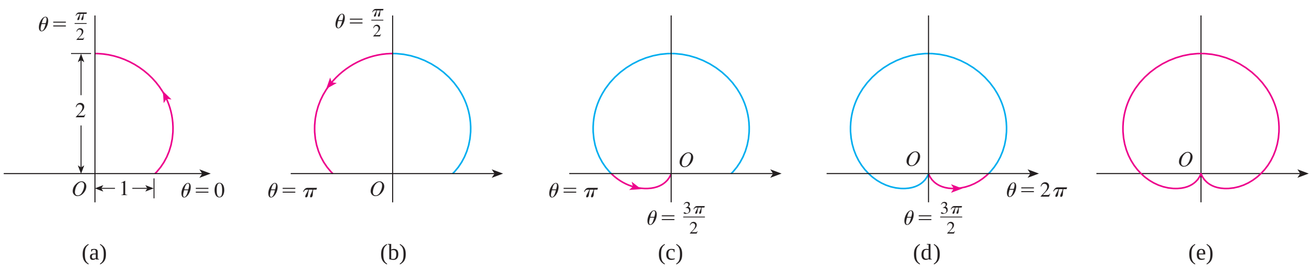


FIGURE 11 Stages in sketching the cardioid $r = 1 + \sin\theta$

EXAMPLE 8 Sketch the curve $r = \cos 2\theta$.

SOLUTION As in Example 7, we first sketch $r = \cos 2\theta$, $0 \leqslant \theta \leqslant 2\pi$, in Cartesian coordinates in Figure 12. As θ increases from 0 to $\pi/4$, Figure 12 shows that r decreases from 1 to 0 and so we draw the corresponding portion of the polar curve in Figure 13 (indicated by ①). As θ increases from $\pi/4$ to $\pi/2$, r goes from 0 to -1. This means that the distance from O increases from 0 to 1, but instead of being in the first quadrant this portion of the polar curve (indicated by ②) lies on the opposite side of the pole in the third quadrant. The remainder of the curve is drawn in a similar fashion, with the arrows and numbers indicating the order in which the portions are traced out. The resulting curve has four loops and is called a **four-leaved rose**.

TEC Module 10.3 helps you see how polar curves are traced out by showing animations similar to Figures 10–13.

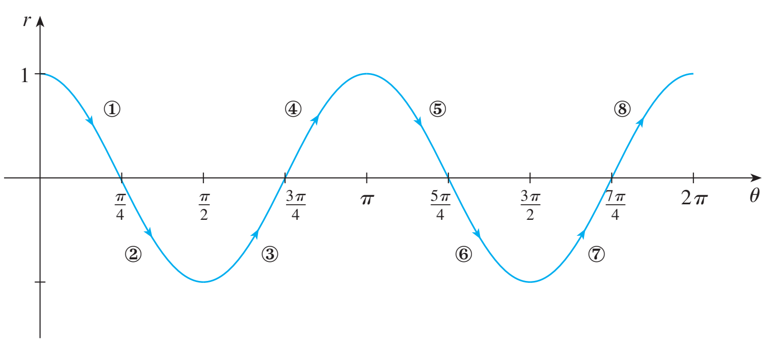


FIGURE 12
$r = \cos 2\theta$ in Cartesian coordinates

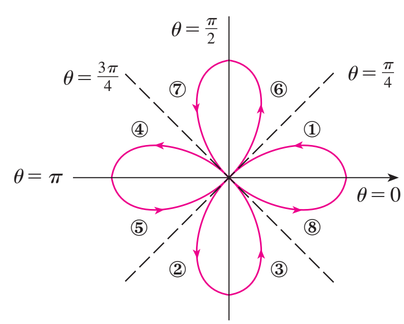


FIGURE 13
Four-leaved rose $r = \cos 2\theta$

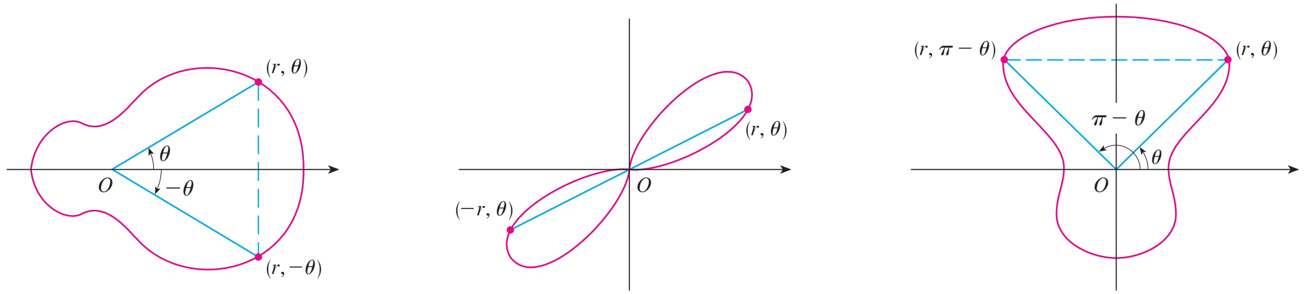
Symmetry

When we sketch polar curves it is sometimes helpful to take advantage of symmetry. The following three rules are explained by Figure 14.

(a) If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis.

(b) If the equation is unchanged when r is replaced by $-r$, or when θ is replaced by $\theta + \pi$, the curve is symmetric about the pole. (This means that the curve remains unchanged if we rotate it through 180° about the origin.)

(c) If the equation is unchanged when θ is replaced by $\pi - \theta$, the curve is symmetric about the vertical line $\theta = \pi/2$.



(a) (b) (c)

FIGURE 14

The curves sketched in Examples 6 and 8 are symmetric about the polar axis, since $\cos(-\theta) = \cos\theta$. The curves in Examples 7 and 8 are symmetric about $\theta = \pi/2$ because $\sin(\pi - \theta) = \sin\theta$ and $\cos 2(\pi - \theta) = \cos 2\theta$. The four-leaved rose is also symmetric about the pole. These symmetry properties could have been used in sketching the curves. For instance, in Example 6 we need only have plotted points for $0 \leqslant \theta \leqslant \pi/2$ and then reflected about the polar axis to obtain the complete circle.

Tangents to Polar Curves

To find a tangent line to a polar curve $r = f(\theta)$, we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Then, using the method for finding slopes of parametric curves (Equation 10.2.1) and the Product Rule, we have

$$\boxed{3} \qquad \frac{dy}{dx} = \frac{\dfrac{dy}{d\theta}}{\dfrac{dx}{d\theta}} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We locate horizontal tangents by finding the points where $dy/d\theta = 0$ (provided that $dx/d\theta \neq 0$). Likewise, we locate vertical tangents at the points where $dx/d\theta = 0$ (provided that $dy/d\theta \neq 0$).

Notice that if we are looking for tangent lines at the pole, then $r = 0$ and Equation 3 simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \quad \frac{dr}{d\theta} \neq 0$$

For instance, in Example 8 we found that $r = \cos 2\theta = 0$ when $\theta = \pi/4$ or $3\pi/4$. This means that the lines $\theta = \pi/4$ and $\theta = 3\pi/4$ (or $y = x$ and $y = -x$) are tangent lines to $r = \cos 2\theta$ at the origin.

EXAMPLE 9

(a) For the cardioid $r = 1 + \sin\theta$ of Example 7, find the slope of the tangent line when $\theta = \pi/3$.

(b) Find the points on the cardioid where the tangent line is horizontal or vertical.

SOLUTION Using Equation 3 with $r = 1 + \sin\theta$, we have

$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\,\sin\theta + (1 + \sin\theta)\cos\theta}{\cos\theta\,\cos\theta - (1 + \sin\theta)\sin\theta}$$

$$= \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{(1 + \sin\theta)(1 - 2\sin\theta)}$$

(a) The slope of the tangent at the point where $\theta = \pi/3$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/3} = \frac{\cos(\pi/3)(1 + 2\sin(\pi/3))}{(1 + \sin(\pi/3))(1 - 2\sin(\pi/3))} = \frac{\frac{1}{2}\left(1 + \sqrt{3}\right)}{\left(1 + \sqrt{3}/2\right)\left(1 - \sqrt{3}\right)}$$

$$= \frac{1 + \sqrt{3}}{\left(2 + \sqrt{3}\right)\left(1 - \sqrt{3}\right)} = \frac{1 + \sqrt{3}}{-1 - \sqrt{3}} = -1$$

(b) Observe that

$$\frac{dy}{d\theta} = \cos\theta\,(1 + 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{dx}{d\theta} = (1 + \sin\theta)(1 - 2\sin\theta) = 0 \qquad \text{when } \theta = \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore there are horizontal tangents at the points $(2, \pi/2)$, $\left(\frac{1}{2}, 7\pi/6\right)$, $\left(\frac{1}{2}, 11\pi/6\right)$ and vertical tangents at $\left(\frac{3}{2}, \pi/6\right)$ and $\left(\frac{3}{2}, 5\pi/6\right)$. When $\theta = 3\pi/2$, both $dy/d\theta$ and $dx/d\theta$ are 0, so we must be careful. Using l'Hospital's Rule, we have

$$\lim_{\theta\to(3\pi/2)^-}\frac{dy}{dx} = \left(\lim_{\theta\to(3\pi/2)^-}\frac{1 + 2\sin\theta}{1 - 2\sin\theta}\right)\left(\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta}\right)$$

$$= -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{\cos\theta}{1 + \sin\theta} = -\frac{1}{3}\lim_{\theta\to(3\pi/2)^-}\frac{-\sin\theta}{\cos\theta} = \infty$$

By symmetry, $$\lim_{\theta\to(3\pi/2)^+}\frac{dy}{dx} = -\infty$$

Thus there is a vertical tangent line at the pole (see Figure 15).

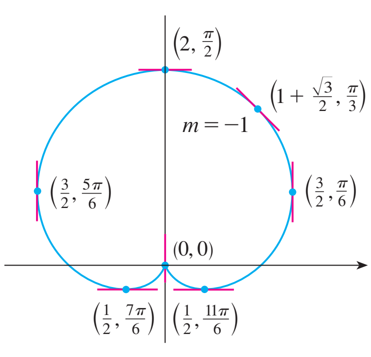


FIGURE 15
Tangent lines for $r = 1 + \sin\theta$

NOTE Instead of having to remember Equation 3, we could employ the method used to derive it. For instance, in Example 9 we could have written

$$x = r\cos\theta = (1 + \sin\theta)\cos\theta = \cos\theta + \tfrac{1}{2}\sin 2\theta$$

$$y = r\sin\theta = (1 + \sin\theta)\sin\theta = \sin\theta + \sin^2\theta$$

Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

Graphing Polar Curves with Graphing Devices

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the ones shown in Figures 16 and 17.

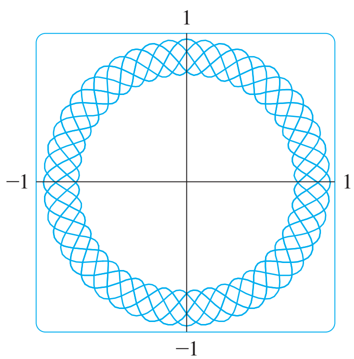


FIGURE 16
$r = \sin^2(2.4\theta) + \cos^4(2.4\theta)$

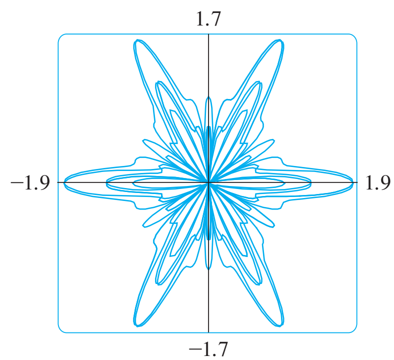


FIGURE 17
$r = \sin^2(1.2\theta) + \cos^3(6\theta)$

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Some machines require that the parameter be called t rather than θ.

EXAMPLE 10 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin(8\theta/5)\cos\theta \qquad y = r\sin\theta = \sin(8\theta/5)\sin\theta$$

In any case we need to determine the domain for θ. So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$.

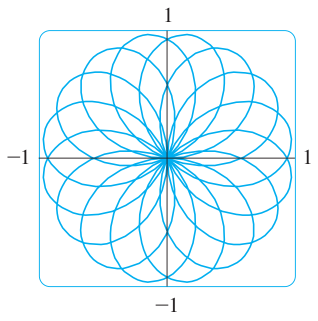


FIGURE 18
$r = \sin(8\theta/5)$

In Exercise 53 you are asked to prove analytically what we have discovered from the graphs in Figure 19.

Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 18 shows the resulting curve. Notice that this rose has 16 loops.

V EXAMPLE 11 Investigate the family of polar curves given by $r = 1 + c\sin\theta$. How does the shape change as c changes? (These curves are called **limaçons**, after a French word for snail, because of the shape of the curves for certain values of c.)

SOLUTION Figure 19 shows computer-drawn graphs for various values of c. For $c > 1$ there is a loop that decreases in size as c decreases. When $c = 1$ the loop disappears and the curve becomes the cardioid that we sketched in Example 7. For c between 1 and $\frac{1}{2}$ the cardioid's cusp is smoothed out and becomes a "dimple." When c decreases from $\frac{1}{2}$ to 0, the limaçon is shaped like an oval. This oval becomes more circular as $c \to 0$, and when $c = 0$ the curve is just the circle $r = 1$.

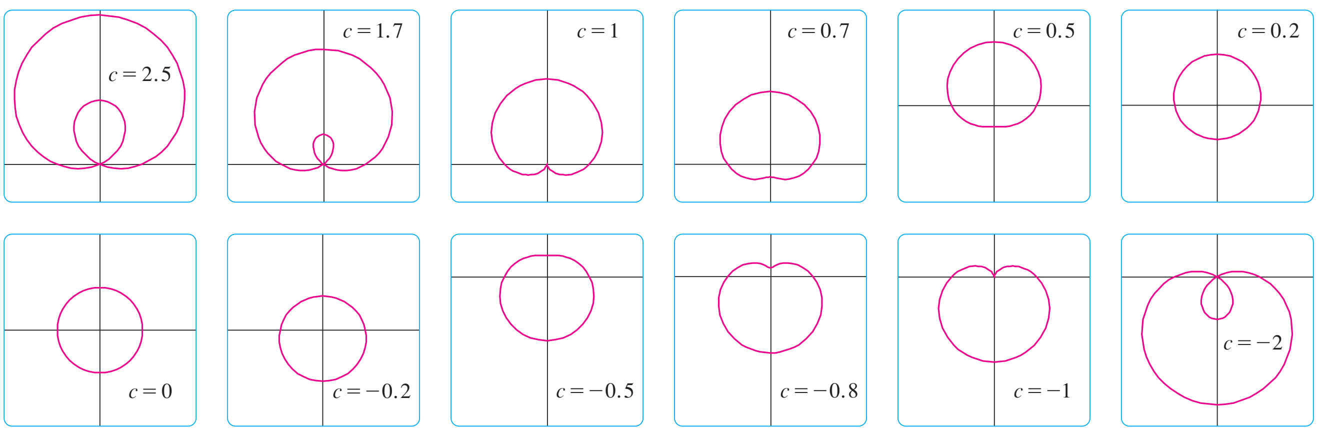


FIGURE 19
Members of the family of limaçons $r = 1 + c\sin\theta$

The remaining parts of Figure 19 show that as c becomes negative, the shapes change in reverse order. In fact, these curves are reflections about the horizontal axis of the corresponding curves with positive c.

Limaçons arise in the study of planetary motion. In particular, the trajectory of Mars, as viewed from the planet Earth, has been modeled by a limaçon with a loop, as in the parts of Figure 19 with $|c| > 1$.

10.3 Exercises

1–2 Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(2, \pi/3)$ (b) $(1, -3\pi/4)$ (c) $(-1, \pi/2)$

2. (a) $(1, 7\pi/4)$ (b) $(-3, \pi/6)$ (c) $(1, -1)$

3–4 Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(1, \pi)$ (b) $(2, -2\pi/3)$ (c) $(-2, 3\pi/4)$

4. (a) $(-\sqrt{2}, 5\pi/4)$ (b) $(1, 5\pi/2)$ (c) $(2, -7\pi/6)$

5–6 The Cartesian coordinates of a point are given.
(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.
(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(2, -2)$ (b) $(-1, \sqrt{3})$

6. (a) $(3\sqrt{3}, 3)$ (b) $(1, -2)$

7–12 Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $r \geqslant 1$

8. $0 \leqslant r < 2, \quad \pi \leqslant \theta \leqslant 3\pi/2$

9. $r \geqslant 0, \quad \pi/4 \leqslant \theta \leqslant 3\pi/4$

10. $1 \leqslant r \leqslant 3, \quad \pi/6 < \theta < 5\pi/6$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $r \geqslant 1, \quad \pi \leqslant \theta \leqslant 2\pi$

13. Find the distance between the points with polar coordinates $(2, \pi/3)$ and $(4, 2\pi/3)$.

14. Find a formula for the distance between the points with polar coordinates (r_1, θ_1) and (r_2, θ_2).

15–20 Identify the curve by finding a Cartesian equation for the curve.

15. $r^2 = 5$

16. $r = 4 \sec \theta$

17. $r = 2 \cos \theta$

18. $\theta = \pi/3$

19. $r^2 \cos 2\theta = 1$

20. $r = \tan \theta \sec \theta$

21–26 Find a polar equation for the curve represented by the given Cartesian equation.

21. $y = 2$

22. $y = x$

23. $y = 1 + 3x$

24. $4y^2 = x$

25. $x^2 + y^2 = 2cx$

26. $xy = 4$

27–28 For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

27. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point $(3, 3)$

28. (a) A circle with radius 5 and center $(2, 3)$
(b) A circle centered at the origin with radius 4

29–46 Sketch the curve with the given polar equation by first sketching the graph of r as a function of θ in Cartesion coordinates.

29. $r = -2 \sin \theta$

30. $r = 1 - \cos \theta$

31. $r = 2(1 + \cos \theta)$

32. $r = 1 + 2 \cos \theta$

33. $r = \theta, \ \theta \geqslant 0$

34. $r = \ln \theta, \ \theta \geqslant 1$

35. $r = 4 \sin 3\theta$

36. $r = \cos 5\theta$

37. $r = 2 \cos 4\theta$

38. $r = 3 \cos 6\theta$

39. $r = 1 - 2 \sin \theta$

40. $r = 2 + \sin \theta$

41. $r^2 = 9 \sin 2\theta$

42. $r^2 = \cos 4\theta$

43. $r = 2 + \sin 3\theta$

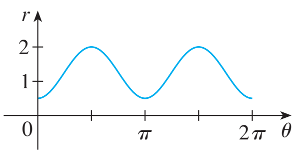
44. $r^2\theta = 1$

45. $r = 1 + 2 \cos 2\theta$

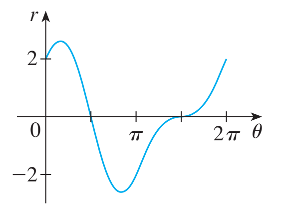
46. $r = 3 + 4 \cos \theta$

47–48 The figure shows a graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

47.



48.



49. Show that the polar curve $r = 4 + 2 \sec \theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \to \pm\infty} x = 2$. Use this fact to help sketch the conchoid.

50. Show that the curve $r = 2 - \csc \theta$ (also a conchoid) has the line $y = -1$ as a horizontal asymptote by showing that $\lim_{r \to \pm\infty} y = -1$. Use this fact to help sketch the conchoid.

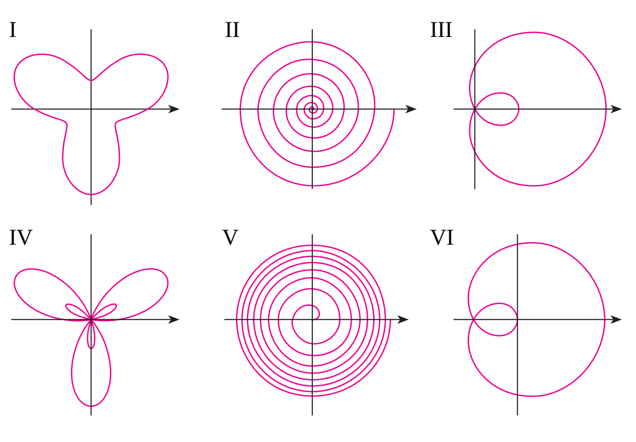
51. Show that the curve $r = \sin \theta \ \tan \theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

52. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

53. (a) In Example 11 the graphs suggest that the limaçon $r = 1 + c \sin \theta$ has an inner loop when $|c| > 1$. Prove that this is true, and find the values of θ that correspond to the inner loop.
(b) From Figure 19 it appears that the limaçon loses its dimple when $c = \frac{1}{2}$. Prove this.

54. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sqrt{\theta}, \quad 0 \leqslant \theta \leqslant 16\pi$ (b) $r = \theta^2, \quad 0 \leqslant \theta \leqslant 16\pi$
(c) $r = \cos(\theta/3)$ (d) $r = 1 + 2 \cos \theta$
(e) $r = 2 + \sin 3\theta$ (f) $r = 1 + 2 \sin 3\theta$



55–60 Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

55. $r = 2\sin\theta, \quad \theta = \pi/6$

56. $r = 2 - \sin\theta, \quad \theta = \pi/3$

57. $r = 1/\theta, \quad \theta = \pi$

58. $r = \cos(\theta/3), \quad \theta = \pi$

59. $r = \cos 2\theta, \quad \theta = \pi/4$

60. $r = 1 + 2\cos\theta, \quad \theta = \pi/3$

61–64 Find the points on the given curve where the tangent line is horizontal or vertical.

61. $r = 3\cos\theta$

62. $r = 1 - \sin\theta$

63. $r = 1 + \cos\theta$

64. $r = e^{\theta}$

65. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

66. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

67–72 Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

67. $r = 1 + 2\sin(\theta/2)$ (nephroid of Freeth)

68. $r = \sqrt{1 - 0.8\sin^2\theta}$ (hippopede)

69. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

70. $r = |\tan\theta|^{|\cot\theta|}$ (valentine curve)

71. $r = 1 + \cos^{999}\theta$ (PacMan curve)

72. $r = \sin^2(4\theta) + \cos(4\theta)$

73. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

74. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

75. Investigate the family of curves with polar equations $r = 1 + c\cos\theta$, where c is a real number. How does the shape change as c changes?

76. Investigate the family of polar curves

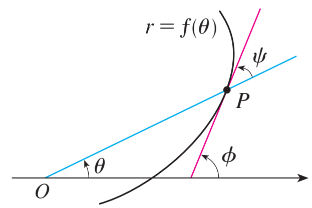
$$r = 1 + \cos^n\theta$$

where n is a positive integer. How does the shape change as n increases? What happens as n becomes large? Explain the shape for large n by considering the graph of r as a function of θ in Cartesian coordinates.

77. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



78. (a) Use Exercise 77 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.

(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.

(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

LABORATORY PROJECT FAMILIES OF POLAR CURVES

In this project you will discover the interesting and beautiful shapes that members of families of polar curves can take. You will also see how the shape of the curve changes when you vary the constants.

1. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?

(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

2. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

Graphing calculator or computer required

3. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

4. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

10.4 Areas and Lengths in Polar Coordinates

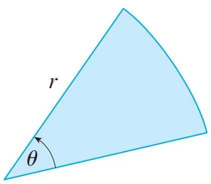


FIGURE 1

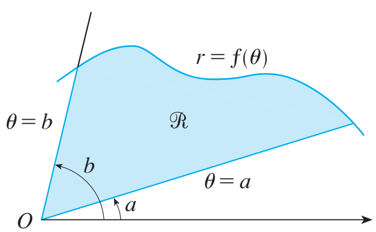


FIGURE 2

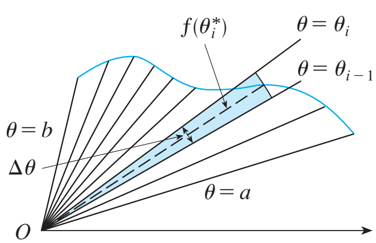


FIGURE 3

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle:

$$\boxed{1} \qquad A = \tfrac{1}{2}r^2\theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle. Formula 1 follows from the fact that the area of a sector is proportional to its central angle: $A = (\theta/2\pi)\pi r^2 = \frac{1}{2}r^2\theta$. (See also Exercise 35 in Section 7.3.)

Let $\mathcal{R}$ be the region, illustrated in Figure 2, bounded by the polar curve $r = f(\theta)$ and by the rays $\theta = a$ and $\theta = b$, where f is a positive continuous function and where $0 < b - a \leqslant 2\pi$. We divide the interval $[a, b]$ into subintervals with endpoints θ_0, θ_1, θ_2, . . . , θ_n and equal width $\Delta\theta$. The rays $\theta = \theta_i$ then divide $\mathcal{R}$ into n smaller regions with central angle $\Delta\theta = \theta_i - \theta_{i-1}$. If we choose θ_i^* in the ith subinterval $[\theta_{i-1}, \theta_i]$, then the area ΔA_i of the ith region is approximated by the area of the sector of a circle with central angle $\Delta\theta$ and radius $f(\theta_i^*)$. (See Figure 3.)

Thus from Formula 1 we have

$$\Delta A_i \approx \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

and so an approximation to the total area A of $\mathcal{R}$ is

$$\boxed{2} \qquad A \approx \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta$$

It appears from Figure 3 that the approximation in $\boxed{2}$ improves as $n \to \infty$. But the sums in $\boxed{2}$ are Riemann sums for the function $g(\theta) = \frac{1}{2}[f(\theta)]^2$, so

$$\lim_{n\to\infty} \sum_{i=1}^{n} \tfrac{1}{2}[f(\theta_i^*)]^2\,\Delta\theta = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta$$

It therefore appears plausible (and can in fact be proved) that the formula for the area A of the polar region $\mathcal{R}$ is

3
$$\boxed{A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\, d\theta}$$

Formula 3 is often written as

4
$$\boxed{A = \int_a^b \tfrac{1}{2} r^2\, d\theta}$$

with the understanding that $r = f(\theta)$. Note the similarity between Formulas 1 and 4.

When we apply Formula 3 or 4 it is helpful to think of the area as being swept out by a rotating ray through O that starts with angle a and ends with angle b.

V EXAMPLE 1 Find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.

SOLUTION The curve $r = \cos 2\theta$ was sketched in Example 8 in Section 10.3. Notice from Figure 4 that the region enclosed by the right loop is swept out by a ray that rotates from $\theta = -\pi/4$ to $\theta = \pi/4$. Therefore Formula 4 gives

$$A = \int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2\, d\theta = \tfrac{1}{2}\int_{-\pi/4}^{\pi/4} \cos^2 2\theta\, d\theta = \int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$A = \int_0^{\pi/4} \tfrac{1}{2}(1 + \cos 4\theta)\, d\theta = \tfrac{1}{2}\Big[\theta + \tfrac{1}{4}\sin 4\theta\Big]_0^{\pi/4} = \frac{\pi}{8}$$

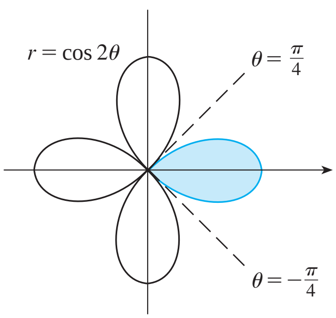


FIGURE 4

V EXAMPLE 2 Find the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid (see Example 7 in Section 10.3) and the circle are sketched in Figure 5 and the desired region is shaded. The values of a and b in Formula 4 are determined by finding the points of intersection of the two curves. They intersect when $3\sin\theta = 1 + \sin\theta$, which gives $\sin\theta = \frac{1}{2}$, so $\theta = \pi/6, 5\pi/6$. The desired area can be found by subtracting the area inside the cardioid between $\theta = \pi/6$ and $\theta = 5\pi/6$ from the area inside the circle from $\pi/6$ to $5\pi/6$. Thus

$$A = \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (3\sin\theta)^2\, d\theta - \tfrac{1}{2}\int_{\pi/6}^{5\pi/6} (1 + \sin\theta)^2\, d\theta$$

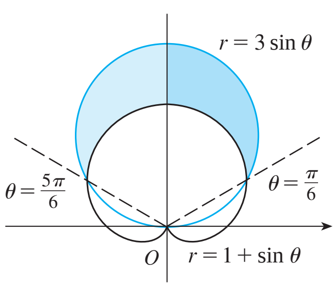


FIGURE 5

Since the region is symmetric about the vertical axis $\theta = \pi/2$, we can write

$$\begin{aligned} A &= 2\left[\tfrac{1}{2}\int_{\pi/6}^{\pi/2} 9\sin^2\theta\, d\theta - \tfrac{1}{2}\int_{\pi/6}^{\pi/2} (1 + 2\sin\theta + \sin^2\theta)\, d\theta\right] \\ &= \int_{\pi/6}^{\pi/2} (8\sin^2\theta - 1 - 2\sin\theta)\, d\theta \\ &= \int_{\pi/6}^{\pi/2} (3 - 4\cos 2\theta - 2\sin\theta)\, d\theta \qquad \left[\text{because } \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)\right] \\ &= 3\theta - 2\sin 2\theta + 2\cos\theta \Big]_{\pi/6}^{\pi/2} = \pi \end{aligned}$$

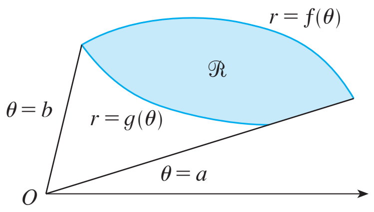


FIGURE 6

Example 2 illustrates the procedure for finding the area of the region bounded by two polar curves. In general, let $\mathcal{R}$ be a region, as illustrated in Figure 6, that is bounded by curves with polar equations $r = f(\theta)$, $r = g(\theta)$, $\theta = a$, and $\theta = b$, where $f(\theta) \geqslant g(\theta) \geqslant 0$ and $0 < b - a \leqslant 2\pi$. The area A of $\mathcal{R}$ is found by subtracting the area inside $r = g(\theta)$ from the area inside $r = f(\theta)$, so using Formula 3 we have

$$A = \int_a^b \tfrac{1}{2}[f(\theta)]^2\,d\theta - \int_a^b \tfrac{1}{2}[g(\theta)]^2\,d\theta$$

$$= \tfrac{1}{2}\int_a^b \left([f(\theta)]^2 - [g(\theta)]^2\right) d\theta$$

CAUTION The fact that a single point has many representations in polar coordinates sometimes makes it difficult to find all the points of intersection of two polar curves. For instance, it is obvious from Figure 5 that the circle and the cardioid have three points of intersection; however, in Example 2 we solved the equations $r = 3\sin\theta$ and $r = 1 + \sin\theta$ and found only two such points, $\left(\tfrac{3}{2}, \pi/6\right)$ and $\left(\tfrac{3}{2}, 5\pi/6\right)$. The origin is also a point of intersection, but we can't find it by solving the equations of the curves because the origin has no single representation in polar coordinates that satisfies both equations. Notice that, when represented as $(0, 0)$ or $(0, \pi)$, the origin satisfies $r = 3\sin\theta$ and so it lies on the circle; when represented as $(0, 3\pi/2)$, it satisfies $r = 1 + \sin\theta$ and so it lies on the cardioid. Think of two points moving along the curves as the parameter value θ increases from 0 to 2π. On one curve the origin is reached at $\theta = 0$ and $\theta = \pi$; on the other curve it is reached at $\theta = 3\pi/2$. The points don't collide at the origin because they reach the origin at different times, but the curves intersect there nonetheless.

Thus, to find *all* points of intersection of two polar curves, it is recommended that you draw the graphs of both curves. It is especially convenient to use a graphing calculator or computer to help with this task.

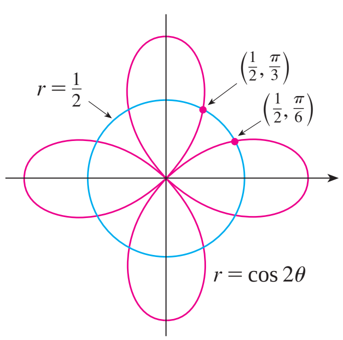


FIGURE 7

EXAMPLE 3 Find all points of intersection of the curves $r = \cos 2\theta$ and $r = \tfrac{1}{2}$.

SOLUTION If we solve the equations $r = \cos 2\theta$ and $r = \tfrac{1}{2}$, we get $\cos 2\theta = \tfrac{1}{2}$ and, therefore, $2\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3$. Thus the values of θ between 0 and 2π that satisfy both equations are $\theta = \pi/6, 5\pi/6, 7\pi/6, 11\pi/6$. We have found four points of intersection: $\left(\tfrac{1}{2}, \pi/6\right)$, $\left(\tfrac{1}{2}, 5\pi/6\right)$, $\left(\tfrac{1}{2}, 7\pi/6\right)$, and $\left(\tfrac{1}{2}, 11\pi/6\right)$.

However, you can see from Figure 7 that the curves have four other points of intersection—namely, $\left(\tfrac{1}{2}, \pi/3\right)$, $\left(\tfrac{1}{2}, 2\pi/3\right)$, $\left(\tfrac{1}{2}, 4\pi/3\right)$, and $\left(\tfrac{1}{2}, 5\pi/3\right)$. These can be found using symmetry or by noticing that another equation of the circle is $r = -\tfrac{1}{2}$ and then solving the equations $r = \cos 2\theta$ and $r = -\tfrac{1}{2}$.

Arc Length

To find the length of a polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Using the Product Rule and differentiating with respect to θ, we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\begin{aligned}\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\,\frac{dr}{d\theta}\cos\theta\,\sin\theta + r^2\sin^2\theta \\ &\qquad + \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\,\frac{dr}{d\theta}\sin\theta\,\cos\theta + r^2\cos^2\theta \\ &= \left(\frac{dr}{d\theta}\right)^2 + r^2\end{aligned}$$

Assuming that f' is continuous, we can use Theorem 10.2.5 to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore the length of a curve with polar equation $r = f(\theta)$, $a \leqslant \theta \leqslant b$, is

5
$$\boxed{L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta}$$

V EXAMPLE 4 Find the length of the cardioid $r = 1 + \sin\theta$.

SOLUTION The cardioid is shown in Figure 8. (We sketched it in Example 7 in Section 10.3.) Its full length is given by the parameter interval $0 \leqslant \theta \leqslant 2\pi$, so Formula 5 gives

$$\begin{aligned}L &= \int_0^{2\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta\end{aligned}$$

We could evaluate this integral by multiplying and dividing the integrand by $\sqrt{2 - 2\sin\theta}$, or we could use a computer algebra system. In any event, we find that the length of the cardioid is $L = 8$.

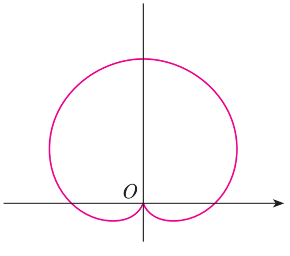


FIGURE 8
$r = 1 + \sin\theta$

10.4 Exercises

1–4 Find the area of the region that is bounded by the given curve and lies in the specified sector.

1. $r = e^{-\theta/4}$, $\pi/2 \leqslant \theta \leqslant \pi$

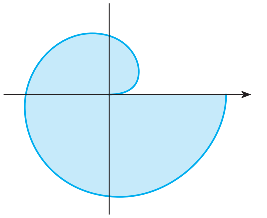
2. $r = \cos\theta$, $0 \leqslant \theta \leqslant \pi/6$

3. $r^2 = 9\sin 2\theta$, $r \geqslant 0$, $0 \leqslant \theta \leqslant \pi/2$

4. $r = \tan\theta$, $\pi/6 \leqslant \theta \leqslant \pi/3$

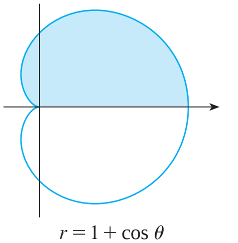
5–8 Find the area of the shaded region.

5.



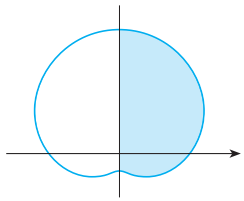
$r = \sqrt{\theta}$

6.



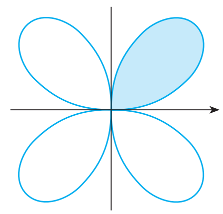
$r = 1 + \cos\theta$

7.



$r = 4 + 3\sin\theta$

8.



$r = \sin 2\theta$

9–12 Sketch the curve and find the area that it encloses.

9. $r = 2\sin\theta$

10. $r = 1 - \sin\theta$

11. $r = 3 + 2\cos\theta$

12. $r = 4 + 3\sin\theta$

13–16 Graph the curve and find the area that it encloses.

13. $r = 2 + \sin 4\theta$

14. $r = 3 - 2\cos 4\theta$

15. $r = \sqrt{1 + \cos^2(5\theta)}$

16. $r = 1 + 5\sin 6\theta$

17–21 Find the area of the region enclosed by one loop of the curve.

17. $r = 4\cos 3\theta$

18. $r^2 = \sin 2\theta$

19. $r = \sin 4\theta$

20. $r = 2\sin 5\theta$

21. $r = 1 + 2\sin\theta$ (inner loop)

22. Find the area enclosed by the loop of the **strophoid** $r = 2\cos\theta - \sec\theta$.

23–28 Find the area of the region that lies inside the first curve and outside the second curve.

23. $r = 2\cos\theta, \quad r = 1$

24. $r = 1 - \sin\theta, \quad r = 1$

25. $r^2 = 8\cos 2\theta, \quad r = 2$

26. $r = 2 + \sin\theta, \quad r = 3\sin\theta$

27. $r = 3\cos\theta, \quad r = 1 + \cos\theta$

28. $r = 3\sin\theta, \quad r = 2 - \sin\theta$

29–34 Find the area of the region that lies inside both curves.

29. $r = \sqrt{3}\cos\theta, \quad r = \sin\theta$

30. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

31. $r = \sin 2\theta, \quad r = \cos 2\theta$

32. $r = 3 + 2\cos\theta, \quad r = 3 + 2\sin\theta$

33. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

34. $r = a\sin\theta, \quad r = b\cos\theta, \quad a > 0, \; b > 0$

35. Find the area inside the larger loop and outside the smaller loop of the limaçon $r = \frac{1}{2} + \cos\theta$.

36. Find the area between a large loop and the enclosed small loop of the curve $r = 1 + 2\cos 3\theta$.

37–42 Find all points of intersection of the given curves.

37. $r = 1 + \sin\theta, \quad r = 3\sin\theta$

38. $r = 1 - \cos\theta, \quad r = 1 + \sin\theta$

39. $r = 2\sin 2\theta, \quad r = 1$

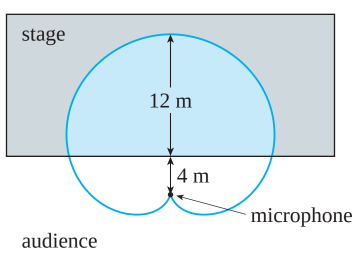
40. $r = \cos 3\theta, \quad r = \sin 3\theta$

41. $r = \sin\theta, \quad r = \sin 2\theta$

42. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$

43. The points of intersection of the cardioid $r = 1 + \sin\theta$ and the spiral loop $r = 2\theta$, $-\pi/2 \leqslant \theta \leqslant \pi/2$, can't be found exactly. Use a graphing device to find the approximate values of θ at which they intersect. Then use these values to estimate the area that lies inside both curves.

44. When recording live performances, sound engineers often use a microphone with a cardioid pickup pattern because it suppresses noise from the audience. Suppose the microphone is placed 4 m from the front of the stage (as in the figure) and the boundary of the optimal pickup region is given by the cardioid $r = 8 + 8\sin\theta$, where r is measured in meters and the microphone is at the pole. The musicians want to know the area they will have on stage within the optimal pickup range of the microphone. Answer their question.



45–48 Find the exact length of the polar curve.

45. $r = 2\cos\theta, \quad 0 \leqslant \theta \leqslant \pi$

46. $r = 5^\theta, \quad 0 \leqslant \theta \leqslant 2\pi$

47. $r = \theta^2, \quad 0 \leqslant \theta \leqslant 2\pi$

48. $r = 2(1 + \cos\theta)$

49–50 Find the exact length of the curve. Use a graph to determine the parameter interval.

49. $r = \cos^4(\theta/4)$

50. $r = \cos^2(\theta/2)$

51–54 Use a calculator to find the length of the curve correct to four decimal places. If necessary, graph the curve to determine the parameter interval.

51. One loop of the curve $r = \cos 2\theta$

52. $r = \tan\theta, \quad \pi/6 \leqslant \theta \leqslant \pi/3$

53. $r = \sin(6 \sin\theta)$

54. $r = \sin(\theta/4)$

55. (a) Use Formula 10.2.6 to show that the area of the surface generated by rotating the polar curve

$$r = f(\theta) \qquad a \leqslant \theta \leqslant b$$

(where f' is continuous and $0 \leqslant a < b \leqslant \pi$) about the polar axis is

$$S = \int_a^b 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

(b) Use the formula in part (a) to find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the polar axis.

56. (a) Find a formula for the area of the surface generated by rotating the polar curve $r = f(\theta)$, $a \leqslant \theta \leqslant b$ (where f' is continuous and $0 \leqslant a < b \leqslant \pi$), about the line $\theta = \pi/2$.

(b) Find the surface area generated by rotating the lemniscate $r^2 = \cos 2\theta$ about the line $\theta = \pi/2$.

10.5 Conic Sections

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

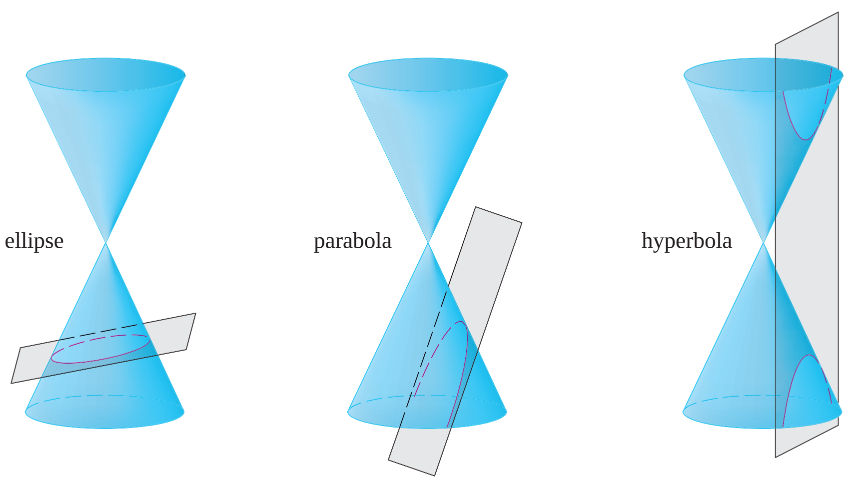


FIGURE 1
Conics

Parabolas

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

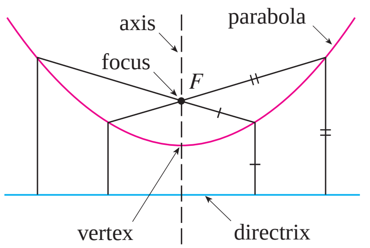


FIGURE 2

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Problem 16 on page 196 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola,

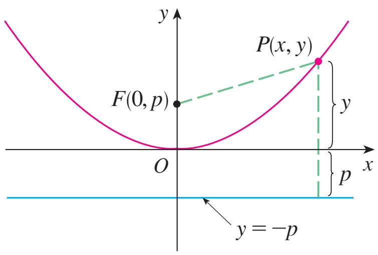


FIGURE 3

then the distance from P to the focus is

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

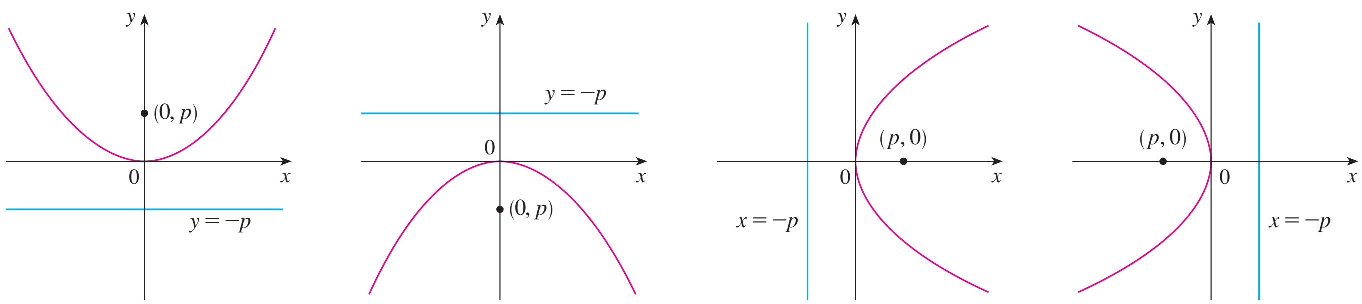
We get an equivalent equation by squaring and simplifying:

$$\begin{aligned} x^2 + (y - p)^2 &= |y + p|^2 = (y + p)^2 \\ x^2 + y^2 - 2py + p^2 &= y^2 + 2py + p^2 \\ x^2 &= 4py \end{aligned}$$

1 An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If we write $a = 1/(4p)$, then the standard equation of a parabola $\boxed{1}$ becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because $\boxed{1}$ is unchanged when x is replaced by $-x$.



(a) $x^2 = 4py,\ p > 0$ (b) $x^2 = 4py,\ p < 0$ (c) $y^2 = 4px,\ p > 0$ (d) $y^2 = 4px,\ p < 0$

FIGURE 4

If we interchange x and y in $\boxed{1}$, we obtain

2
$$\boxed{y^2 = 4px}$$

which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the x-axis, which is the axis of the parabola.

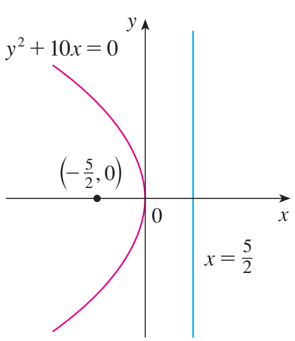


FIGURE 5

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

SOLUTION If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = \left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5.

Ellipses

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the sun at one focus.

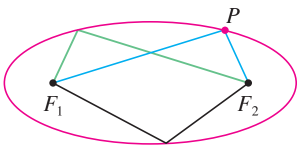


FIGURE 6

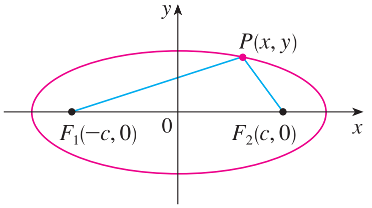


FIGURE 7

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is, $$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

or $$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to $$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes $$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From triangle F_1F_2P in Figure 7 we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

3 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

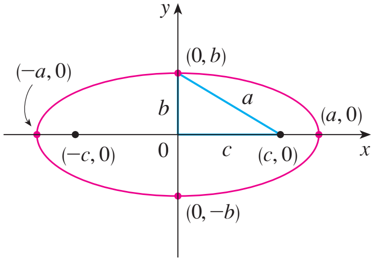


FIGURE 8
$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1,\ a \geqslant b$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. The line segment joining $(0, b)$ and $(0, -b)$ is the **minor axis**. Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

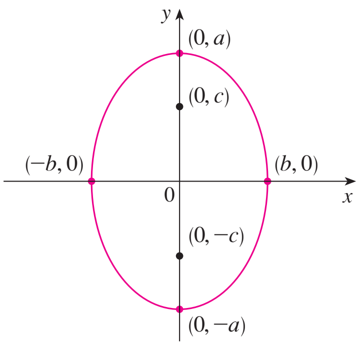


FIGURE 9
$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \ a \geqslant b$

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in $\boxed{4}$. (See Figure 9.)

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

V EXAMPLE 2 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 10.

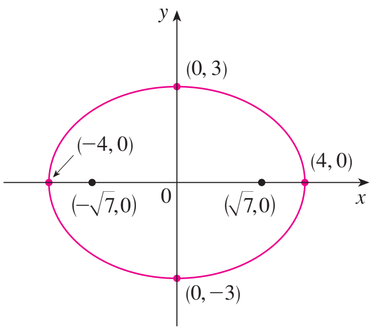


FIGURE 10
$9x^2 + 16y^2 = 144$

V EXAMPLE 3 Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

SOLUTION Using the notation of $\boxed{5}$, we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$.

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see Exercise 65). This principle is used in *lithotripsy*, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

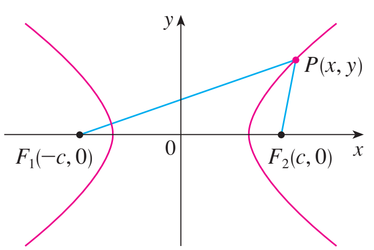


FIGURE 11
P is on the hyperbola when $|PF_1| - |PF_2| = \pm 2a$.

Hyperbolas

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle's Law, Ohm's Law, supply and demand curves). A particularly significant application of hyperbolas is found in the navigation systems developed in World Wars I and II (see Exercise 51).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 52 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

$$\boxed{6} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

This shows that $x^2 \geqslant a^2$, so $|x| = \sqrt{x^2} \geqslant a$. Therefore we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches.*

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes. [See Exercise 73 in Section 4.5, where these lines are shown to be slant asymptotes.]

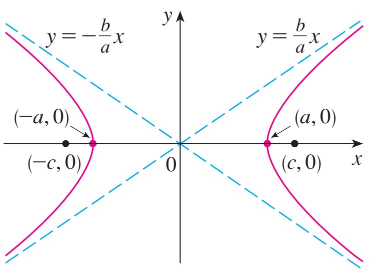


FIGURE 12

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

7 The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information, which is illustrated in Figure 13.

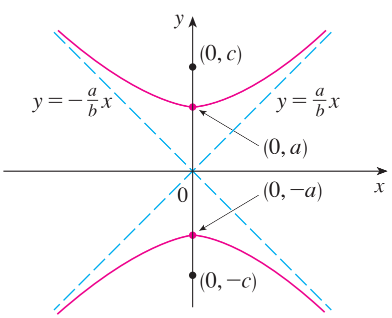


FIGURE 13

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

8 The hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

EXAMPLE 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

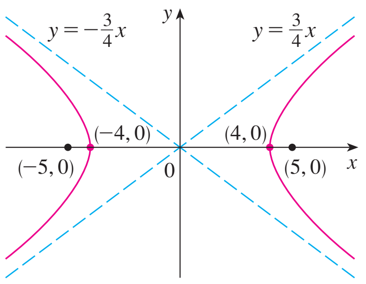


FIGURE 14
$9x^2 - 16y^2 = 144$

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in $\boxed{7}$ with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14.

EXAMPLE 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

SOLUTION From $\boxed{8}$ and the given information, we see that $a = 1$ and $a/b = 2$. Thus $b = a/2 = \frac{1}{2}$ and $c^2 = a^2 + b^2 = \frac{5}{4}$. The foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is

$$y^2 - 4x^2 = 1$$

Shifted Conics

As discussed in Appendix C, we shift conics by taking the standard equations $\boxed{1}$, $\boxed{2}$, $\boxed{4}$, $\boxed{5}$, $\boxed{7}$, and $\boxed{8}$ and replacing x and y by $x - h$ and $y - k$.

EXAMPLE 6 Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$ and vertices $(1, -2)$, $(5, -2)$.

SOLUTION The major axis is the line segment that joins the vertices $(1, -2)$, $(5, -2)$ and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is $(3, -2)$, we replace x and y in $\boxed{4}$ by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse.

V EXAMPLE 7 Sketch the conic $9x^2 - 4y^2 - 72x + 8y + 176 = 0$ and find its foci.

SOLUTION We complete the squares as follows:

$$\begin{aligned} 4(y^2 - 2y) - 9(x^2 - 8x) &= 176 \\ 4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) &= 176 + 4 - 144 \\ 4(y-1)^2 - 9(x-4)^2 &= 36 \\ \frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} &= 1 \end{aligned}$$

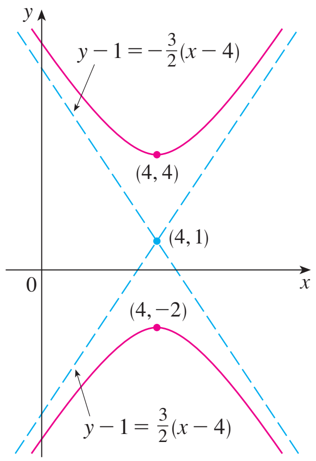


FIGURE 15
$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

This is in the form $\boxed{8}$ except that x and y are replaced by $x - 4$ and $y - 1$. Thus $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $\left(4, 1 + \sqrt{13}\right)$ and $\left(4, 1 - \sqrt{13}\right)$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15.

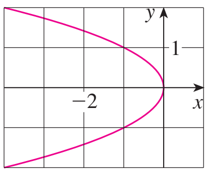
10.5 Exercises

1–8 Find the vertex, focus, and directrix of the parabola and sketch its graph.

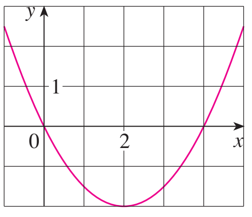
1. $x^2 = 6y$
2. $2y^2 = 5x$
3. $2x = -y^2$
4. $3x^2 + 8y = 0$
5. $(x + 2)^2 = 8(y - 3)$
6. $x - 1 = (y + 5)^2$
7. $y^2 + 2y + 12x + 25 = 0$
8. $y + 12x - 2x^2 = 16$

9–10 Find an equation of the parabola. Then find the focus and directrix.

9.



10.

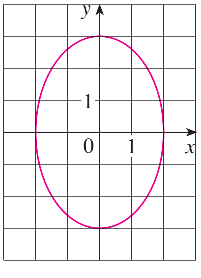


11–16 Find the vertices and foci of the ellipse and sketch its graph.

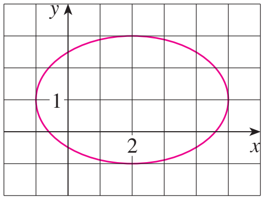
11. $\dfrac{x^2}{2} + \dfrac{y^2}{4} = 1$
12. $\dfrac{x^2}{36} + \dfrac{y^2}{8} = 1$
13. $x^2 + 9y^2 = 9$
14. $100x^2 + 36y^2 = 225$
15. $9x^2 - 18x + 4y^2 = 27$
16. $x^2 + 3y^2 + 2x - 12y + 10 = 0$

17–18 Find an equation of the ellipse. Then find its foci.

17.



18.



19–24 Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{y^2}{25} - \dfrac{x^2}{9} = 1$
20. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1$
21. $x^2 - y^2 = 100$
22. $y^2 - 16x^2 = 16$
23. $4x^2 - y^2 - 24x - 4y + 28 = 0$
24. $y^2 - 4x^2 - 2y + 16x = 31$

25–30 Identify the type of conic section whose equation is given and find the vertices and foci.

25. $x^2 = y + 1$
26. $x^2 = y^2 + 1$
27. $x^2 = 4y - 2y^2$
28. $y^2 - 8y = 6x - 16$
29. $y^2 + 2y = 4x^2 + 3$
30. $4x^2 + 4x + y^2 = 0$

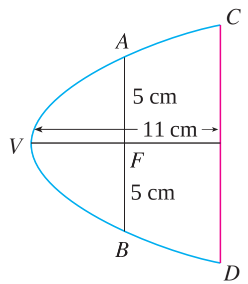
31–48 Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex $(0, 0)$, focus $(1, 0)$
32. Parabola, focus $(0, 0)$, directrix $y = 6$
33. Parabola, focus $(-4, 0)$, directrix $x = 2$
34. Parabola, focus $(3, 6)$, vertex $(3, 2)$
35. Parabola, vertex $(2, 3)$, vertical axis, passing through $(1, 5)$
36. Parabola, horizontal axis, passing through $(-1, 0)$, $(1, -1)$, and $(3, 1)$
37. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$
38. Ellipse, foci $(0, \pm 5)$, vertices $(0, \pm 13)$
39. Ellipse, foci $(0, 2)$, $(0, 6)$, vertices $(0, 0)$, $(0, 8)$
40. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$
41. Ellipse, center $(-1, 4)$, vertex $(-1, 0)$, focus $(-1, 6)$
42. Ellipse, foci $(\pm 4, 0)$, passing through $(-4, 1.8)$
43. Hyperbola, vertices $(\pm 3, 0)$, foci $(\pm 5, 0)$
44. Hyperbola, vertices $(0, \pm 2)$, foci $(0, \pm 5)$
45. Hyperbola, vertices $(-3, -4)$, $(-3, 6)$, foci $(-3, -7)$, $(-3, 9)$
46. Hyperbola, vertices $(-1, 2)$, $(7, 2)$, foci $(-2, 2)$, $(8, 2)$
47. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$
48. Hyperbola, foci $(2, 0)$, $(2, 8)$, asymptotes $y = 3 + \frac{1}{2}x$ and $y = 5 - \frac{1}{2}x$

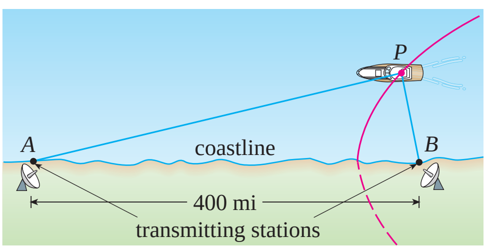
1. Homework Hints available at stewartcalculus.com

49. The point in a lunar orbit nearest the surface of the moon is called *perilune* and the point farthest from the surface is called *apolune*. The *Apollo 11* spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

50. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.
(a) Find an equation of the parabola.
(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.



51. In the LORAN (LOng RAnge Navigation) radio navigation system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 400 mi due east of station A on a coastline. A ship received the signal from B 1200 microseconds (μs) before it received the signal from A.
(a) Assuming that radio signals travel at a speed of 980 ft/μs, find an equation of the hyperbola on which the ship lies.
(b) If the ship is due north of B, how far off the coastline is the ship?



52. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

53. Show that the function defined by the upper branch of the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is concave upward.

54. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

55. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k - 16} = 1$$

in each of the following cases: (a) $k > 16$, (b) $0 < k < 16$, and (c) $k < 0$.
(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of k is.

56. (a) Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (x_0, y_0) can be written as

$$y_0 y = 2p(x + x_0)$$

(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

57. Show that the tangent lines to the parabola $x^2 = 4py$ drawn from any point on the directrix are perpendicular.

58. Show that if an ellipse and a hyperbola have the same foci, then their tangent lines at each point of intersection are perpendicular.

59. Use parametric equations and Simpson's Rule with $n = 8$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$.

60. The planet Pluto travels in an elliptical orbit around the sun (at one focus). The length of the major axis is 1.18×10^{10} km and the length of the minor axis is 1.14×10^{10} km. Use Simpson's Rule with $n = 10$ to estimate the distance traveled by the planet during one complete orbit around the sun.

61. Find the area of the region enclosed by the hyperbola $x^2/a^2 - y^2/b^2 = 1$ and the vertical line through a focus.

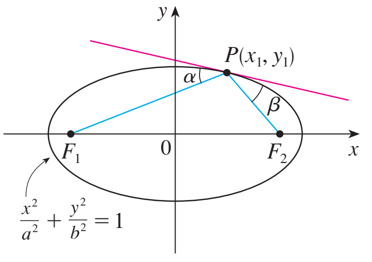
62. (a) If an ellipse is rotated about its major axis, find the volume of the resulting solid.
(b) If it is rotated about its minor axis, find the resulting volume.

63. Find the centroid of the region enclosed by the x-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

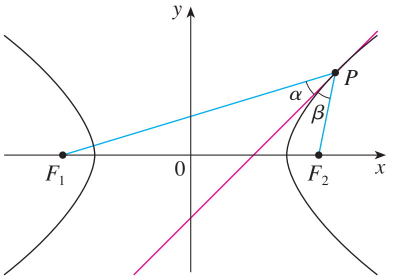
64. (a) Calculate the surface area of the ellipsoid that is generated by rotating an ellipse about its major axis.
(b) What is the surface area if the ellipse is rotated about its minor axis?

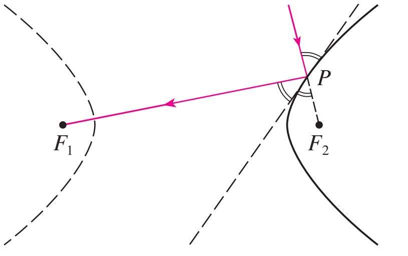
65. Let $P(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines

PF_1, PF_2 and the ellipse as shown in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula in Problem 15 on page 195 to show that $\tan \alpha = \tan \beta$.]



66. Let $P(x_1, y_1)$ be a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the hyperbola as shown in the figure. Prove that $\alpha = \beta$. (This is the reflection property of the hyperbola. It shows that light aimed at a focus F_2 of a hyperbolic mirror is reflected toward the other focus F_1.)





10.6 Conic Sections in Polar Coordinates

In the preceding section we defined the parabola in terms of a focus and directrix, but we defined the ellipse and hyperbola in terms of two foci. In this section we give a more unified treatment of all three types of conic sections in terms of a focus and directrix. Furthermore, if we place the focus at the origin, then a conic section has a simple polar equation, which provides a convenient description of the motion of planets, satellites, and comets.

1 Theorem Let F be a fixed point (called the **focus**) and l be a fixed line (called the **directrix**) in a plane. Let e be a fixed positive number (called the **eccentricity**). The set of all points P in the plane such that

$$\frac{|PF|}{|Pl|} = e$$

(that is, the ratio of the distance from F to the distance from l is the constant e) is a conic section. The conic is

(a) an ellipse if $e < 1$

(b) a parabola if $e = 1$

(c) a hyperbola if $e > 1$

PROOF Notice that if the eccentricity is $e = 1$, then $|PF| = |Pl|$ and so the given condition simply becomes the definition of a parabola as given in Section 10.5.

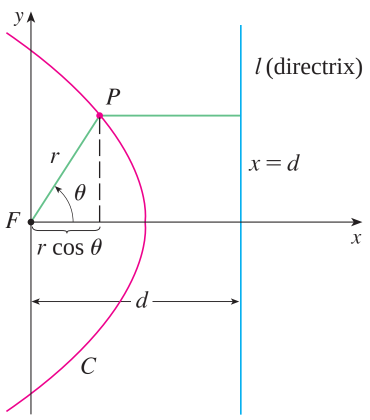


FIGURE 1

Let us place the focus F at the origin and the directrix parallel to the y-axis and d units to the right. Thus the directrix has equation $x = d$ and is perpendicular to the polar axis. If the point P has polar coordinates (r, θ), we see from Figure 1 that

$$|PF| = r \qquad |Pl| = d - r\cos\theta$$

Thus the condition $|PF|/|Pl| = e$, or $|PF| = e|Pl|$, becomes

$$\boxed{2} \qquad r = e(d - r\cos\theta)$$

If we square both sides of this polar equation and convert to rectangular coordinates, we get

$$x^2 + y^2 = e^2(d - x)^2 = e^2(d^2 - 2dx + x^2)$$

or

$$(1 - e^2)x^2 + 2de^2x + y^2 = e^2d^2$$

After completing the square, we have

$$\boxed{3} \qquad \left(x + \frac{e^2d}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2d^2}{(1 - e^2)^2}$$

If $e < 1$, we recognize Equation 3 as the equation of an ellipse. In fact, it is of the form

$$\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1$$

where

$$\boxed{4} \qquad h = -\frac{e^2d}{1 - e^2} \qquad a^2 = \frac{e^2d^2}{(1 - e^2)^2} \qquad b^2 = \frac{e^2d^2}{1 - e^2}$$

In Section 10.5 we found that the foci of an ellipse are at a distance c from the center, where

$$\boxed{5} \qquad c^2 = a^2 - b^2 = \frac{e^4d^2}{(1 - e^2)^2}$$

This shows that

$$c = \frac{e^2d}{1 - e^2} = -h$$

and confirms that the focus as defined in Theorem 1 means the same as the focus defined in Section 10.5. It also follows from Equations 4 and 5 that the eccentricity is given by

$$e = \frac{c}{a}$$

If $e > 1$, then $1 - e^2 < 0$ and we see that Equation 3 represents a hyperbola. Just as we did before, we could rewrite Equation 3 in the form

$$\frac{(x - h)^2}{a^2} - \frac{y^2}{b^2} = 1$$

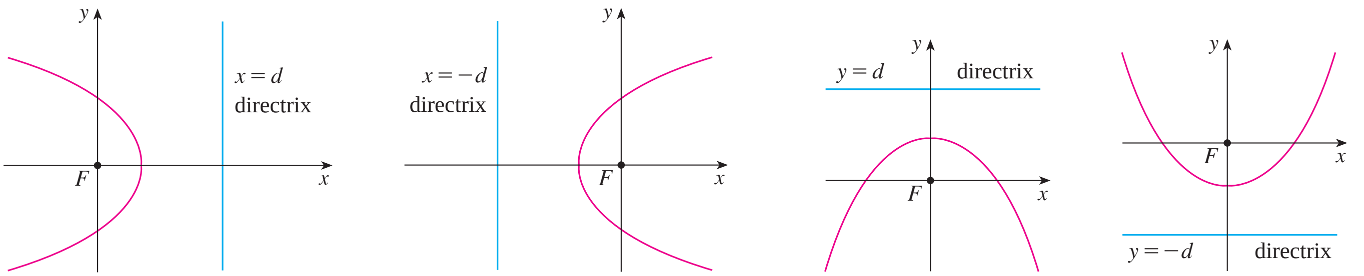
and see that

$$e = \frac{c}{a} \qquad \text{where} \quad c^2 = a^2 + b^2$$

By solving Equation 2 for r, we see that the polar equation of the conic shown in Figure 1 can be written as

$$r = \frac{ed}{1 + e\cos\theta}$$

If the directrix is chosen to be to the left of the focus as $x = -d$, or if the directrix is chosen to be parallel to the polar axis as $y = \pm d$, then the polar equation of the conic is given by the following theorem, which is illustrated by Figure 2. (See Exercises 21–23.)



(a) $r = \dfrac{ed}{1 + e\cos\theta}$ (b) $r = \dfrac{ed}{1 - e\cos\theta}$ (c) $r = \dfrac{ed}{1 + e\sin\theta}$ (d) $r = \dfrac{ed}{1 - e\sin\theta}$

FIGURE 2
Polar equations of conics

6 Theorem A polar equation of the form

$$r = \frac{ed}{1 \pm e\cos\theta} \qquad \text{or} \qquad r = \frac{ed}{1 \pm e\sin\theta}$$

represents a conic section with eccentricity e. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

V EXAMPLE 1 Find a polar equation for a parabola that has its focus at the origin and whose directrix is the line $y = -6$.

SOLUTION Using Theorem 6 with $e = 1$ and $d = 6$, and using part (d) of Figure 2, we see that the equation of the parabola is

$$r = \frac{6}{1 - \sin\theta}$$

V EXAMPLE 2 A conic is given by the polar equation

$$r = \frac{10}{3 - 2\cos\theta}$$

Find the eccentricity, identify the conic, locate the directrix, and sketch the conic.

SOLUTION Dividing numerator and denominator by 3, we write the equation as

$$r = \frac{\frac{10}{3}}{1 - \frac{2}{3}\cos\theta}$$

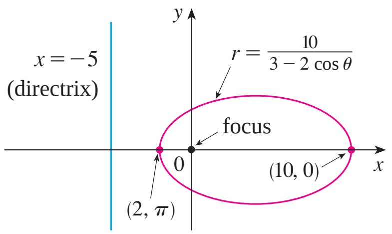


FIGURE 3

From Theorem 6 we see that this represents an ellipse with $e = \frac{2}{3}$. Since $ed = \frac{10}{3}$, we have

$$d = \frac{\frac{10}{3}}{e} = \frac{\frac{10}{3}}{\frac{2}{3}} = 5$$

so the directrix has Cartesian equation $x = -5$. When $\theta = 0$, $r = 10$; when $\theta = \pi$, $r = 2$. So the vertices have polar coordinates $(10, 0)$ and $(2, \pi)$. The ellipse is sketched in Figure 3.

EXAMPLE 3 Sketch the conic $r = \dfrac{12}{2 + 4\sin\theta}$.

SOLUTION Writing the equation in the form

$$r = \frac{6}{1 + 2\sin\theta}$$

we see that the eccentricity is $e = 2$ and the equation therefore represents a hyperbola. Since $ed = 6$, $d = 3$ and the directrix has equation $y = 3$. The vertices occur when $\theta = \pi/2$ and $3\pi/2$, so they are $(2, \pi/2)$ and $(-6, 3\pi/2) = (6, \pi/2)$. It is also useful to plot the x-intercepts. These occur when $\theta = 0, \pi$; in both cases $r = 6$. For additional accuracy we could draw the asymptotes. Note that $r \to \pm\infty$ when $1 + 2\sin\theta \to 0^+$ or 0^- and $1 + 2\sin\theta = 0$ when $\sin\theta = -\frac{1}{2}$. Thus the asymptotes are parallel to the rays $\theta = 7\pi/6$ and $\theta = 11\pi/6$. The hyperbola is sketched in Figure 4.

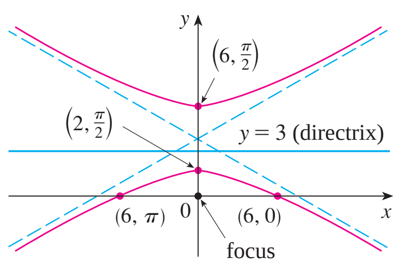


FIGURE 4
$r = \dfrac{12}{2 + 4\sin\theta}$

When rotating conic sections, we find it much more convenient to use polar equations than Cartesian equations. We just use the fact (see Exercise 73 in Section 10.3) that the graph of $r = f(\theta - \alpha)$ is the graph of $r = f(\theta)$ rotated counterclockwise about the origin through an angle α.

V EXAMPLE 4 If the ellipse of Example 2 is rotated through an angle $\pi/4$ about the origin, find a polar equation and graph the resulting ellipse.

SOLUTION We get the equation of the rotated ellipse by replacing θ with $\theta - \pi/4$ in the equation given in Example 2. So the new equation is

$$r = \frac{10}{3 - 2\cos(\theta - \pi/4)}$$

We use this equation to graph the rotated ellipse in Figure 5. Notice that the ellipse has been rotated about its left focus.

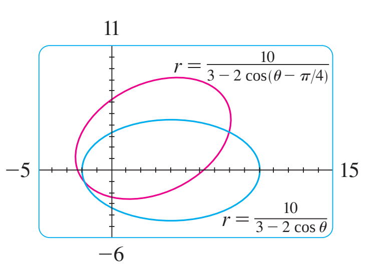


FIGURE 5

In Figure 6 we use a computer to sketch a number of conics to demonstrate the effect of varying the eccentricity e. Notice that when e is close to 0 the ellipse is nearly circular, whereas it becomes more elongated as $e \to 1^-$. When $e = 1$, of course, the conic is a parabola.

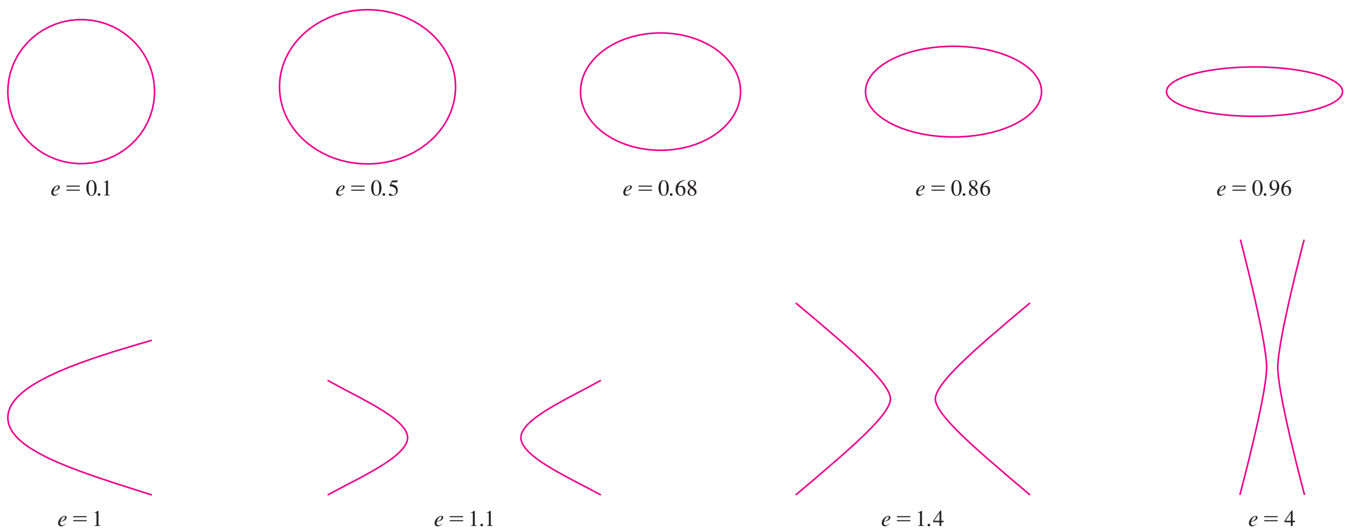


FIGURE 6

Kepler's Laws

In 1609 the German mathematician and astronomer Johannes Kepler, on the basis of huge amounts of astronomical data, published the following three laws of planetary motion.

Kepler's Laws

1. A planet revolves around the sun in an elliptical orbit with the sun at one focus.
2. The line joining the sun to a planet sweeps out equal areas in equal times.
3. The square of the period of revolution of a planet is proportional to the cube of the length of the major axis of its orbit.

Although Kepler formulated his laws in terms of the motion of planets around the sun, they apply equally well to the motion of moons, comets, satellites, and other bodies that orbit subject to a single gravitational force. In Section 13.4 we will show how to deduce Kepler's Laws from Newton's Laws. Here we use Kepler's First Law, together with the polar equation of an ellipse, to calculate quantities of interest in astronomy.

For purposes of astronomical calculations, it's useful to express the equation of an ellipse in terms of its eccentricity e and its semimajor axis a. We can write the distance d from the focus to the directrix in terms of a if we use $\boxed{4}$:

$$a^2 = \frac{e^2d^2}{(1 - e^2)^2} \qquad \Rightarrow \qquad d^2 = \frac{a^2(1 - e^2)^2}{e^2} \qquad \Rightarrow \qquad d = \frac{a(1 - e^2)}{e}$$

So $ed = a(1 - e^2)$. If the directrix is $x = d$, then the polar equation is

$$r = \frac{ed}{1 + e\cos\theta} = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

7 The polar equation of an ellipse with focus at the origin, semimajor axis a, eccentricity e, and directrix $x = d$ can be written in the form

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta}$$

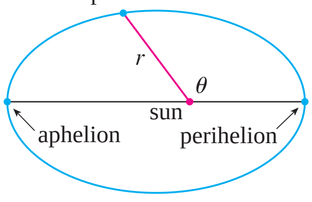


FIGURE 7

The positions of a planet that are closest to and farthest from the sun are called its **perihelion** and **aphelion**, respectively, and correspond to the vertices of the ellipse. (See Figure 7.) The distances from the sun to the perihelion and aphelion are called the **perihelion distance** and **aphelion distance**, respectively. In Figure 1 the sun is at the focus F, so at perihelion we have $\theta = 0$ and, from Equation 7,

$$r = \frac{a(1 - e^2)}{1 + e \cos 0} = \frac{a(1 - e)(1 + e)}{1 + e} = a(1 - e)$$

Similarly, at aphelion $\theta = \pi$ and $r = a(1 + e)$.

8 The perihelion distance from a planet to the sun is $a(1 - e)$ and the aphelion distance is $a(1 + e)$.

EXAMPLE 5

(a) Find an approximate polar equation for the elliptical orbit of the earth around the sun (at one focus) given that the eccentricity is about 0.017 and the length of the major axis is about 2.99×10^8 km.

(b) Find the distance from the earth to the sun at perihelion and at aphelion.

SOLUTION

(a) The length of the major axis is $2a = 2.99 \times 10^8$, so $a = 1.495 \times 10^8$. We are given that $e = 0.017$ and so, from Equation 7, an equation of the earth's orbit around the sun is

$$r = \frac{a(1 - e^2)}{1 + e \cos\theta} = \frac{(1.495 \times 10^8)[1 - (0.017)^2]}{1 + 0.017 \cos\theta}$$

or, approximately,

$$r = \frac{1.49 \times 10^8}{1 + 0.017 \cos\theta}$$

(b) From $\boxed{8}$, the perihelion distance from the earth to the sun is

$$a(1 - e) \approx (1.495 \times 10^8)(1 - 0.017) \approx 1.47 \times 10^8 \text{ km}$$

and the aphelion distance is

$$a(1 + e) \approx (1.495 \times 10^8)(1 + 0.017) \approx 1.52 \times 10^8 \text{ km}$$

10.6 Exercises

1–8 Write a polar equation of a conic with the focus at the origin and the given data.

1. Ellipse, eccentricity $\frac{1}{2}$, directrix $x = 4$

2. Parabola, directrix $x = -3$

3. Hyperbola, eccentricity 1.5, directrix $y = 2$

4. Hyperbola, eccentricity 3, directrix $x = 3$

5. Parabola, vertex $(4, 3\pi/2)$

6. Ellipse, eccentricity 0.8, vertex $(1, \pi/2)$

7. Ellipse, eccentricity $\frac{1}{2}$, directrix $r = 4 \sec\theta$

8. Hyperbola, eccentricity 3, directrix $r = -6 \csc\theta$

9–16 (a) Find the eccentricity, (b) identify the conic, (c) give an equation of the directrix, and (d) sketch the conic.

9. $r = \dfrac{4}{5 - 4\sin\theta}$

10. $r = \dfrac{12}{3 - 10\cos\theta}$

11. $r = \dfrac{2}{3 + 3\sin\theta}$

12. $r = \dfrac{3}{2 + 2\cos\theta}$

13. $r = \dfrac{9}{6 + 2\cos\theta}$

14. $r = \dfrac{8}{4 + 5\sin\theta}$

15. $r = \dfrac{3}{4 - 8\cos\theta}$

16. $r = \dfrac{10}{5 - 6\sin\theta}$

17. (a) Find the eccentricity and directrix of the conic $r = 1/(1 - 2\sin\theta)$ and graph the conic and its directrix.
(b) If this conic is rotated counterclockwise about the origin through an angle $3\pi/4$, write the resulting equation and graph its curve.

18. Graph the conic $r = 4/(5 + 6\cos\theta)$ and its directrix. Also graph the conic obtained by rotating this curve about the origin through an angle $\pi/3$.

19. Graph the conics $r = e/(1 - e\cos\theta)$ with $e = 0.4, 0.6, 0.8$, and 1.0 on a common screen. How does the value of e affect the shape of the curve?

20. (a) Graph the conics $r = ed/(1 + e\sin\theta)$ for $e = 1$ and various values of d. How does the value of d affect the shape of the conic?
(b) Graph these conics for $d = 1$ and various values of e. How does the value of e affect the shape of the conic?

21. Show that a conic with focus at the origin, eccentricity e, and directrix $x = -d$ has polar equation

$$r = \frac{ed}{1 - e\cos\theta}$$

22. Show that a conic with focus at the origin, eccentricity e, and directrix $y = d$ has polar equation

$$r = \frac{ed}{1 + e\sin\theta}$$

23. Show that a conic with focus at the origin, eccentricity e, and directrix $y = -d$ has polar equation

$$r = \frac{ed}{1 - e\sin\theta}$$

24. Show that the parabolas $r = c/(1 + \cos\theta)$ and $r = d/(1 - \cos\theta)$ intersect at right angles.

25. The orbit of Mars around the sun is an ellipse with eccentricity 0.093 and semimajor axis 2.28×10^8 km. Find a polar equation for the orbit.

26. Jupiter's orbit has eccentricity 0.048 and the length of the major axis is 1.56×10^9 km. Find a polar equation for the orbit.

27. The orbit of Halley's comet, last seen in 1986 and due to return in 2062, is an ellipse with eccentricity 0.97 and one focus at the sun. The length of its major axis is 36.18 AU. [An astronomical unit (AU) is the mean distance between the earth and the sun, about 93 million miles.] Find a polar equation for the orbit of Halley's comet. What is the maximum distance from the comet to the sun?

28. The Hale-Bopp comet, discovered in 1995, has an elliptical orbit with eccentricity 0.9951 and the length of the major axis is 356.5 AU. Find a polar equation for the orbit of this comet. How close to the sun does it come?

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29. The planet Mercury travels in an elliptical orbit with eccentricity 0.206. Its minimum distance from the sun is 4.6×10^7 km. Find its maximum distance from the sun.

30. The distance from the planet Pluto to the sun is 4.43×10^9 km at perihelion and 7.37×10^9 km at aphelion. Find the eccentricity of Pluto's orbit.

31. Using the data from Exercise 29, find the distance traveled by the planet Mercury during one complete orbit around the sun. (If your calculator or computer algebra system evaluates definite integrals, use it. Otherwise, use Simpson's Rule.)

10 Review

Concept Check

1. (a) What is a parametric curve?
   (b) How do you sketch a parametric curve?

2. (a) How do you find the slope of a tangent to a parametric curve?
   (b) How do you find the area under a parametric curve?

3. Write an expression for each of the following:
   (a) The length of a parametric curve
   (b) The area of the surface obtained by rotating a parametric curve about the x-axis

4. (a) Use a diagram to explain the meaning of the polar coordinates (r, θ) of a point.
   (b) Write equations that express the Cartesian coordinates (x, y) of a point in terms of the polar coordinates.
   (c) What equations would you use to find the polar coordinates of a point if you knew the Cartesian coordinates?

5. (a) How do you find the slope of a tangent line to a polar curve?
   (b) How do you find the area of a region bounded by a polar curve?
   (c) How do you find the length of a polar curve?

6. (a) Give a geometric definition of a parabola.
   (b) Write an equation of a parabola with focus $(0, p)$ and directrix $y = -p$. What if the focus is $(p, 0)$ and the directrix is $x = -p$?

7. (a) Give a definition of an ellipse in terms of foci.
   (b) Write an equation for the ellipse with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

8. (a) Give a definition of a hyperbola in terms of foci.
   (b) Write an equation for the hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.
   (c) Write equations for the asymptotes of the hyperbola in part (b).

9. (a) What is the eccentricity of a conic section?
   (b) What can you say about the eccentricity if the conic section is an ellipse? A hyperbola? A parabola?
   (c) Write a polar equation for a conic section with eccentricity e and directrix $x = d$. What if the directrix is $x = -d$? $y = d$? $y = -d$?

True-False Quiz

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If the parametric curve $x = f(t)$, $y = g(t)$ satisfies $g'(1) = 0$, then it has a horizontal tangent when $t = 1$.

2. If $x = f(t)$ and $y = g(t)$ are twice differentiable, then
$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}$$

3. The length of the curve $x = f(t)$, $y = g(t)$, $a \leqslant t \leqslant b$, is $\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$.

4. If a point is represented by (x, y) in Cartesian coordinates (where $x \neq 0$) and (r, θ) in polar coordinates, then $\theta = \tan^{-1}(y/x)$.

5. The polar curves $r = 1 - \sin 2\theta$ and $r = \sin 2\theta - 1$ have the same graph.

6. The equations $r = 2$, $x^2 + y^2 = 4$, and $x = 2 \sin 3t$, $y = 2 \cos 3t$ $(0 \leqslant t \leqslant 2\pi)$ all have the same graph.

7. The parametric equations $x = t^2$, $y = t^4$ have the same graph as $x = t^3$, $y = t^6$.

8. The graph of $y^2 = 2y + 3x$ is a parabola.

9. A tangent line to a parabola intersects the parabola only once.

10. A hyperbola never intersects its directrix.

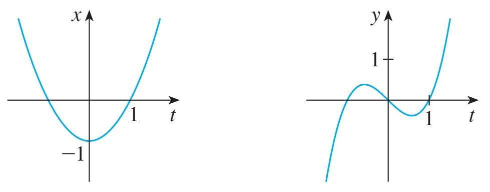
Exercises

1–4 Sketch the parametric curve and eliminate the parameter to find the Cartesian equation of the curve.

1. $x = t^2 + 4t, \quad y = 2 - t, \quad -4 \leqslant t \leqslant 1$
2. $x = 1 + e^{2t}, \quad y = e^t$
3. $x = \cos\theta, \quad y = \sec\theta, \quad 0 \leqslant \theta < \pi/2$
4. $x = 2\cos\theta, \quad y = 1 + \sin\theta$

5. Write three different sets of parametric equations for the curve $y = \sqrt{x}$.

6. Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



7. (a) Plot the point with polar coordinates $(4, 2\pi/3)$. Then find its Cartesian coordinates.
   (b) The Cartesian coordinates of a point are $(-3, 3)$. Find two sets of polar coordinates for the point.

8. Sketch the region consisting of points whose polar coordinates satisfy $1 \leqslant r < 2$ and $\pi/6 \leqslant \theta \leqslant 5\pi/6$.

9–16 Sketch the polar curve.

9. $r = 1 - \cos\theta$
10. $r = \sin 4\theta$
11. $r = \cos 3\theta$
12. $r = 3 + \cos 3\theta$
13. $r = 1 + \cos 2\theta$
14. $r = 2\cos(\theta/2)$
15. $r = \dfrac{3}{1 + 2\sin\theta}$
16. $r = \dfrac{3}{2 - 2\cos\theta}$

17–18 Find a polar equation for the curve represented by the given Cartesian equation.

17. $x + y = 2$
18. $x^2 + y^2 = 2$

19. The curve with polar equation $r = (\sin\theta)/\theta$ is called a **cochleoid**. Use a graph of r as a function of θ in Cartesian coordinates to sketch the cochleoid by hand. Then graph it with a machine to check your sketch.

20. Graph the ellipse $r = 2/(4 - 3\cos\theta)$ and its directrix. Also graph the ellipse obtained by rotation about the origin through an angle $2\pi/3$.

21–24 Find the slope of the tangent line to the given curve at the point corresponding to the specified value of the parameter.

21. $x = \ln t, \; y = 1 + t^2; \quad t = 1$
22. $x = t^3 + 6t + 1, \quad y = 2t - t^2; \quad t = -1$
23. $r = e^{-\theta}; \quad \theta = \pi$
24. $r = 3 + \cos 3\theta; \quad \theta = \pi/2$

25–26 Find dy/dx and d^2y/dx^2.

25. $x = t + \sin t, \quad y = t - \cos t$
26. $x = 1 + t^2, \quad y = t - t^3$

27. Use a graph to estimate the coordinates of the lowest point on the curve $x = t^3 - 3t$, $y = t^2 + t + 1$. Then use calculus to find the exact coordinates.

28. Find the area enclosed by the loop of the curve in Exercise 27.

29. At what points does the curve

$$x = 2a\cos t - a\cos 2t \qquad y = 2a\sin t - a\sin 2t$$

have vertical or horizontal tangents? Use this information to help sketch the curve.

30. Find the area enclosed by the curve in Exercise 29.

31. Find the area enclosed by the curve $r^2 = 9\cos 5\theta$.

32. Find the area enclosed by the inner loop of the curve $r = 1 - 3\sin\theta$.

33. Find the points of intersection of the curves $r = 2$ and $r = 4\cos\theta$.

34. Find the points of intersection of the curves $r = \cot\theta$ and $r = 2\cos\theta$.

35. Find the area of the region that lies inside both of the circles $r = 2\sin\theta$ and $r = \sin\theta + \cos\theta$.

36. Find the area of the region that lies inside the curve $r = 2 + \cos 2\theta$ but outside the curve $r = 2 + \sin\theta$.

37–40 Find the length of the curve.

37. $x = 3t^2, \quad y = 2t^3, \quad 0 \leqslant t \leqslant 2$
38. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$
39. $r = 1/\theta, \quad \pi \leqslant \theta \leqslant 2\pi$
40. $r = \sin^3(\theta/3), \quad 0 \leqslant \theta \leqslant \pi$

Graphing calculator or computer required   CAS Computer algebra system required

41–42 Find the area of the surface obtained by rotating the given curve about the x-axis.

41. $x = 4\sqrt{t}, \quad y = \dfrac{t^3}{3} + \dfrac{1}{2t^2}, \quad 1 \leqslant t \leqslant 4$

42. $x = 2 + 3t, \quad y = \cosh 3t, \quad 0 \leqslant t \leqslant 1$

---

43. The curves defined by the parametric equations

$$x = \frac{t^2 - c}{t^2 + 1} \qquad y = \frac{t(t^2 - c)}{t^2 + 1}$$

are called **strophoids** (from a Greek word meaning "to turn or twist"). Investigate how these curves vary as c varies.

44. A family of curves has polar equations $r^a = |\sin 2\theta|$ where a is a positive number. Investigate how the curves change as a changes.

45–48 Find the foci and vertices and sketch the graph.

45. $\dfrac{x^2}{9} + \dfrac{y^2}{8} = 1$

46. $4x^2 - y^2 = 16$

47. $6y^2 + x - 36y + 55 = 0$

48. $25x^2 + 4y^2 + 50x - 16y = 59$

---

49. Find an equation of the ellipse with foci $(\pm 4, 0)$ and vertices $(\pm 5, 0)$.

50. Find an equation of the parabola with focus $(2, 1)$ and directrix $x = -4$.

51. Find an equation of the hyperbola with foci $(0, \pm 4)$ and asymptotes $y = \pm 3x$.

52. Find an equation of the ellipse with foci $(3, \pm 2)$ and major axis with length 8.

53. Find an equation for the ellipse that shares a vertex and a focus with the parabola $x^2 + y = 100$ and that has its other focus at the origin.

54. Show that if m is any real number, then there are exactly two lines of slope m that are tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ and their equations are $y = mx \pm \sqrt{a^2m^2 + b^2}$.

55. Find a polar equation for the ellipse with focus at the origin, eccentricity $\frac{1}{3}$, and directrix with equation $r = 4 \sec \theta$.

56. Show that the angles between the polar axis and the asymptotes of the hyperbola $r = ed/(1 - e \cos \theta)$, $e > 1$, are given by $\cos^{-1}(\pm 1/e)$.

57. A curve called the **folium of Descartes** is defined by the parametric equations

$$x = \frac{3t}{1 + t^3} \qquad y = \frac{3t^2}{1 + t^3}$$

(a) Show that if (a, b) lies on the curve, then so does (b, a); that is, the curve is symmetric with respect to the line $y = x$. Where does the curve intersect this line?
(b) Find the points on the curve where the tangent lines are horizontal or vertical.
(c) Show that the line $y = -x - 1$ is a slant asymptote.
(d) Sketch the curve.
(e) Show that a Cartesian equation of this curve is $x^3 + y^3 = 3xy$.
(f) Show that the polar equation can be written in the form

$$r = \frac{3 \sec \theta \tan \theta}{1 + \tan^3 \theta}$$

(g) Find the area enclosed by the loop of this curve.
CAS (h) Show that the area of the loop is the same as the area that lies between the asymptote and the infinite branches of the curve. (Use a computer algebra system to evaluate the integral.)

Problems Plus

1. A curve is defined by the parametric equations

$$x = \int_1^t \frac{\cos u}{u}\,du \qquad y = \int_1^t \frac{\sin u}{u}\,du$$

Find the length of the arc of the curve from the origin to the nearest point where there is a vertical tangent line.

2. (a) Find the highest and lowest points on the curve $x^4 + y^4 = x^2 + y^2$.
(b) Sketch the curve. (Notice that it is symmetric with respect to both axes and both of the lines $y = \pm x$, so it suffices to consider $y \geqslant x \geqslant 0$ initially.)
CAS (c) Use polar coordinates and a computer algebra system to find the area enclosed by the curve.

3. What is the smallest viewing rectangle that contains every member of the family of polar curves $r = 1 + c \sin\theta$, where $0 \leqslant c \leqslant 1$? Illustrate your answer by graphing several members of the family in this viewing rectangle.

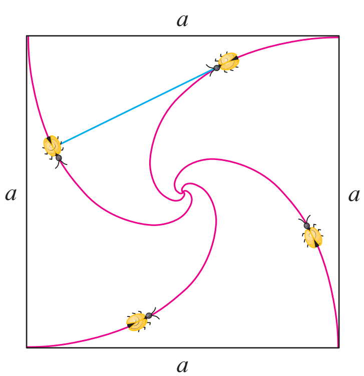


FIGURE FOR PROBLEM 4

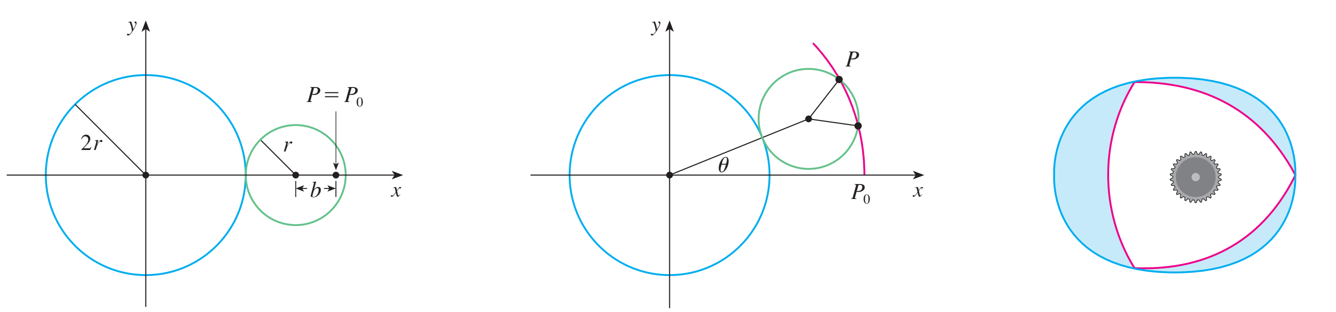
4. Four bugs are placed at the four corners of a square with side length a. The bugs crawl counterclockwise at the same speed and each bug crawls directly toward the next bug at all times. They approach the center of the square along spiral paths.
(a) Find the polar equation of a bug's path assuming the pole is at the center of the square. (Use the fact that the line joining one bug to the next is tangent to the bug's path.)
(b) Find the distance traveled by a bug by the time it meets the other bugs at the center.

5. Show that any tangent line to a hyperbola touches the hyperbola halfway between the points of intersection of the tangent and the asymptotes.

6. A circle C of radius $2r$ has its center at the origin. A circle of radius r rolls without slipping in the counterclockwise direction around C. A point P is located on a fixed radius of the rolling circle at a distance b from its center, $0 < b < r$. [See parts (i) and (ii) of the figure.] Let L be the line from the center of C to the center of the rolling circle and let θ be the angle that L makes with the positive x-axis.
(a) Using θ as a parameter, show that parametric equations of the path traced out by P are

$$x = b\cos 3\theta + 3r\cos\theta \qquad y = b\sin 3\theta + 3r\sin\theta$$

Note: If $b = 0$, the path is a circle of radius $3r$; if $b = r$, the path is an *epicycloid*. The path traced out by P for $0 < b < r$ is called an *epitrochoid*.
(b) Graph the curve for various values of b between 0 and r.
(c) Show that an equilateral triangle can be inscribed in the epitrochoid and that its centroid is on the circle of radius b centered at the origin.
Note: This is the principle of the Wankel rotary engine. When the equilateral triangle rotates with its vertices on the epitrochoid, its centroid sweeps out a circle whose center is at the center of the curve.
(d) In most rotary engines the sides of the equilateral triangles are replaced by arcs of circles centered at the opposite vertices as in part (iii) of the figure. (Then the diameter of the rotor is constant.) Show that the rotor will fit in the epitrochoid if $b \leqslant \frac{3}{2}(2 - \sqrt{3})r$.



(i) (ii) (iii)

11 Infinite Sequences and Series

In the last section of this chapter you are asked to use a series to derive a formula for the velocity of an ocean wave.

Infinite sequences and series were introduced briefly in *A Preview of Calculus* in connection with Zeno's paradoxes and the decimal representation of numbers. Their importance in calculus stems from Newton's idea of representing functions as sums of infinite series. For instance, in finding areas he often integrated a function by first expressing it as a series and then integrating each term of the series. We will pursue his idea in Section 11.10 in order to integrate such functions as e^{-x^2}. (Recall that we have previously been unable to do this.) Many of the functions that arise in mathematical physics and chemistry, such as Bessel functions, are defined as sums of series, so it is important to be familiar with the basic concepts of convergence of infinite sequences and series.

Physicists also use series in another way, as we will see in Section 11.11. In studying fields as diverse as optics, special relativity, and electromagnetism, they analyze phenomena by replacing a function with the first few terms in the series that represents it.

11.1 Sequences

A **sequence** can be thought of as a list of numbers written in a definite order:

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ldots$$

The number a_1 is called the *first term*, a_2 is the *second term*, and in general a_n is the *nth term*. We will deal exclusively with infinite sequences and so each term a_n will have a successor a_{n+1}.

Notice that for every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers. But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

NOTATION The sequence $\{a_1, a_2, a_3, \ldots\}$ is also denoted by

$$\{a_n\} \qquad \text{or} \qquad \{a_n\}_{n=1}^{\infty}$$

EXAMPLE 1 Some sequences can be defined by giving a formula for the nth term. In the following examples we give three descriptions of the sequence: one by using the preceding notation, another by using the defining formula, and a third by writing out the terms of the sequence. Notice that n doesn't have to start at 1.

(a) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ $\qquad a_n = \dfrac{n}{n+1}$ $\qquad \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots, \dfrac{n}{n+1}, \ldots\right\}$

(b) $\left\{\dfrac{(-1)^n(n+1)}{3^n}\right\}$ $\qquad a_n = \dfrac{(-1)^n(n+1)}{3^n}$ $\qquad \left\{-\dfrac{2}{3}, \dfrac{3}{9}, -\dfrac{4}{27}, \dfrac{5}{81}, \ldots, \dfrac{(-1)^n(n+1)}{3^n}, \ldots\right\}$

(c) $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ $\qquad a_n = \sqrt{n-3},\ n \geqslant 3$ $\qquad \{0, 1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n-3}, \ldots\}$

(d) $\left\{\cos\dfrac{n\pi}{6}\right\}_{n=0}^{\infty}$ $\qquad a_n = \cos\dfrac{n\pi}{6},\ n \geqslant 0$ $\qquad \left\{1, \dfrac{\sqrt{3}}{2}, \dfrac{1}{2}, 0, \ldots, \cos\dfrac{n\pi}{6}, \ldots\right\}$

V EXAMPLE 2 Find a formula for the general term a_n of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots\right\}$$

assuming that the pattern of the first few terms continues.

SOLUTION We are given that

$$a_1 = \frac{3}{5} \qquad a_2 = -\frac{4}{25} \qquad a_3 = \frac{5}{125} \qquad a_4 = -\frac{6}{625} \qquad a_5 = \frac{7}{3125}$$

Notice that the numerators of these fractions start with 3 and increase by 1 whenever we go to the next term. The second term has numerator 4, the third term has numerator 5; in general, the nth term will have numerator $n + 2$. The denominators are the powers of 5, so a_n has denominator 5^n. The signs of the terms are alternately positive and negative, so we need to multiply by a power of -1. In Example 1(b) the factor $(-1)^n$ meant we started with a negative term. Here we want to start with a positive term and so we use $(-1)^{n-1}$ or $(-1)^{n+1}$. Therefore

$$a_n = (-1)^{n-1}\frac{n+2}{5^n}$$

EXAMPLE 3 Here are some sequences that don't have a simple defining equation.

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the nth decimal place of the number e, then $\{a_n\}$ is a well-defined sequence whose first few terms are

$$\{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, \ldots\}$$

(c) The **Fibonacci sequence** $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \qquad f_2 = 1 \qquad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Each term is the sum of the two preceding terms. The first few terms are

$$\{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$$

This sequence arose when the 13th-century Italian mathematician known as Fibonacci solved a problem concerning the breeding of rabbits (see Exercise 83).

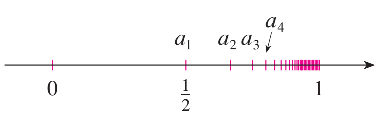


FIGURE 1

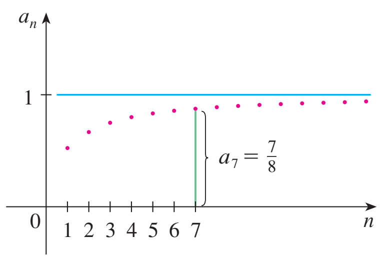


FIGURE 2

A sequence such as the one in Example 1(a), $a_n = n/(n+1)$, can be pictured either by plotting its terms on a number line, as in Figure 1, or by plotting its graph, as in Figure 2. Note that, since a sequence is a function whose domain is the set of positive integers, its graph consists of isolated points with coordinates

$$(1, a_1) \qquad (2, a_2) \qquad (3, a_3) \qquad \ldots \qquad (n, a_n) \qquad \ldots$$

From Figure 1 or Figure 2 it appears that the terms of the sequence $a_n = n/(n+1)$ are approaching 1 as n becomes large. In fact, the difference

$$1 - \frac{n}{n+1} = \frac{1}{n+1}$$

can be made as small as we like by taking n sufficiently large. We indicate this by writing

$$\lim_{n\to\infty} \frac{n}{n+1} = 1$$

In general, the notation

$$\lim_{n\to\infty} a_n = L$$

means that the terms of the sequence $\{a_n\}$ approach L as n becomes large. Notice that the following definition of the limit of a sequence is very similar to the definition of a limit of a function at infinity given in Section 3.4.

1 Definition A sequence $\{a_n\}$ has the **limit** L and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large. If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).

Figure 3 illustrates Definition 1 by showing the graphs of two sequences that have the limit L.

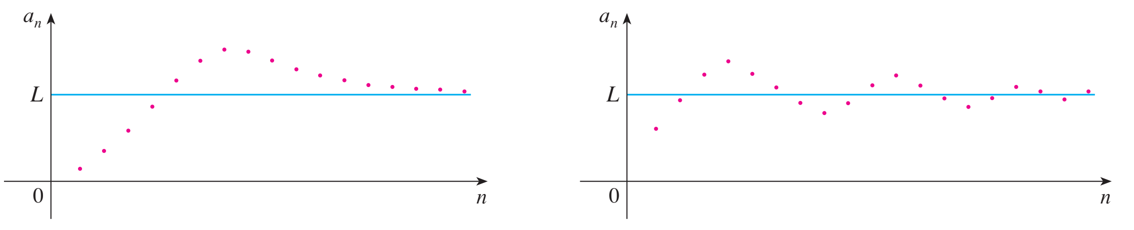


FIGURE 3
Graphs of two sequences with $\lim_{n\to\infty} a_n = L$

A more precise version of Definition 1 is as follows.

Compare this definition with Definition 3.4.5.

2 Definition A sequence $\{a_n\}$ has the **limit** L and we write

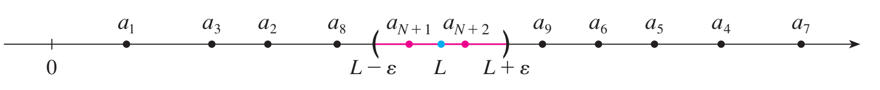
$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if for every $\varepsilon > 0$ there is a corresponding integer N such that

$$\text{if} \qquad n > N \qquad \text{then} \qquad |a_n - L| < \varepsilon$$

Definition 2 is illustrated by Figure 4, in which the terms $a_1, a_2, a_3, \ldots$ are plotted on a number line. No matter how small an interval $(L - \varepsilon, L + \varepsilon)$ is chosen, there exists an N such that all terms of the sequence from a_{N+1} onward must lie in that interval.

FIGURE 4



Another illustration of Definition 2 is given in Figure 5. The points on the graph of $\{a_n\}$ must lie between the horizontal lines $y = L + \varepsilon$ and $y = L - \varepsilon$ if $n > N$. This picture must be valid no matter how small ε is chosen, but usually a smaller ε requires a larger N.

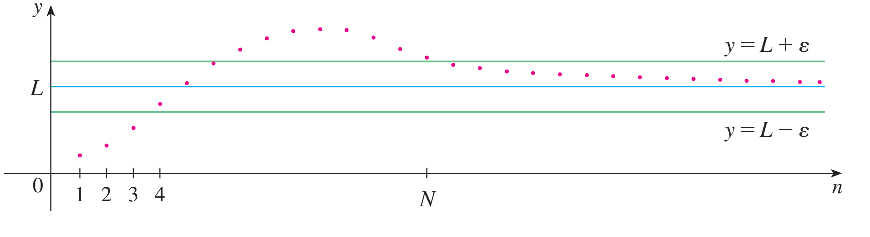


FIGURE 5

If you compare Definition 2 with Definition 3.4.5 you will see that the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{x\to\infty} f(x) = L$ is that n is required to be an integer. Thus we have the following theorem, which is illustrated by Figure 6.

3 Theorem If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

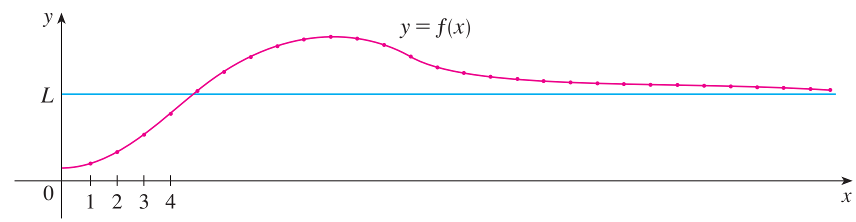


FIGURE 6

In particular, since we know that $\lim_{x\to\infty} (1/x^r) = 0$ when $r > 0$ (Theorem 3.4.4), we have

4
$$\lim_{n\to\infty} \frac{1}{n^r} = 0 \qquad \text{if } r > 0$$

If a_n becomes large as n becomes large, we use the notation $\lim_{n\to\infty} a_n = \infty$. The following precise definition is similar to Definition 3.4.7.

5 Definition $\lim_{n\to\infty} a_n = \infty$ means that for every positive number M there is an integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad a_n > M$$

If $\lim_{n\to\infty} a_n = \infty$, then the sequence $\{a_n\}$ is divergent but in a special way. We say that $\{a_n\}$ diverges to ∞.

The Limit Laws given in Section 1.6 also hold for the limits of sequences and their proofs are similar.

Limit Laws for Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \quad \text{if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p \quad \text{if } p > 0 \text{ and } a_n > 0$$

The Squeeze Theorem can also be adapted for sequences as follows (see Figure 7).

Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

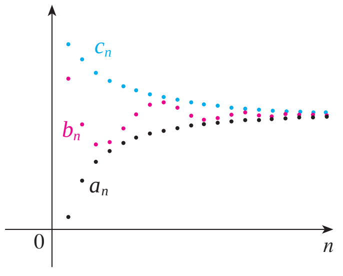


FIGURE 7
The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Another useful fact about limits of sequences is given by the following theorem, whose proof is left as Exercise 87.

6 Theorem If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 4 Find $\lim_{n\to\infty} \dfrac{n}{n+1}$.

SOLUTION The method is similar to the one we used in Section 3.4: Divide numerator and denominator by the highest power of n that occurs in the denominator and then use the Limit Laws.

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\dfrac{1}{n}} = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \dfrac{1}{n}}$$

$$= \frac{1}{1+0} = 1$$

This shows that the guess we made earlier from Figures 1 and 2 was correct.

Here we used Equation 4 with $r = 1$.

EXAMPLE 5 Is the sequence $a_n = \dfrac{n}{\sqrt{10+n}}$ convergent or divergent?

SOLUTION As in Example 4, we divide numerator and denominator by n:

$$\lim_{n\to\infty} \frac{n}{\sqrt{10+n}} = \lim_{n\to\infty} \frac{1}{\sqrt{\dfrac{10}{n^2} + \dfrac{1}{n}}} = \infty$$

because the numerator is constant and the denominator approaches 0. So $\{a_n\}$ is divergent.

EXAMPLE 6 Calculate $\lim_{n\to\infty} \dfrac{\ln n}{n}$.

SOLUTION Notice that both numerator and denominator approach infinity as $n \to \infty$. We can't apply l'Hospital's Rule directly because it applies not to sequences but to functions of a real variable. However, we can apply l'Hospital's Rule to the related function $f(x) = (\ln x)/x$ and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

Therefore, by Theorem 3, we have

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0$$

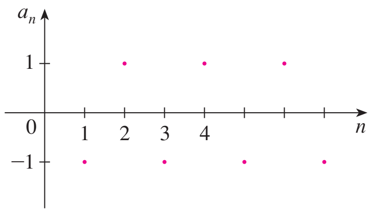


FIGURE 8

EXAMPLE 7 Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

SOLUTION If we write out the terms of the sequence, we obtain

$$\{-1, 1, -1, 1, -1, 1, -1, \ldots\}$$

The graph of this sequence is shown in Figure 8. Since the terms oscillate between 1 and -1 infinitely often, a_n does not approach any number. Thus $\lim_{n\to\infty} (-1)^n$ does not exist; that is, the sequence $\{(-1)^n\}$ is divergent.

The graph of the sequence in Example 8 is shown in Figure 9 and supports our answer.

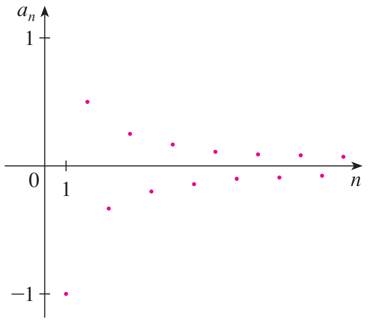


FIGURE 9

EXAMPLE 8 Evaluate $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

SOLUTION We first calculate the limit of the absolute value:

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, by Theorem 6,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

The following theorem says that if we apply a continuous function to the terms of a convergent sequence, the result is also convergent. The proof is left as Exercise 88.

7 Theorem If $\displaystyle\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

EXAMPLE 9 Find $\displaystyle\lim_{n\to\infty} \sin(\pi/n)$.

SOLUTION Because the sine function is continuous at 0, Theorem 7 enables us to write

$$\lim_{n\to\infty} \sin(\pi/n) = \sin\left(\lim_{n\to\infty} (\pi/n)\right) = \sin 0 = 0$$

Creating Graphs of Sequences

Some computer algebra systems have special commands that enable us to create sequences and graph them directly. With most graphing calculators, however, sequences can be graphed by using parametric equations. For instance, the sequence in Example 10 can be graphed by entering the parametric equations

$$x = t \qquad y = t!/t^t$$

and graphing in dot mode, starting with $t = 1$ and setting the t-step equal to 1. The result is shown in Figure 10.

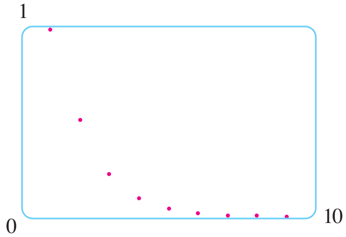


FIGURE 10

V EXAMPLE 10 Discuss the convergence of the sequence $a_n = n!/n^n$, where $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$.

SOLUTION Both numerator and denominator approach infinity as $n \to \infty$ but here we have no corresponding function for use with l'Hospital's Rule ($x!$ is not defined when x is not an integer). Let's write out a few terms to get a feeling for what happens to a_n as n gets large:

$$a_1 = 1 \qquad a_2 = \frac{1 \cdot 2}{2 \cdot 2} \qquad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}$$

8
$$a_n = \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot n \cdot \cdots \cdot n}$$

It appears from these expressions and the graph in Figure 10 that the terms are decreasing and perhaps approach 0. To confirm this, observe from Equation 8 that

$$a_n = \frac{1}{n}\left(\frac{2 \cdot 3 \cdot \cdots \cdot n}{n \cdot n \cdot \cdots \cdot n}\right)$$

Notice that the expression in parentheses is at most 1 because the numerator is less than (or equal to) the denominator. So

$$0 < a_n \leqslant \frac{1}{n}$$

We know that $1/n \to 0$ as $n \to \infty$. Therefore $a_n \to 0$ as $n \to \infty$ by the Squeeze Theorem.

V EXAMPLE 11 For what values of r is the sequence $\{r^n\}$ convergent?

SOLUTION We know from Section 3.4 and the graphs of the exponential functions in Section 6.2 (or Section 6.4*) that $\lim_{x\to\infty} a^x = \infty$ for $a > 1$ and $\lim_{x\to\infty} a^x = 0$ for $0 < a < 1$. Therefore, putting $a = r$ and using Theorem 3, we have

$$\lim_{n\to\infty} r^n = \begin{cases} \infty & \text{if } r > 1 \\ 0 & \text{if } 0 < r < 1 \end{cases}$$

It is obvious that

$$\lim_{n\to\infty} 1^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} 0^n = 0$$

If $-1 < r < 0$, then $0 < |r| < 1$, so

$$\lim_{n\to\infty} |r^n| = \lim_{n\to\infty} |r|^n = 0$$

and therefore $\lim_{n\to\infty} r^n = 0$ by Theorem 6. If $r \leqslant -1$, then $\{r^n\}$ diverges as in Example 7. Figure 11 shows the graphs for various values of r. (The case $r = -1$ is shown in Figure 8.)

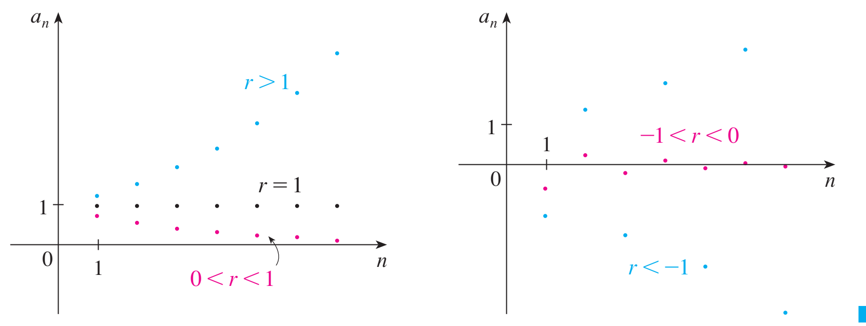


FIGURE 11
The sequence $a_n = r^n$

The results of Example 11 are summarized for future use as follows.

9 The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

10 Definition A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geqslant 1$, that is, $a_1 < a_2 < a_3 < \cdots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geqslant 1$. A sequence is **monotonic** if it is either increasing or decreasing.

EXAMPLE 12 The sequence $\left\{\dfrac{3}{n+5}\right\}$ is decreasing because

The right side is smaller because it has a larger denominator.

$$\frac{3}{n+5} > \frac{3}{(n+1)+5} = \frac{3}{n+6}$$

and so $a_n > a_{n+1}$ for all $n \geqslant 1$.

EXAMPLE 13 Show that the sequence $a_n = \dfrac{n}{n^2+1}$ is decreasing.

SOLUTION 1 We must show that $a_{n+1} < a_n$, that is,

$$\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

This inequality is equivalent to the one we get by cross-multiplication:

$$\begin{aligned}
\frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1} \quad &\iff \quad (n+1)(n^2+1) < n[(n+1)^2+1] \\
&\iff \quad n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n \\
&\iff \quad 1 < n^2 + n
\end{aligned}$$

Since $n \geqslant 1$, we know that the inequality $n^2 + n > 1$ is true. Therefore $a_{n+1} < a_n$ and so $\{a_n\}$ is decreasing.

SOLUTION 2 Consider the function $f(x) = \dfrac{x}{x^2+1}$:

$$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} < 0 \qquad \text{whenever } x^2 > 1$$

Thus f is decreasing on $(1, \infty)$ and so $f(n) > f(n+1)$. Therefore $\{a_n\}$ is decreasing.

11 Definition A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1$$

It is **bounded below** if there is a number m such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1$$

If it is bounded above and below, then $\{a_n\}$ is a **bounded sequence**.

For instance, the sequence $a_n = n$ is bounded below $(a_n > 0)$ but not above. The sequence $a_n = n/(n+1)$ is bounded because $0 < a_n < 1$ for all n.

We know that not every bounded sequence is convergent [for instance, the sequence $a_n = (-1)^n$ satisfies $-1 \leqslant a_n \leqslant 1$ but is divergent from Example 7] and not every monotonic sequence is convergent ($a_n = n \to \infty$). But if a sequence is both bounded *and* monotonic, then it must be convergent. This fact is proved as Theorem 12, but intuitively you can understand why it is true by looking at Figure 12. If $\{a_n\}$ is increasing and $a_n \leqslant M$ for all n, then the terms are forced to crowd together and approach some number L.

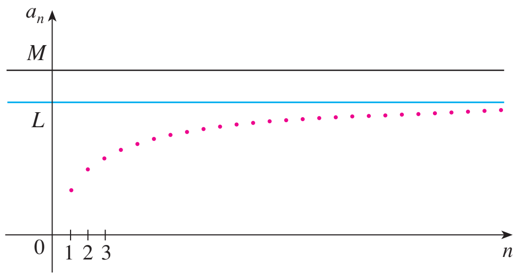


FIGURE 12

The proof of Theorem 12 is based on the **Completeness Axiom** for the set $\mathbb{R}$ of real numbers, which says that if S is a nonempty set of real numbers that has an upper bound M ($x \leqslant M$ for all x in S), then S has a **least upper bound** b. (This means that b is an upper bound for S, but if M is any other upper bound, then $b \leqslant M$.) The Completeness Axiom is an expression of the fact that there is no gap or hole in the real number line.

12 Monotonic Sequence Theorem Every bounded, monotonic sequence is convergent.

PROOF Suppose $\{a_n\}$ is an increasing sequence. Since $\{a_n\}$ is bounded, the set $S = \{a_n \mid n \geqslant 1\}$ has an upper bound. By the Completeness Axiom it has a least upper bound L. Given $\varepsilon > 0$, $L - \varepsilon$ is *not* an upper bound for S (since L is the *least* upper bound). Therefore

$$a_N > L - \varepsilon \qquad \text{for some integer } N$$

But the sequence is increasing so $a_n \geqslant a_N$ for every $n > N$. Thus if $n > N$, we have

$$a_n > L - \varepsilon$$

so

$$0 \leqslant L - a_n < \varepsilon$$

since $a_n \leqslant L$. Thus

$$|L - a_n| < \varepsilon \qquad \text{whenever } n > N$$

so $\lim_{n\to\infty} a_n = L$.

A similar proof (using the greatest lower bound) works if $\{a_n\}$ is decreasing.

The proof of Theorem 12 shows that a sequence that is increasing and bounded above is convergent. (Likewise, a decreasing sequence that is bounded below is convergent.) This fact is used many times in dealing with infinite series.

EXAMPLE 14 Investigate the sequence $\{a_n\}$ defined by the *recurrence relation*

$$a_1 = 2 \qquad a_{n+1} = \tfrac{1}{2}(a_n + 6) \qquad \text{for } n = 1, 2, 3, \ldots$$

SOLUTION We begin by computing the first several terms:

$$a_1 = 2 \qquad a_2 = \tfrac{1}{2}(2 + 6) = 4 \qquad a_3 = \tfrac{1}{2}(4 + 6) = 5$$

$$a_4 = \tfrac{1}{2}(5 + 6) = 5.5 \qquad a_5 = 5.75 \qquad a_6 = 5.875$$

$$a_7 = 5.9375 \qquad a_8 = 5.96875 \qquad a_9 = 5.984375$$

Mathematical induction is often used in dealing with recursive sequences. See page 98 for a discussion of the Principle of Mathematical Induction.

These initial terms suggest that the sequence is increasing and the terms are approaching 6. To confirm that the sequence is increasing, we use mathematical induction to show that $a_{n+1} > a_n$ for all $n \geqslant 1$. This is true for $n = 1$ because $a_2 = 4 > a_1$. If we assume that it is true for $n = k$, then we have

$$a_{k+1} > a_k$$

so $$a_{k+1} + 6 > a_k + 6$$

and $$\tfrac{1}{2}(a_{k+1} + 6) > \tfrac{1}{2}(a_k + 6)$$

Thus $$a_{k+2} > a_{k+1}$$

We have deduced that $a_{n+1} > a_n$ is true for $n = k + 1$. Therefore the inequality is true for all n by induction.

Next we verify that $\{a_n\}$ is bounded by showing that $a_n < 6$ for all n. (Since the sequence is increasing, we already know that it has a lower bound: $a_n \geqslant a_1 = 2$ for all n.) We know that $a_1 < 6$, so the assertion is true for $n = 1$. Suppose it is true for $n = k$. Then

$$a_k < 6$$

so $$a_k + 6 < 12$$

and $$\tfrac{1}{2}(a_k + 6) < \tfrac{1}{2}(12) = 6$$

Thus $$a_{k+1} < 6$$

This shows, by mathematical induction, that $a_n < 6$ for all n.

Since the sequence $\{a_n\}$ is increasing and bounded, Theorem 12 guarantees that it has a limit. The theorem doesn't tell us what the value of the limit is. But now that we know $L = \lim_{n\to\infty} a_n$ exists, we can use the given recurrence relation to write

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \tfrac{1}{2}(a_n + 6) = \tfrac{1}{2}\left(\lim_{n\to\infty} a_n + 6\right) = \tfrac{1}{2}(L + 6)$$

A proof of this fact is requested in Exercise 70.

Since $a_n \to L$, it follows that $a_{n+1} \to L$ too (as $n \to \infty$, $n + 1 \to \infty$ also). So we have

$$L = \tfrac{1}{2}(L + 6)$$

Solving this equation for L, we get $L = 6$, as we predicted.

11.1 Exercises

1. (a) What is a sequence?
(b) What does it mean to say that $\lim_{n\to\infty} a_n = 8$?
(c) What does it mean to say that $\lim_{n\to\infty} a_n = \infty$?

2. (a) What is a convergent sequence? Give two examples.
(b) What is a divergent sequence? Give two examples.

3–12 List the first five terms of the sequence.

3. $a_n = \dfrac{2n}{n^2+1}$

4. $a_n = \dfrac{3^n}{1+2^n}$

5. $a_n = \dfrac{(-1)^{n-1}}{5^n}$

6. $a_n = \cos\dfrac{n\pi}{2}$

7. $a_n = \dfrac{1}{(n+1)!}$

8. $a_n = \dfrac{(-1)^n n}{n!+1}$

9. $a_1 = 1, \quad a_{n+1} = 5a_n - 3$

10. $a_1 = 6, \quad a_{n+1} = \dfrac{a_n}{n}$

11. $a_1 = 2, \quad a_{n+1} = \dfrac{a_n}{1+a_n}$

12. $a_1 = 2, \quad a_2 = 1, \quad a_{n+1} = a_n - a_{n-1}$

13–18 Find a formula for the general term a_n of the sequence, assuming that the pattern of the first few terms continues.

13. $\left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots\right\}$

14. $\left\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots\right\}$

15. $\left\{-3, 2, -\frac{4}{3}, \frac{8}{9}, -\frac{16}{27}, \dots\right\}$

16. $\{5, 8, 11, 14, 17, \dots\}$

17. $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots\right\}$

18. $\{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

19–22 Calculate, to four decimal places, the first ten terms of the sequence and use them to plot the graph of the sequence by hand. Does the sequence appear to have a limit? If so, calculate it. If not, explain why.

19. $a_n = \dfrac{3n}{1+6n}$

20. $a_n = 2 + \dfrac{(-1)^n}{n}$

21. $a_n = 1 + \left(-\frac{1}{2}\right)^n$

22. $a_n = 1 + \dfrac{10^n}{9^n}$

23–56 Determine whether the sequence converges or diverges. If it converges, find the limit.

23. $a_n = 1 - (0.2)^n$

24. $a_n = \dfrac{n^3}{n^3+1}$

25. $a_n = \dfrac{3+5n^2}{n+n^2}$

26. $a_n = \dfrac{n^3}{n+1}$

27. $a_n = e^{1/n}$

28. $a_n = \dfrac{3^{n+2}}{5^n}$

29. $a_n = \tan\left(\dfrac{2n\pi}{1+8n}\right)$

30. $a_n = \sqrt{\dfrac{n+1}{9n+1}}$

31. $a_n = \dfrac{n^2}{\sqrt{n^3+4n}}$

32. $a_n = e^{2n/(n+2)}$

33. $a_n = \dfrac{(-1)^n}{2\sqrt{n}}$

34. $a_n = \dfrac{(-1)^{n+1} n}{n+\sqrt{n}}$

35. $a_n = \cos(n/2)$

36. $a_n = \cos(2/n)$

37. $\left\{\dfrac{(2n-1)!}{(2n+1)!}\right\}$

38. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$

39. $\left\{\dfrac{e^n + e^{-n}}{e^{2n}-1}\right\}$

40. $a_n = \dfrac{\tan^{-1} n}{n}$

41. $\{n^2 e^{-n}\}$

42. $a_n = \ln(n+1) - \ln n$

43. $a_n = \dfrac{\cos^2 n}{2^n}$

44. $a_n = \sqrt[n]{2^{1+3n}}$

45. $a_n = n\sin(1/n)$

46. $a_n = 2^{-n}\cos n\pi$

47. $a_n = \left(1 + \dfrac{2}{n}\right)^n$

48. $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$

49. $a_n = \ln(2n^2+1) - \ln(n^2+1)$

50. $a_n = \dfrac{(\ln n)^2}{n}$

51. $a_n = \arctan(\ln n)$

52. $a_n = n - \sqrt{n+1}\sqrt{n+3}$

53. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \dots\}$

54. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$

55. $a_n = \dfrac{n!}{2^n}$

56. $a_n = \dfrac{(-3)^n}{n!}$

57–63 Use a graph of the sequence to decide whether the sequence is convergent or divergent. If the sequence is convergent, guess the value of the limit from the graph and then prove your guess. (See the margin note on page 719 for advice on graphing sequences.)

57. $a_n = 1 + (-2/e)^n$

58. $a_n = \sqrt{n}\,\sin(\pi/\sqrt{n})$

59. $a_n = \sqrt{\dfrac{3 + 2n^2}{8n^2 + n}}$

60. $a_n = \sqrt[n]{3^n + 5^n}$

61. $a_n = \dfrac{n^2 \cos n}{1 + n^2}$

62. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{n!}$

63. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n)^n}$

64. (a) Determine whether the sequence defined as follows is convergent or divergent:

$$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n \geqslant 1$$

(b) What happens if the first term is $a_1 = 2$?

65. If \$1000 is invested at 6% interest, compounded annually, then after n years the investment is worth $a_n = 1000(1.06)^n$ dollars.
(a) Find the first five terms of the sequence $\{a_n\}$.
(b) Is the sequence convergent or divergent? Explain.

66. If you deposit \$100 at the end of every month into an account that pays 3% interest per year compounded monthly, the amount of interest accumulated after n months is given by the sequence

$$I_n = 100\left(\frac{1.0025^n - 1}{0.0025} - n\right)$$

(a) Find the first six terms of the sequence.
(b) How much interest will you have earned after two years?

67. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
(a) Show that the catfish population P_n after n months is given recursively by

$$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000$$

(b) How many catfish are in the pond after six months?

68. Find the first 40 terms of the sequence defined by

$$a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$$

and $a_1 = 11$. Do the same if $a_1 = 25$. Make a conjecture about this type of sequence.

69. For what values of r is the sequence $\{nr^n\}$ convergent?

70. (a) If $\{a_n\}$ is convergent, show that

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

(b) A sequence $\{a_n\}$ is defined by $a_1 = 1$ and $a_{n+1} = 1/(1 + a_n)$ for $n \geqslant 1$. Assuming that $\{a_n\}$ is convergent, find its limit.

71. Suppose you know that $\{a_n\}$ is a decreasing sequence and all its terms lie between the numbers 5 and 8. Explain why the sequence has a limit. What can you say about the value of the limit?

72–78 Determine whether the sequence is increasing, decreasing, or not monotonic. Is the sequence bounded?

72. $a_n = (-2)^{n+1}$

73. $a_n = \dfrac{1}{2n + 3}$

74. $a_n = \dfrac{2n - 3}{3n + 4}$

75. $a_n = n(-1)^n$

76. $a_n = ne^{-n}$

77. $a_n = \dfrac{n}{n^2 + 1}$

78. $a_n = n + \dfrac{1}{n}$

79. Find the limit of the sequence

$$\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$$

80. A sequence $\{a_n\}$ is given by $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$.
(a) By induction or otherwise, show that $\{a_n\}$ is increasing and bounded above by 3. Apply the Monotonic Sequence Theorem to show that $\lim_{n\to\infty} a_n$ exists.
(b) Find $\lim_{n\to\infty} a_n$.

81. Show that the sequence defined by

$$a_1 = 1 \qquad a_{n+1} = 3 - \frac{1}{a_n}$$

is increasing and $a_n < 3$ for all n. Deduce that $\{a_n\}$ is convergent and find its limit.

82. Show that the sequence defined by

$$a_1 = 2 \qquad a_{n+1} = \frac{1}{3 - a_n}$$

satisfies $0 < a_n \leqslant 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.

83. (a) Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the nth month? Show that the answer is f_n, where $\{f_n\}$ is the Fibonacci sequence defined in Example 3(c).
(b) Let $a_n = f_{n+1}/f_n$ and show that $a_{n-1} = 1 + 1/a_{n-2}$. Assuming that $\{a_n\}$ is convergent, find its limit.

84. (a) Let $a_1 = a$, $a_2 = f(a)$, $a_3 = f(a_2) = f(f(a))$, $\dots$, $a_{n+1} = f(a_n)$, where f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.
(b) Illustrate part (a) by taking $f(x) = \cos x$, $a = 1$, and estimating the value of L to five decimal places.

85. (a) Use a graph to guess the value of the limit

$$\lim_{n\to\infty} \frac{n^5}{n!}$$

(b) Use a graph of the sequence in part (a) to find the smallest values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.001$ in Definition 2.

86. Use Definition 2 directly to prove that $\lim_{n\to\infty} r^n = 0$ when $|r| < 1$.

87. Prove Theorem 6.
[*Hint:* Use either Definition 2 or the Squeeze Theorem.]

88. Prove Theorem 7.

89. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\lim_{n\to\infty} (a_n b_n) = 0$.

90. Let $a_n = \left(1 + \dfrac{1}{n}\right)^n$.
(a) Show that if $0 \leqslant a < b$, then

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

(b) Deduce that $b^n[(n + 1)a - nb] < a^{n+1}$.
(c) Use $a = 1 + 1/(n + 1)$ and $b = 1 + 1/n$ in part (b) to show that $\{a_n\}$ is increasing.
(d) Use $a = 1$ and $b = 1 + 1/(2n)$ in part (b) to show that $a_{2n} < 4$.
(e) Use parts (c) and (d) to show that $a_n < 4$ for all n.
(f) Use Theorem 12 to show that $\lim_{n\to\infty} (1 + 1/n)^n$ exists. (The limit is e. See Equation 6.4.9 or 6.4*.9.)

91. Let a and b be positive numbers with $a > b$. Let a_1 be their arithmetic mean and b_1 their geometric mean:

$$a_1 = \frac{a + b}{2} \qquad b_1 = \sqrt{ab}$$

Repeat this process so that, in general,

$$a_{n+1} = \frac{a_n + b_n}{2} \qquad b_{n+1} = \sqrt{a_n b_n}$$

(a) Use mathematical induction to show that

$$a_n > a_{n+1} > b_{n+1} > b_n$$

(b) Deduce that both $\{a_n\}$ and $\{b_n\}$ are convergent.
(c) Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Gauss called the common value of these limits the **arithmetic-geometric mean** of the numbers a and b.

92. (a) Show that if $\lim_{n\to\infty} a_{2n} = L$ and $\lim_{n\to\infty} a_{2n+1} = L$, then $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = L$.
(b) If $a_1 = 1$ and

$$a_{n+1} = 1 + \frac{1}{1 + a_n}$$

find the first eight terms of the sequence $\{a_n\}$. Then use part (a) to show that $\lim_{n\to\infty} a_n = \sqrt{2}$. This gives the **continued fraction expansion**

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}$$

93. The size of an undisturbed fish population has been modeled by the formula

$$p_{n+1} = \frac{bp_n}{a + p_n}$$

where p_n is the fish population after n years and a and b are positive constants that depend on the species and its environment. Suppose that the population in year 0 is $p_0 > 0$.
(a) Show that if $\{p_n\}$ is convergent, then the only possible values for its limit are 0 and $b - a$.
(b) Show that $p_{n+1} < (b/a)p_n$.
(c) Use part (b) to show that if $a > b$, then $\lim_{n\to\infty} p_n = 0$; in other words, the population dies out.
(d) Now assume that $a < b$. Show that if $p_0 < b - a$, then $\{p_n\}$ is increasing and $0 < p_n < b - a$. Show also that if $p_0 > b - a$, then $\{p_n\}$ is decreasing and $p_n > b - a$. Deduce that if $a < b$, then $\lim_{n\to\infty} p_n = b - a$.

LABORATORY PROJECT CAS LOGISTIC SEQUENCES

A sequence that arises in ecology as a model for population growth is defined by the **logistic difference equation**

$$p_{n+1} = kp_n(1 - p_n)$$

where p_n measures the size of the population of the nth generation of a single species. To keep the numbers manageable, p_n is a fraction of the maximal size of the population, so $0 \leqslant p_n \leqslant 1$. Notice that the form of this equation is similar to the logistic differential equation in Section 9.4. The discrete model—with sequences instead of continuous functions—is preferable for modeling insect populations, where mating and death occur in a periodic fashion.

An ecologist is interested in predicting the size of the population as time goes on, and asks these questions: Will it stabilize at a limiting value? Will it change in a cyclical fashion? Or will it exhibit random behavior?

Write a program to compute the first n terms of this sequence starting with an initial population p_0, where $0 < p_0 < 1$. Use this program to do the following.

1. Calculate 20 or 30 terms of the sequence for $p_0 = \frac{1}{2}$ and for two values of k such that $1 < k < 3$. Graph each sequence. Do the sequences appear to converge? Repeat for a different value of p_0 between 0 and 1. Does the limit depend on the choice of p_0? Does it depend on the choice of k?

2. Calculate terms of the sequence for a value of k between 3 and 3.4 and plot them. What do you notice about the behavior of the terms?

3. Experiment with values of k between 3.4 and 3.5. What happens to the terms?

4. For values of k between 3.6 and 4, compute and plot at least 100 terms and comment on the behavior of the sequence. What happens if you change p_0 by 0.001? This type of behavior is called *chaotic* and is exhibited by insect populations under certain conditions.

CAS Computer algebra system required

11.2 Series

What do we mean when we express a number as an infinite decimal? For instance, what does it mean to write

The current record is that π has been computed to 2,576,980,370,000 (more than two trillion) decimal places by T. Daisuke and his team.

$$\pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ldots$$

The convention behind our decimal notation is that any number can be written as an infinite sum. Here it means that

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

where the three dots $(\cdots)$ indicate that the sum continues forever, and the more terms we add, the closer we get to the actual value of π.

In general, if we try to add the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an expression of the form

$$\boxed{1} \qquad a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

which is called an **infinite series** (or just a **series**) and is denoted, for short, by the symbol

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \qquad \sum a_n$$

Does it make sense to talk about the sum of infinitely many terms?

It would be impossible to find a finite sum for the series

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

because if we start adding the terms we get the cumulative sums 1, 3, 6, 10, 15, 21, . . . and, after the nth term, we get $n(n + 1)/2$, which becomes very large as n increases.

However, if we start to add the terms of the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots + \frac{1}{2^n} + \cdots$$

n	Sum of first n terms
1	0.50000000
2	0.75000000
3	0.87500000
4	0.93750000
5	0.96875000
6	0.98437500
7	0.99218750
10	0.99902344
15	0.99996948
20	0.99999905
25	0.99999997

we get $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \frac{63}{64}, \ldots, 1 - 1/2^n, \ldots$. The table shows that as we add more and more terms, these *partial sums* become closer and closer to 1. (See also Figure 11 in *A Preview of Calculus,* page 6.) In fact, by adding sufficiently many terms of the series we can make the partial sums as close as we like to 1. So it seems reasonable to say that the sum of this infinite series is 1 and to write

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

We use a similar idea to determine whether or not a general series $\boxed{1}$ has a sum. We consider the **partial sums**

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_4 = a_1 + a_2 + a_3 + a_4$$

and, in general,

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit. If $\lim_{n\to\infty} s_n = s$ exists (as a finite number), then, as in the preceding example, we call it the sum of the infinite series $\Sigma\, a_n$.

2 Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$, let s_n denote its nth partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\Sigma\, a_n$ is called **convergent** and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

Thus the sum of a series is the limit of the sequence of partial sums. So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s. Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Compare with the improper integral

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t f(x)\,dx$$

To find this integral we integrate from 1 to t and then let $t \to \infty$. For a series, we sum from 1 to n and then let $n \to \infty$.

EXAMPLE 1 Suppose we know that the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = a_1 + a_2 + \cdots + a_n = \frac{2n}{3n+5}$$

Then the sum of the series is the limit of the sequence $\{s_n\}$:

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n}{3n+5} = \lim_{n\to\infty} \frac{2}{3 + \dfrac{5}{n}} = \frac{2}{3}$$

In Example 1 we were *given* an expression for the sum of the first n terms, but it's usually not easy to *find* such an expression. In Example 2, however, we look at a famous series for which we *can* find an explicit formula for s_n.

EXAMPLE 2 An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (We have already considered the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$ on page 728.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \qquad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Figure 1 provides a geometric demonstration of the result in Example 2. If the triangles are constructed as shown and s is the sum of the series, then, by similar triangles,

$$\frac{s}{a} = \frac{a}{a - ar} \quad \text{so} \quad s = \frac{a}{1 - r}$$

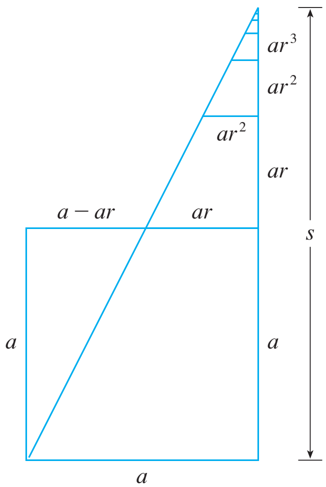


FIGURE 1

In words: The sum of a convergent geometric series is

$$\frac{\text{first term}}{1 - \text{common ratio}}$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

$$\boxed{3} \qquad s_n = \frac{a(1 - r^n)}{1 - r}$$

If $-1 < r < 1$, we know from (11.1.9) that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r}\lim_{n\to\infty} r^n = \frac{a}{1 - r}$$

Thus when $|r| < 1$ the geometric series is convergent and its sum is $a/(1 - r)$.

If $r \leqslant -1$ or $r > 1$, the sequence $\{r^n\}$ is divergent by (11.1.9) and so, by Equation 3, $\lim_{n\to\infty} s_n$ does not exist. Therefore the geometric series diverges in those cases.

We summarize the results of Example 2 as follows.

$\boxed{4}$ The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

V EXAMPLE 3 Find the sum of the geometric series

$$5 - \tfrac{10}{3} + \tfrac{20}{9} - \tfrac{40}{27} + \cdots$$

SOLUTION The first term is $a = 5$ and the common ratio is $r = -\frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series is convergent by $\boxed{4}$ and its sum is

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots = \frac{5}{1 - \left(-\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

What do we really mean when we say that the sum of the series in Example 3 is 3? Of course, we can't literally add an infinite number of terms, one by one. But, according to Definition 2, the total sum is the limit of the sequence of partial sums. So, by taking the sum of sufficiently many terms, we can get as close as we like to the number 3. The table shows the first ten partial sums s_n and the graph in Figure 2 shows how the sequence of partial sums approaches 3.

n	s_n
1	5.000000
2	1.666667
3	3.888889
4	2.407407
5	3.395062
6	2.736626
7	3.175583
8	2.882945
9	3.078037
10	2.947975

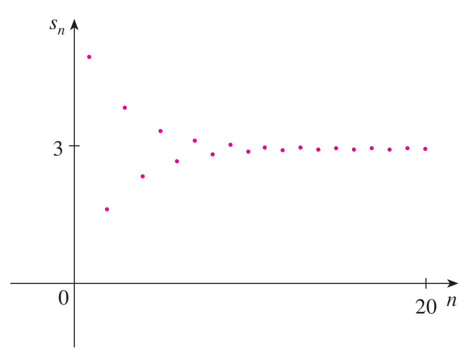


FIGURE 2

EXAMPLE 4 Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

SOLUTION Let's rewrite the nth term of the series in the form ar^{n-1}:

$$\sum_{n=1}^{\infty} 2^{2n}3^{1-n} = \sum_{n=1}^{\infty} (2^2)^n 3^{-(n-1)} = \sum_{n=1}^{\infty} \frac{4^n}{3^{n-1}} = \sum_{n=1}^{\infty} 4\left(\tfrac{4}{3}\right)^{n-1}$$

Another way to identify a and r is to write out the first few terms:

$$4 + \tfrac{16}{3} + \tfrac{64}{9} + \cdots$$

We recognize this series as a geometric series with $a = 4$ and $r = \frac{4}{3}$. Since $r > 1$, the series diverges by $\boxed{4}$.

V EXAMPLE 5 Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

SOLUTION

$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \cdots$$

After the first term we have a geometric series with $a = 17/10^3$ and $r = 1/10^2$. Therefore

$$2.3\overline{17} = 2.3 + \frac{\dfrac{17}{10^3}}{1 - \dfrac{1}{10^2}} = 2.3 + \frac{\dfrac{17}{1000}}{\dfrac{99}{100}}$$

$$= \frac{23}{10} + \frac{17}{990} = \frac{1147}{495}$$

EXAMPLE 6 Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

SOLUTION Notice that this series starts with $n = 0$ and so the first term is $x^0 = 1$. (With series, we adopt the convention that $x^0 = 1$ even when $x = 0$.) Thus

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

TEC Module 11.2 explores a series that depends on an angle θ in a triangle and enables you to see how rapidly the series converges when θ varies.

This is a geometric series with $a = 1$ and $r = x$. Since $|r| = |x| < 1$, it converges and $\boxed{4}$ gives

$$\boxed{5} \qquad \sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}$$

EXAMPLE 7 Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, and find its sum.

SOLUTION This is not a geometric series, so we go back to the definition of a convergent series and compute the partial sums.

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

We can simplify this expression if we use the partial fraction decomposition

$$\frac{1}{i(i+1)} = \frac{1}{i} - \frac{1}{i+1}$$

(see Section 7.4). Thus we have

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right) \\ &= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Notice that the terms cancel in pairs. This is an example of a **telescoping sum**: Because of all the cancellations, the sum collapses (like a pirate's collapsing telescope) into just two terms.

and so
$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1 - 0 = 1$$

Therefore the given series is convergent and

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

Figure 3 illustrates Example 7 by showing the graphs of the sequence of terms $a_n = 1/[n(n+1)]$ and the sequence $\{s_n\}$ of partial sums. Notice that $a_n \to 0$ and $s_n \to 1$. See Exercises 76 and 77 for two geometric interpretations of Example 7.

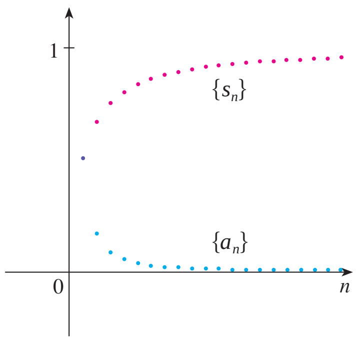


FIGURE 3

V EXAMPLE 8 Show that the **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

is divergent.

SOLUTION For this particular series it's convenient to consider the partial sums $s_2, s_4, s_8, s_{16}, s_{32}, \ldots$ and show that they become large.

$$s_2 = 1 + \tfrac{1}{2}$$

$$s_4 = 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) > 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = 1 + \tfrac{2}{2}$$

$$\begin{aligned} s_8 &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{3}{2} \end{aligned}$$

$$\begin{aligned} s_{16} &= 1 + \tfrac{1}{2} + \left(\tfrac{1}{3} + \tfrac{1}{4}\right) + \left(\tfrac{1}{5} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{9} + \cdots + \tfrac{1}{16}\right) \\ &> 1 + \tfrac{1}{2} + \left(\tfrac{1}{4} + \tfrac{1}{4}\right) + \left(\tfrac{1}{8} + \cdots + \tfrac{1}{8}\right) + \left(\tfrac{1}{16} + \cdots + \tfrac{1}{16}\right) \\ &= 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} = 1 + \tfrac{4}{2} \end{aligned}$$

Similarly, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$

The method used in Example 8 for showing that the harmonic series diverges is due to the French scholar Nicole Oresme (1323–1382).

This shows that $s_{2^n} \to \infty$ as $n \to \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges.

6 Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\Sigma\, a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= s - s = 0 \end{aligned}$$

NOTE 1 With any *series* $\Sigma\, a_n$ we associate two *sequences:* the sequence $\{s_n\}$ of its partial sums and the sequence $\{a_n\}$ of its terms. If $\Sigma\, a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is s (the sum of the series) and, as Theorem 6 asserts, the limit of the sequence $\{a_n\}$ is 0.

NOTE 2 The converse of Theorem 6 is not true in general. If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\Sigma\, a_n$ is convergent. Observe that for the harmonic series $\Sigma\, 1/n$ we have $a_n = 1/n \to 0$ as $n \to \infty$, but we showed in Example 8 that $\Sigma\, 1/n$ is divergent.

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

The Test for Divergence follows from Theorem 6 because, if the series is not divergent, then it is convergent, and so $\lim_{n\to\infty} a_n = 0$.

EXAMPLE 9 Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 4}$ diverges.

SOLUTION

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2 + 4} = \lim_{n\to\infty} \frac{1}{5 + 4/n^2} = \frac{1}{5} \neq 0$$

So the series diverges by the Test for Divergence.

NOTE 3 If we find that $\lim_{n\to\infty} a_n \neq 0$, we know that $\Sigma\, a_n$ is divergent. If we find that $\lim_{n\to\infty} a_n = 0$, we know *nothing* about the convergence or divergence of $\Sigma\, a_n$. Remember the warning in Note 2: If $\lim_{n\to\infty} a_n = 0$, the series $\Sigma\, a_n$ might converge or it might diverge.

8 Theorem If $\Sigma\, a_n$ and $\Sigma\, b_n$ are convergent series, then so are the series $\Sigma\, ca_n$ (where c is a constant), $\Sigma\, (a_n + b_n)$, and $\Sigma\, (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ (ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

These properties of convergent series follow from the corresponding Limit Laws for Sequences in Section 11.1. For instance, here is how part (ii) of Theorem 8 is proved:

Let

$$s_n = \sum_{i=1}^{n} a_i \qquad s = \sum_{n=1}^{\infty} a_n \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

The nth partial sum for the series $\Sigma\,(a_n + b_n)$ is

$$u_n = \sum_{i=1}^{n} (a_i + b_i)$$

and, using Equation 4.2.10, we have

$$\begin{aligned}
\lim_{n\to\infty} u_n = \lim_{n\to\infty} \sum_{i=1}^{n} (a_i + b_i) &= \lim_{n\to\infty} \left(\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} a_i + \lim_{n\to\infty} \sum_{i=1}^{n} b_i \\
&= \lim_{n\to\infty} s_n + \lim_{n\to\infty} t_n = s + t
\end{aligned}$$

Therefore $\Sigma\,(a_n + b_n)$ is convergent and its sum is

$$\sum_{n=1}^{\infty} (a_n + b_n) = s + t = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

EXAMPLE 10 Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$.

SOLUTION The series $\Sigma\, 1/2^n$ is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$, so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

In Example 7 we found that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

So, by Theorem 8, the given series is convergent and

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right) &= 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} \\
&= 3 \cdot 1 + 1 = 4
\end{aligned}$$

NOTE 4 A finite number of terms doesn't affect the convergence or divergence of a series. For instance, suppose that we were able to show that the series

$$\sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

is convergent. Since

$$\sum_{n=1}^{\infty} \frac{n}{n^3 + 1} = \frac{1}{2} + \frac{2}{9} + \frac{3}{28} + \sum_{n=4}^{\infty} \frac{n}{n^3 + 1}$$

it follows that the entire series $\Sigma_{n=1}^{\infty}\, n/(n^3 + 1)$ is convergent. Similarly, if it is known that the series $\Sigma_{n=N+1}^{\infty}\, a_n$ converges, then the full series

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

is also convergent.

11.2 Exercises

1. (a) What is the difference between a sequence and a series?
 (b) What is a convergent series? What is a divergent series?

2. Explain what it means to say that $\sum_{n=1}^{\infty} a_n = 5$.

3–4 Calculate the sum of the series $\sum_{n=1}^{\infty} a_n$ whose partial sums are given.

3. $s_n = 2 - 3(0.8)^n$

4. $s_n = \dfrac{n^2 - 1}{4n^2 + 1}$

5–8 Calculate the first eight terms of the sequence of partial sums correct to four decimal places. Does it appear that the series is convergent or divergent?

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\ln(n+1)}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{1+\sqrt{n}}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$

9–14 Find at least 10 partial sums of the series. Graph both the sequence of terms and the sequence of partial sums on the same screen. Does it appear that the series is convergent or divergent? If it is convergent, find the sum. If it is divergent, explain why.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{12}{(-5)^n}$

10. $\displaystyle\sum_{n=1}^{\infty} \cos n$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^2+4}}$

12. $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{10^n}$

13. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \right)$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n+2)}$

15. Let $a_n = \dfrac{2n}{3n+1}$.
 (a) Determine whether $\{a_n\}$ is convergent.
 (b) Determine whether $\sum_{n=1}^{\infty} a_n$ is convergent.

16. (a) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{j=1}^{n} a_j$$

 (b) Explain the difference between

$$\sum_{i=1}^{n} a_i \quad \text{and} \quad \sum_{i=1}^{n} a_j$$

17–26 Determine whether the geometric series is convergent or divergent. If it is convergent, find its sum.

17. $3 - 4 + \frac{16}{3} - \frac{64}{9} + \cdots$

18. $4 + 3 + \frac{9}{4} + \frac{27}{16} + \cdots$

19. $10 - 2 + 0.4 - 0.08 + \cdots$

20. $2 + 0.5 + 0.125 + 0.03125 + \cdots$

21. $\displaystyle\sum_{n=1}^{\infty} 6(0.9)^{n-1}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(-9)^{n-1}}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n}$

24. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(\sqrt{2})^n}$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n}{3^{n+1}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}}$

27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

27. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$

28. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{3n-1}$

30. $\displaystyle\sum_{k=1}^{\infty} \frac{k(k+2)}{(k+3)^2}$

31. $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{3^n}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1+3^n}{2^n}$

33. $\displaystyle\sum_{n=1}^{\infty} \sqrt[n]{2}$

34. $\displaystyle\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n]$

35. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1} \right)$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

37. $\displaystyle\sum_{k=0}^{\infty} \left(\frac{\pi}{3} \right)^k$

38. $\displaystyle\sum_{k=1}^{\infty} (\cos 1)^k$

39. $\displaystyle\sum_{n=1}^{\infty} \arctan n$

40. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{5^n} + \frac{2}{n} \right)$

41. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{e^n} + \frac{1}{n(n+1)} \right)$

42. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

43–48 Determine whether the series is convergent or divergent by expressing s_n as a telescoping sum (as in Example 7). If it is convergent, find its sum.

43. $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2-1}$

44. $\displaystyle\sum_{n=1}^{\infty} \ln \frac{n}{n+1}$

45. $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$

46. $\displaystyle\sum_{n=1}^{\infty} \left(\cos \frac{1}{n^2} - \cos \frac{1}{(n+1)^2} \right)$

47. $\displaystyle\sum_{n=1}^{\infty} \left(e^{1/n} - e^{1/(n+1)} \right)$

48. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^3 - n}$

49. Let $x = 0.99999\ldots$.
(a) Do you think that $x < 1$ or $x = 1$?
(b) Sum a geometric series to find the value of x.
(c) How many decimal representations does the number 1 have?
(d) Which numbers have more than one decimal representation?

50. A sequence of terms is defined by

$$a_1 = 1 \qquad a_n = (5 - n)a_{n-1}$$

Calculate $\sum_{n=1}^{\infty} a_n$.

51–56 Express the number as a ratio of integers.

51. $0.\overline{8} = 0.8888\ldots$

52. $0.\overline{46} = 0.46464646\ldots$

53. $2.\overline{516} = 2.516516516\ldots$

54. $10.1\overline{35} = 10.135353535\ldots$

55. $1.53\overline{42}$

56. $7.\overline{12345}$

57–63 Find the values of x for which the series converges. Find the sum of the series for those values of x.

57. $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n$

58. $\displaystyle\sum_{n=1}^{\infty} (x + 2)^n$

59. $\displaystyle\sum_{n=0}^{\infty} \frac{(x-2)^n}{3^n}$

60. $\displaystyle\sum_{n=0}^{\infty} (-4)^n (x-5)^n$

61. $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{x^n}$

62. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin^n x}{3^n}$

63. $\displaystyle\sum_{n=0}^{\infty} e^{nx}$

64. We have seen that the harmonic series is a divergent series whose terms approach 0. Show that

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{1}{n}\right)$$

is another series with this property.

CAS **65–66** Use the partial fraction command on your CAS to find a convenient expression for the partial sum, and then use this expression to find the sum of the series. Check your answer by using the CAS to sum the series directly.

65. $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2 + 3n + 1}{(n^2 + n)^3}$

66. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^5 - 5n^3 + 4n}$

67. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = \frac{n-1}{n+1}$$

find a_n and $\sum_{n=1}^{\infty} a_n$.

68. If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is $s_n = 3 - n2^{-n}$, find a_n and $\sum_{n=1}^{\infty} a_n$.

69. A patient takes 150 mg of a drug at the same time every day. Just before each tablet is taken, 5% of the drug remains in the body.
(a) What quantity of the drug is in the body after the third tablet? After the nth tablet?
(b) What quantity of the drug remains in the body in the long run?

70. After injection of a dose D of insulin, the concentration of insulin in a patient's system decays exponentially and so it can be written as De^{-at}, where t represents time in hours and a is a positive constant.
(a) If a dose D is injected every T hours, write an expression for the sum of the residual concentrations just before the $(n + 1)$st injection.
(b) Determine the limiting pre-injection concentration.
(c) If the concentration of insulin must always remain at or above a critical value C, determine a minimal dosage D in terms of C, a, and T.

71. When money is spent on goods and services, those who receive the money also spend some of it. The people receiving some of the twice-spent money will spend some of that, and so on. Economists call this chain reaction the *multiplier effect*. In a hypothetical isolated community, the local government begins the process by spending D dollars. Suppose that each recipient of spent money spends $100c\%$ and saves $100s\%$ of the money that he or she receives. The values c and s are called the *marginal propensity to consume* and the *marginal propensity to save* and, of course, $c + s = 1$.
(a) Let S_n be the total spending that has been generated after n transactions. Find an equation for S_n.
(b) Show that $\lim_{n\to\infty} S_n = kD$, where $k = 1/s$. The number k is called the *multiplier*. What is the multiplier if the marginal propensity to consume is 80%?

Note: The federal government uses this principle to justify deficit spending. Banks use this principle to justify lending a large percentage of the money that they receive in deposits.

72. A certain ball has the property that each time it falls from a height h onto a hard, level surface, it rebounds to a height rh, where $0 < r < 1$. Suppose that the ball is dropped from an initial height of H meters.
(a) Assuming that the ball continues to bounce indefinitely, find the total distance that it travels.
(b) Calculate the total time that the ball travels. (Use the fact that the ball falls $\frac{1}{2}gt^2$ meters in t seconds.)
(c) Suppose that each time the ball strikes the surface with velocity v it rebounds with velocity $-kv$, where $0 < k < 1$. How long will it take for the ball to come to rest?

73. Find the value of c if

$$\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$$

74. Find the value of c such that

$$\sum_{n=0}^{\infty} e^{nc} = 10$$

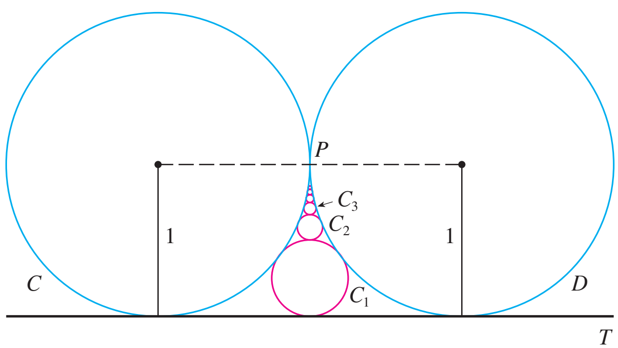
75. In Example 8 we showed that the harmonic series is divergent. Here we outline another method, making use of the fact that $e^x > 1 + x$ for any $x > 0$. (See Exercise 6.2.103.)

If s_n is the nth partial sum of the harmonic series, show that $e^{s_n} > n + 1$. Why does this imply that the harmonic series is divergent?

76. Graph the curves $y = x^n$, $0 \leqslant x \leqslant 1$, for $n = 0, 1, 2, 3, 4, \dots$ on a common screen. By finding the areas between successive curves, give a geometric demonstration of the fact, shown in Example 7, that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

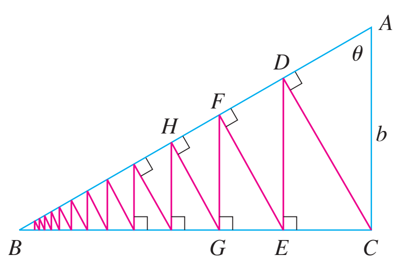
77. The figure shows two circles C and D of radius 1 that touch at P. T is a common tangent line; C_1 is the circle that touches C, D, and T; C_2 is the circle that touches C, D, and C_1; C_3 is the circle that touches C, D, and C_2. This procedure can be continued indefinitely and produces an infinite sequence of circles $\{C_n\}$. Find an expression for the diameter of C_n and thus provide another geometric demonstration of Example 7.



78. A right triangle ABC is given with $\angle A = \theta$ and $|AC| = b$. CD is drawn perpendicular to AB, DE is drawn perpendicular to BC, $EF \perp AB$, and this process is continued indefinitely, as shown in the figure. Find the total length of all the perpendiculars

$$|CD| + |DE| + |EF| + |FG| + \cdots$$

in terms of b and θ.



79. What is wrong with the following calculation?

$$\begin{aligned} 0 &= 0 + 0 + 0 + \cdots \\ &= (1-1) + (1-1) + (1-1) + \cdots \\ &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\ &= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots \\ &= 1 + 0 + 0 + 0 + \cdots = 1 \end{aligned}$$

(Guido Ubaldus thought that this proved the existence of God because "something has been created out of nothing.")

80. Suppose that $\Sigma_{n=1}^{\infty} a_n$ $(a_n \neq 0)$ is known to be a convergent series. Prove that $\Sigma_{n=1}^{\infty} 1/a_n$ is a divergent series.

81. Prove part (i) of Theorem 8.

82. If $\Sigma\, a_n$ is divergent and $c \neq 0$, show that $\Sigma\, ca_n$ is divergent.

83. If $\Sigma\, a_n$ is convergent and $\Sigma\, b_n$ is divergent, show that the series $\Sigma\,(a_n + b_n)$ is divergent. [*Hint:* Argue by contradiction.]

84. If $\Sigma\, a_n$ and $\Sigma\, b_n$ are both divergent, is $\Sigma\,(a_n + b_n)$ necessarily divergent?

85. Suppose that a series $\Sigma\, a_n$ has positive terms and its partial sums s_n satisfy the inequality $s_n \leqslant 1000$ for all n. Explain why $\Sigma\, a_n$ must be convergent.

86. The Fibonacci sequence was defined in Section 11.1 by the equations

$$f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2} \qquad n \geqslant 3$$

Show that each of the following statements is true.

(a) $\dfrac{1}{f_{n-1} f_{n+1}} = \dfrac{1}{f_{n-1} f_n} - \dfrac{1}{f_n f_{n+1}}$

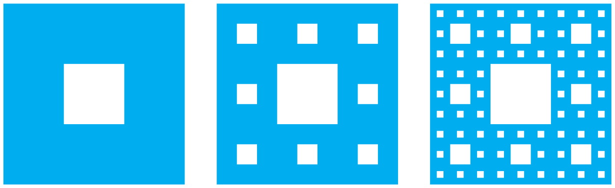
(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{f_{n-1} f_{n+1}} = 1$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1} f_{n+1}} = 2$

87. The **Cantor set**, named after the German mathematician Georg Cantor (1845–1918), is constructed as follows. We start with the closed interval $[0, 1]$ and remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. That leaves the two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ and we remove the open middle third of each. Four intervals remain and again we remove the open middle third of each of them. We continue this procedure indefinitely, at each step removing the open middle third of every interval that remains from the preceding step. The Cantor set consists of the numbers that remain in $[0, 1]$ after all those intervals have been removed.

(a) Show that the total length of all the intervals that are removed is 1. Despite that, the Cantor set contains infinitely many numbers. Give examples of some numbers in the Cantor set.

(b) The **Sierpinski carpet** is a two-dimensional counterpart of the Cantor set. It is constructed by removing the center one-ninth of a square of side 1, then removing the centers of the eight smaller remaining squares, and so on. (The figure shows the first three steps of the construction.) Show that the sum of the areas of the removed squares is 1. This implies that the Sierpinski carpet has area 0.



88. (a) A sequence $\{a_n\}$ is defined recursively by the equation $a_n = \frac{1}{2}(a_{n-1} + a_{n-2})$ for $n \geqslant 3$, where a_1 and a_2 can be any real numbers. Experiment with various values of a_1 and a_2 and use your calculator to guess the limit of the sequence.

(b) Find $\lim_{n \to \infty} a_n$ in terms of a_1 and a_2 by expressing $a_{n+1} - a_n$ in terms of $a_2 - a_1$ and summing a series.

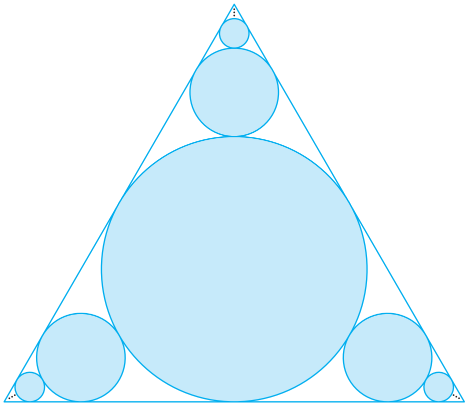
89. Consider the series $\sum_{n=1}^{\infty} n/(n + 1)!$.

(a) Find the partial sums s_1, s_2, s_3, and s_4. Do you recognize the denominators? Use the pattern to guess a formula for s_n.

(b) Use mathematical induction to prove your guess.

(c) Show that the given infinite series is convergent, and find its sum.

90. In the figure there are infinitely many circles approaching the vertices of an equilateral triangle, each circle touching other circles and sides of the triangle. If the triangle has sides of length 1, find the total area occupied by the circles.



11.3 The Integral Test and Estimates of Sums

In general, it is difficult to find the exact sum of a series. We were able to accomplish this for geometric series and the series $\sum 1/[n(n + 1)]$ because in each of those cases we could find a simple formula for the nth partial sum s_n. But usually it isn't easy to discover such a formula. Therefore, in the next few sections, we develop several tests that enable us to determine whether a series is convergent or divergent without explicitly finding its sum. (In some cases, however, our methods will enable us to find good estimates of the sum.) Our first test involves improper integrals.

We begin by investigating the series whose terms are the reciprocals of the squares of the positive integers:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{i^2}$
5	1.4636
10	1.5498
50	1.6251
100	1.6350
500	1.6429
1000	1.6439
5000	1.6447

There's no simple formula for the sum s_n of the first n terms, but the computer-generated table of approximate values given in the margin suggests that the partial sums are approaching a number near 1.64 as $n \to \infty$ and so it looks as if the series is convergent.

We can confirm this impression with a geometric argument. Figure 1 shows the curve $y = 1/x^2$ and rectangles that lie below the curve. The base of each rectangle is an interval of length 1; the height is equal to the value of the function $y = 1/x^2$ at the right endpoint of the interval.

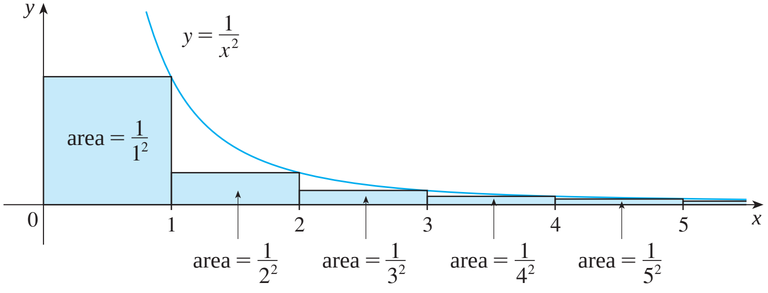


FIGURE 1

So the sum of the areas of the rectangles is

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

If we exclude the first rectangle, the total area of the remaining rectangles is smaller than the area under the curve $y = 1/x^2$ for $x \geqslant 1$, which is the value of the integral $\int_1^\infty (1/x^2)\,dx$. In Section 7.8 we discovered that this improper integral is convergent and has value 1. So the picture shows that all the partial sums are less than

$$\frac{1}{1^2} + \int_1^\infty \frac{1}{x^2}\,dx = 2$$

Thus the partial sums are bounded. We also know that the partial sums are increasing (because all the terms are positive). Therefore the partial sums converge (by the Monotonic Sequence Theorem) and so the series is convergent. The sum of the series (the limit of the partial sums) is also less than 2:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots < 2$$

[The exact sum of this series was found by the Swiss mathematician Leonhard Euler (1707–1783) to be $\pi^2/6$, but the proof of this fact is quite difficult. (See Problem 6 in the Problems Plus following Chapter 15.)]

Now let's look at the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots$$

n	$s_n = \sum_{i=1}^{n} \frac{1}{\sqrt{i}}$
5	3.2317
10	5.0210
50	12.7524
100	18.5896
500	43.2834
1000	61.8010
5000	139.9681

The table of values of s_n suggests that the partial sums aren't approaching a finite number, so we suspect that the given series may be divergent. Again we use a picture for confirmation. Figure 2 shows the curve $y = 1/\sqrt{x}$, but this time we use rectangles whose tops lie *above* the curve.

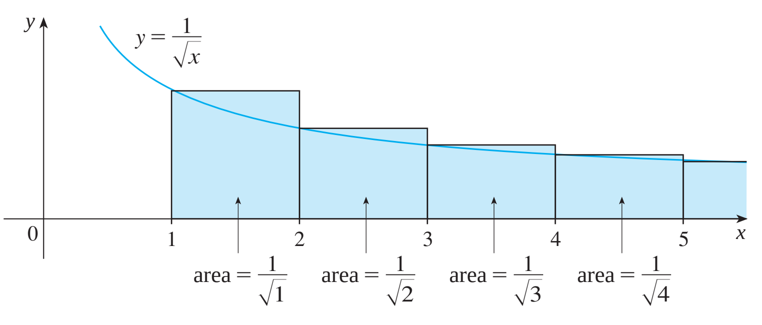


FIGURE 2

The base of each rectangle is an interval of length 1. The height is equal to the value of the function $y = 1/\sqrt{x}$ at the *left* endpoint of the interval. So the sum of the areas of all the rectangles is

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

This total area is greater than the area under the curve $y = 1/\sqrt{x}$ for $x \geqslant 1$, which is equal to the integral $\int_1^\infty (1/\sqrt{x})\,dx$. But we know from Section 7.8 that this improper integral is divergent. In other words, the area under the curve is infinite. So the sum of the series must be infinite; that is, the series is divergent.

The same sort of geometric reasoning that we used for these two series can be used to prove the following test. (The proof is given at the end of this section.)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^\infty a_n$ is convergent if and only if the improper integral $\int_1^\infty f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^\infty f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is convergent.

(ii) If $\displaystyle\int_1^\infty f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^\infty a_n$ is divergent.

NOTE When we use the Integral Test, it is not necessary to start the series or the integral at $n = 1$. For instance, in testing the series

$$\sum_{n=4}^\infty \frac{1}{(n-3)^2} \qquad \text{we use} \qquad \int_4^\infty \frac{1}{(x-3)^2}\,dx$$

Also, it is not necessary that f be *always* decreasing. What is important is that f be *ultimately* decreasing, that is, decreasing for x larger than some number N. Then $\sum_{n=N}^\infty a_n$ is convergent, so $\sum_{n=1}^\infty a_n$ is convergent by Note 4 of Section 11.2.

EXAMPLE 1 Test the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ for convergence or divergence.

SOLUTION The function $f(x) = 1/(x^2 + 1)$ is continuous, positive, and decreasing on $[1, \infty)$ so we use the Integral Test:

$$\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \tan^{-1}x\Big]_1^t$$

$$= \lim_{t\to\infty} \left(\tan^{-1}t - \frac{\pi}{4}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Thus $\int_1^\infty 1/(x^2 + 1)\,dx$ is a convergent integral and so, by the Integral Test, the series $\sum 1/(n^2 + 1)$ is convergent.

V **EXAMPLE 2** For what values of p is the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ convergent?

SOLUTION If $p < 0$, then $\lim_{n\to\infty} (1/n^p) = \infty$. If $p = 0$, then $\lim_{n\to\infty} (1/n^p) = 1$. In either case $\lim_{n\to\infty} (1/n^p) \neq 0$, so the given series diverges by the Test for Divergence (11.2.7).

If $p > 0$, then the function $f(x) = 1/x^p$ is clearly continuous, positive, and decreasing on $[1, \infty)$. We found in Chapter 7 [see (7.8.2)] that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{converges if } p > 1 \text{ and diverges if } p \leqslant 1$$

In order to use the Integral Test we need to be able to evaluate $\int_1^\infty f(x)\,dx$ and therefore we have to be able to find an antiderivative of f. Frequently this is difficult or impossible, so we need other tests for convergence too.

It follows from the Integral Test that the series $\Sigma\, 1/n^p$ converges if $p > 1$ and diverges if $0 < p \leqslant 1$. (For $p = 1$, this series is the harmonic series discussed in Example 8 in Section 11.2.)

The series in Example 2 is called the ***p*-series**. It is important in the rest of this chapter, so we summarize the results of Example 2 for future reference as follows.

1 The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

EXAMPLE 3

(a) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

is convergent because it is a p-series with $p = 3 > 1$.

(b) The series

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/3}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = 1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \cdots$$

is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

NOTE We should *not* infer from the Integral Test that the sum of the series is equal to the value of the integral. In fact,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{whereas} \qquad \int_1^{\infty} \frac{1}{x^2}\, dx = 1$$

Therefore, in general,

$$\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\, dx$$

V EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION The function $f(x) = (\ln x)/x$ is positive and continuous for $x > 1$ because the logarithm function is continuous. But it is not obvious whether or not f is decreasing, so we compute its derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Thus $f'(x) < 0$ when $\ln x > 1$, that is, $x > e$. It follows that f is decreasing when $x > e$ and so we can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\, dx = \lim_{t\to\infty} \frac{(\ln x)^2}{2}\Bigg]_1^t$$

$$= \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

Since this improper integral is divergent, the series $\Sigma\, (\ln n)/n$ is also divergent by the Integral Test.

Estimating the Sum of a Series

Suppose we have been able to use the Integral Test to show that a series $\Sigma\, a_n$ is convergent and we now want to find an approximation to the sum s of the series. Of course, any partial sum s_n is an approximation to s because $\lim_{n\to\infty} s_n = s$. But how good is such an approximation? To find out, we need to estimate the size of the **remainder**

$$R_n = s - s_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

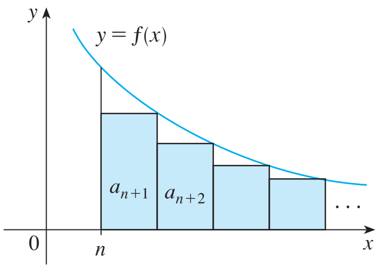


FIGURE 3

The remainder R_n is the error made when s_n, the sum of the first n terms, is used as an approximation to the total sum.

We use the same notation and ideas as in the Integral Test, assuming that f is decreasing on $[n, \infty)$. Comparing the areas of the rectangles with the area under $y = f(x)$ for $x > n$ in Figure 3, we see that

$$R_n = a_{n+1} + a_{n+2} + \cdots \leqslant \int_n^\infty f(x)\,dx$$

Similarly, we see from Figure 4 that

$$R_n = a_{n+1} + a_{n+2} + \cdots \geqslant \int_{n+1}^\infty f(x)\,dx$$

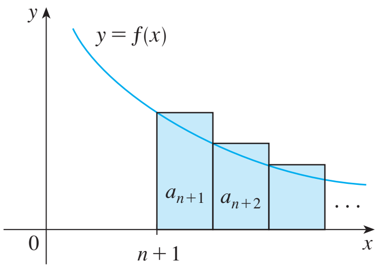


FIGURE 4

So we have proved the following error estimate.

> $\boxed{2}$ **Remainder Estimate for the Integral Test** Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geqslant n$ and $\Sigma\, a_n$ is convergent. If $R_n = s - s_n$, then
>
> $$\int_{n+1}^\infty f(x)\,dx \leqslant R_n \leqslant \int_n^\infty f(x)\,dx$$

V EXAMPLE 5

(a) Approximate the sum of the series $\Sigma\, 1/n^3$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

SOLUTION In both parts (a) and (b) we need to know $\int_n^\infty f(x)\,dx$. With $f(x) = 1/x^3$, which satisfies the conditions of the Integral Test, we have

$$\int_n^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2x^2}\right]_n^t = \lim_{t\to\infty}\left(-\frac{1}{2t^2} + \frac{1}{2n^2}\right) = \frac{1}{2n^2}$$

(a) Approximating the sum of the series by the 10th partial sum, we have

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx s_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{10^3} \approx 1.1975$$

According to the remainder estimate in $\boxed{2}$, we have

$$R_{10} \leqslant \int_{10}^\infty \frac{1}{x^3}\,dx = \frac{1}{2(10)^2} = \frac{1}{200}$$

So the size of the error is at most 0.005.

(b) Accuracy to within 0.0005 means that we have to find a value of n such that $R_n \leqslant 0.0005$. Since

$$R_n \leqslant \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

we want

$$\frac{1}{2n^2} < 0.0005$$

Solving this inequality, we get

$$n^2 > \frac{1}{0.001} = 1000 \qquad \text{or} \qquad n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

If we add s_n to each side of the inequalities in $\boxed{2}$, we get

3

$$\boxed{s_n + \int_{n+1}^\infty f(x)\,dx \leqslant s \leqslant s_n + \int_n^\infty f(x)\,dx}$$

because $s_n + R_n = s$. The inequalities in $\boxed{3}$ give a lower bound and an upper bound for s. They provide a more accurate approximation to the sum of the series than the partial sum s_n does.

Although Euler was able to calculate the exact sum of the p-series for $p = 2$, nobody has been able to find the exact sum for $p = 3$. In Example 6, however, we show how to *estimate* this sum.

EXAMPLE 6 Use $\boxed{3}$ with $n = 10$ to estimate the sum of the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$.

SOLUTION The inequalities in $\boxed{3}$ become

$$s_{10} + \int_{11}^\infty \frac{1}{x^3}\,dx \leqslant s \leqslant s_{10} + \int_{10}^\infty \frac{1}{x^3}\,dx$$

From Example 5 we know that

$$\int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2}$$

so

$$s_{10} + \frac{1}{2(11)^2} \leqslant s \leqslant s_{10} + \frac{1}{2(10)^2}$$

Using $s_{10} \approx 1.197532$, we get

$$1.201664 \leqslant s \leqslant 1.202532$$

If we approximate s by the midpoint of this interval, then the error is at most half the length of the interval. So

$$\sum_{n=1}^\infty \frac{1}{n^3} \approx 1.2021 \qquad \text{with error} < 0.0005$$

If we compare Example 6 with Example 5, we see that the improved estimate in $\boxed{3}$ can be much better than the estimate $s \approx s_n$. To make the error smaller than 0.0005 we had to use 32 terms in Example 5 but only 10 terms in Example 6.

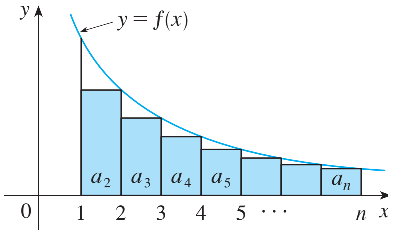


FIGURE 5

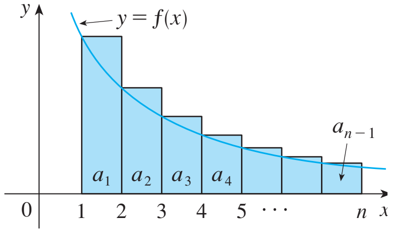


FIGURE 6

Proof of the Integral Test

We have already seen the basic idea behind the proof of the Integral Test in Figures 1 and 2 for the series $\Sigma\, 1/n^2$ and $\Sigma\, 1/\sqrt{n}$. For the general series $\Sigma\, a_n$, look at Figures 5 and 6. The area of the first shaded rectangle in Figure 5 is the value of f at the right endpoint of $[1, 2]$, that is, $f(2) = a_2$. So, comparing the areas of the shaded rectangles with the area under $y = f(x)$ from 1 to n, we see that

$$\boxed{4} \qquad a_2 + a_3 + \cdots + a_n \leqslant \int_1^n f(x)\, dx$$

(Notice that this inequality depends on the fact that f is decreasing.) Likewise, Figure 6 shows that

$$\boxed{5} \qquad \int_1^n f(x)\, dx \leqslant a_1 + a_2 + \cdots + a_{n-1}$$

(i) If $\int_1^\infty f(x)\, dx$ is convergent, then $\boxed{4}$ gives

$$\sum_{i=2}^{n} a_i \leqslant \int_1^n f(x)\, dx \leqslant \int_1^\infty f(x)\, dx$$

since $f(x) \geqslant 0$. Therefore

$$s_n = a_1 + \sum_{i=2}^{n} a_i \leqslant a_1 + \int_1^\infty f(x)\, dx = M, \text{ say}$$

Since $s_n \leqslant M$ for all n, the sequence $\{s_n\}$ is bounded above. Also

$$s_{n+1} = s_n + a_{n+1} \geqslant s_n$$

since $a_{n+1} = f(n + 1) \geqslant 0$. Thus $\{s_n\}$ is an increasing bounded sequence and so it is convergent by the Monotonic Sequence Theorem (11.1.12). This means that $\Sigma\, a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\, dx$ is divergent, then $\int_1^n f(x)\, dx \to \infty$ as $n \to \infty$ because $f(x) \geqslant 0$. But $\boxed{5}$ gives

$$\int_1^n f(x)\, dx \leqslant \sum_{i=1}^{n-1} a_i = s_{n-1}$$

and so $s_{n-1} \to \infty$. This implies that $s_n \to \infty$ and so $\Sigma\, a_n$ diverges.

11.3 Exercises

1. Draw a picture to show that

$$\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^\infty \frac{1}{x^{1.3}}\, dx$$

What can you conclude about the series?

2. Suppose f is a continuous positive decreasing function for $x \geqslant 1$ and $a_n = f(n)$. By drawing a picture, rank the following three quantities in increasing order:

$$\int_1^6 f(x)\, dx \qquad \sum_{i=1}^{5} a_i \qquad \sum_{i=2}^{6} a_i$$

3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^5}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n + 1)^3}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n + 4}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$

8. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

9–26 Determine whether the series is convergent or divergent.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\sqrt{2}}}$

10. $\displaystyle\sum_{n=3}^{\infty} n^{-0.9999}$

11. $1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots$

12. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$

13. $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$

14. $\frac{1}{5} + \frac{1}{8} + \frac{1}{11} + \frac{1}{14} + \frac{1}{17} + \cdots$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n} + 4}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$

18. $\displaystyle\sum_{n=3}^{\infty} \frac{3n - 4}{n^2 - 2n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 6n + 13}$

21. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

24. $\displaystyle\sum_{n=3}^{\infty} \frac{n^2}{e^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + n^3}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4 + 1}$

27–28 Explain why the Integral Test can't be used to determine whether the series is convergent.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos \pi n}{\sqrt{n}}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{1 + n^2}$

29–32 Find the values of p for which the series is convergent.

29. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$

30. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n \ln n\, [\ln(\ln n)]^p}$

31. $\displaystyle\sum_{n=1}^{\infty} n(1 + n^2)^p$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$

33. The Riemann zeta-function ζ is defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

and is used in number theory to study the distribution of prime numbers. What is the domain of ζ?

34. Leonhard Euler was able to calculate the exact sum of the p-series with $p = 2$:

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(See page 739.) Use this fact to find the sum of each series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$ (b) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n + 1)^2}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$

35. Euler also found the sum of the p-series with $p = 4$:

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

Use Euler's result to find the sum of the series.

(a) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^4$ (b) $\displaystyle\sum_{k=5}^{\infty} \frac{1}{(k - 2)^4}$

36. (a) Find the partial sum s_{10} of the series $\sum_{n=1}^{\infty} 1/n^4$. Estimate the error in using s_{10} as an approximation to the sum of the series.
(b) Use $\boxed{3}$ with $n = 10$ to give an improved estimate of the sum.
(c) Compare your estimate in part (b) with the exact value given in Exercise 35.
(d) Find a value of n so that s_n is within 0.00001 of the sum.

37. (a) Use the sum of the first 10 terms to estimate the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. How good is this estimate?
(b) Improve this estimate using $\boxed{3}$ with $n = 10$.
(c) Compare your estimate in part (b) with the exact value given in Exercise 34.
(d) Find a value of n that will ensure that the error in the approximation $s \approx s_n$ is less than 0.001.

38. Find the sum of the series $\sum_{n=1}^{\infty} 1/n^5$ correct to three decimal places.

39. Estimate $\sum_{n=1}^{\infty} (2n + 1)^{-6}$ correct to five decimal places.

40. How many terms of the series $\sum_{n=2}^{\infty} 1/[n(\ln n)^2]$ would you need to add to find its sum to within 0.01?

41. Show that if we want to approximate the sum of the series $\sum_{n=1}^{\infty} n^{-1.001}$ so that the error is less than 5 in the ninth decimal place, then we need to add more than $10^{11,301}$ terms!

CAS **42.** (a) Show that the series $\sum_{n=1}^{\infty} (\ln n)^2/n^2$ is convergent.
(b) Find an upper bound for the error in the approximation $s \approx s_n$.
(c) What is the smallest value of n such that this upper bound is less than 0.05?
(d) Find s_n for this value of n.

43. (a) Use $\boxed{4}$ to show that if s_n is the nth partial sum of the harmonic series, then

$$s_n \leqslant 1 + \ln n$$

(b) The harmonic series diverges, but very slowly. Use part (a) to show that the sum of the first million terms is less than 15 and the sum of the first billion terms is less than 22.

44. Use the following steps to show that the sequence

$$t_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n$$

has a limit. (The value of the limit is denoted by γ and is called Euler's constant.)

(a) Draw a picture like Figure 6 with $f(x) = 1/x$ and interpret t_n as an area [or use $\boxed{5}$] to show that $t_n > 0$ for all n.

(b) Interpret

$$t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1}$$

as a difference of areas to show that $t_n - t_{n+1} > 0$. Therefore $\{t_n\}$ is a decreasing sequence.

(c) Use the Monotonic Sequence Theorem to show that $\{t_n\}$ is convergent.

45. Find all positive values of b for which the series $\sum_{n=1}^{\infty} b^{\ln n}$ converges.

46. Find all values of c for which the following series converges.

$$\sum_{n=1}^{\infty} \left(\frac{c}{n} - \frac{1}{n+1} \right)$$

11.4 The Comparison Tests

In the comparison tests the idea is to compare a given series with a series that is known to be convergent or divergent. For instance, the series

$$\boxed{1} \qquad \sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

reminds us of the series $\sum_{n=1}^{\infty} 1/2^n$, which is a geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$ and is therefore convergent. Because the series $\boxed{1}$ is so similar to a convergent series, we have the feeling that it too must be convergent. Indeed, it is. The inequality

$$\frac{1}{2^n + 1} < \frac{1}{2^n}$$

shows that our given series $\boxed{1}$ has smaller terms than those of the geometric series and therefore all its partial sums are also smaller than 1 (the sum of the geometric series). This means that its partial sums form a bounded increasing sequence, which is convergent. It also follows that the sum of the series is less than the sum of the geometric series:

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1} < 1$$

Similar reasoning can be used to prove the following test, which applies only to series whose terms are positive. The first part says that if we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent. The second part says that if we start with a series whose terms are *larger* than those of a known *divergent* series, then it too is divergent.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

It is important to keep in mind the distinction between a sequence and a series. A sequence is a list of numbers, whereas a series is a sum. With every series $\Sigma\, a_n$ there are associated two sequences: the sequence $\{a_n\}$ of terms and the sequence $\{s_n\}$ of partial sums.

PROOF

(i) Let
$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geqslant s_n$). Also $t_n \to t$, so $t_n \leqslant t$ for all n. Since $a_i \leqslant b_i$, we have $s_n \leqslant t_n$. Thus $s_n \leqslant t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\Sigma\, a_n$ converges.

(ii) If $\Sigma\, b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geqslant b_i$ so $s_n \geqslant t_n$. Thus $s_n \to \infty$. Therefore $\Sigma\, a_n$ diverges.

Standard Series for Use with the Comparison Test

In using the Comparison Test we must, of course, have some known series $\Sigma\, b_n$ for the purpose of comparison. Most of the time we use one of these series:

- A p-series $\left[\Sigma\, 1/n^p \text{ converges if } p > 1 \text{ and diverges if } p \leqslant 1; \text{ see (11.3.1)}\right]$
- A geometric series $\left[\Sigma\, ar^{n-1} \text{ converges if } |r| < 1 \text{ and diverges if } |r| \geqslant 1;\right.$ see (11.2.4)$]$

V EXAMPLE 1 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

SOLUTION For large n the dominant term in the denominator is $2n^2$, so we compare the given series with the series $\Sigma\, 5/(2n^2)$. Observe that

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

because the left side has a bigger denominator. (In the notation of the Comparison Test, a_n is the left side and b_n is the right side.) We know that

$$\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent because it's a constant times a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

is convergent by part (i) of the Comparison Test.

NOTE 1 Although the condition $a_n \leqslant b_n$ or $a_n \geqslant b_n$ in the Comparison Test is given for all n, we need verify only that it holds for $n \geqslant N$, where N is some fixed integer, because the convergence of a series is not affected by a finite number of terms. This is illustrated in the next example.

V EXAMPLE 2 Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$ for convergence or divergence.

SOLUTION We used the Integral Test to test this series in Example 4 of Section 11.3, but we can also test it by comparing it with the harmonic series. Observe that $\ln k > 1$ for $k \geqslant 3$ and so

$$\frac{\ln k}{k} > \frac{1}{k} \qquad k \geqslant 3$$

We know that $\Sigma\, 1/k$ is divergent (p-series with $p = 1$). Thus the given series is divergent by the Comparison Test.

NOTE 2 The terms of the series being tested must be smaller than those of a convergent series or larger than those of a divergent series. If the terms are larger than the terms of a convergent series or smaller than those of a divergent series, then the Comparison Test doesn't apply. Consider, for instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

The inequality

$$\frac{1}{2^n - 1} > \frac{1}{2^n}$$

is useless as far as the Comparison Test is concerned because $\Sigma\, b_n = \Sigma\left(\frac{1}{2}\right)^n$ is convergent and $a_n > b_n$. Nonetheless, we have the feeling that $\Sigma\, 1/(2^n - 1)$ ought to be convergent because it is very similar to the convergent geometric series $\Sigma\left(\frac{1}{2}\right)^n$. In such cases the following test can be used.

Exercises 40 and 41 deal with the cases $c = 0$ and $c = \infty$.

The Limit Comparison Test Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N$$

If $\Sigma\, b_n$ converges, so does $\Sigma\, Mb_n$. Thus $\Sigma\, a_n$ converges by part (i) of the Comparison Test. If $\Sigma\, b_n$ diverges, so does $\Sigma\, mb_n$ and part (ii) of the Comparison Test shows that $\Sigma\, a_n$ diverges.

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

SOLUTION We use the Limit Comparison Test with

$$a_n = \frac{1}{2^n - 1} \qquad b_n = \frac{1}{2^n}$$

and obtain

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - 1)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - 1/2^n} = 1 > 0$$

Since this limit exists and $\Sigma\, 1/2^n$ is a convergent geometric series, the given series converges by the Limit Comparison Test.

EXAMPLE 4 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ converges or diverges.

SOLUTION The dominant part of the numerator is $2n^2$ and the dominant part of the denominator is $\sqrt{n^5} = n^{5/2}$. This suggests taking

$$a_n = \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \qquad b_n = \frac{2n^2}{n^{5/2}} = \frac{2}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \cdot \frac{n^{1/2}}{2} = \lim_{n\to\infty} \frac{2n^{5/2} + 3n^{3/2}}{2\sqrt{5 + n^5}}$$

$$= \lim_{n\to\infty} \frac{2 + \dfrac{3}{n}}{2\sqrt{\dfrac{5}{n^5} + 1}} = \frac{2 + 0}{2\sqrt{0 + 1}} = 1$$

Since $\Sigma\, b_n = 2\,\Sigma\, 1/n^{1/2}$ is divergent $\left(p\text{-series with } p = \frac{1}{2} < 1\right)$, the given series diverges by the Limit Comparison Test.

Notice that in testing many series we find a suitable comparison series $\Sigma\, b_n$ by keeping only the highest powers in the numerator and denominator.

Estimating Sums

If we have used the Comparison Test to show that a series $\Sigma\, a_n$ converges by comparison with a series $\Sigma\, b_n$, then we may be able to estimate the sum $\Sigma\, a_n$ by comparing remainders. As in Section 11.3, we consider the remainder

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

For the comparison series $\Sigma\, b_n$ we consider the corresponding remainder

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \cdots$$

Since $a_n \leqslant b_n$ for all n, we have $R_n \leqslant T_n$. If $\Sigma\, b_n$ is a p-series, we can estimate its remainder T_n as in Section 11.3. If $\Sigma\, b_n$ is a geometric series, then T_n is the sum of a geometric series and we can sum it exactly (see Exercises 35 and 36). In either case we know that R_n is smaller than T_n.

V EXAMPLE 5 Use the sum of the first 100 terms to approximate the sum of the series $\Sigma\, 1/(n^3 + 1)$. Estimate the error involved in this approximation.

SOLUTION Since

$$\frac{1}{n^3 + 1} < \frac{1}{n^3}$$

the given series is convergent by the Comparison Test. The remainder T_n for the comparison series $\Sigma\, 1/n^3$ was estimated in Example 5 in Section 11.3 using the Remainder Estimate for the Integral Test. There we found that

$$T_n \leqslant \int_n^{\infty} \frac{1}{x^3}\, dx = \frac{1}{2n^2}$$

Therefore the remainder R_n for the given series satisfies

$$R_n \leqslant T_n \leqslant \frac{1}{2n^2}$$

With $n = 100$ we have

$$R_{100} \leqslant \frac{1}{2(100)^2} = 0.00005$$

Using a programmable calculator or a computer, we find that

$$\sum_{n=1}^{\infty} \frac{1}{n^3 + 1} \approx \sum_{n=1}^{100} \frac{1}{n^3 + 1} \approx 0.6864538$$

with error less than 0.00005.

11.4 Exercises

1. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be convergent.
   (a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
   (b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

2. Suppose $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is known to be divergent.
   (a) If $a_n > b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?
   (b) If $a_n < b_n$ for all n, what can you say about $\Sigma\, a_n$? Why?

3–32 Determine whether the series converges or diverges.

3. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^3 + 1}$

4. $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 1}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 1}{n\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n - 1}{n^2\sqrt{n}}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{6^n}{5^n - 1}$

9. $\displaystyle\sum_{k=1}^{\infty} \frac{\ln k}{k}$

10. $\displaystyle\sum_{k=1}^{\infty} \frac{k \sin^2 k}{1 + k^3}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3 + 4k + 3}}$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{(2k - 1)(k^2 - 1)}{(k + 1)(k^2 + 4)^2}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan n}{n^{1.2}}$

14. $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n - 1}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n - 2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4 + 1}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n + 3}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{1 + 4^n}{1 + 3^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{n + 4^n}{n + 6^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n + 2}}{2n^2 + n + 1}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{n + 2}{(n + 1)^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{5 + 2n}{(1 + n^2)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 - 5n}{n^3 + n + 1}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 + n^2}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2 - 1}}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^2 e^{-n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

33–36 Use the sum of the first 10 terms to approximate the sum of the series. Estimate the error.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^4 + 1}}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3}$

35. $\displaystyle\sum_{n=1}^{\infty} 5^{-n} \cos^2 n$

36. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3^n + 4^n}$

37. The meaning of the decimal representation of a number $0.d_1d_2d_3\ldots$ (where the digit d_i is one of the numbers 0, 1, 2, . . . , 9) is that

$$0.d_1d_2d_3d_4\ldots = \frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \frac{d_4}{10^4} + \cdots$$

Show that this series always converges.

1. Homework Hints available at stewartcalculus.com

38. For what values of p does the series $\Sigma_{n=2}^{\infty} 1/(n^p \ln n)$ converge?

39. Prove that if $a_n \geqslant 0$ and $\Sigma\, a_n$ converges, then $\Sigma\, a_n^2$ also converges.

40. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is convergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0$$

then $\Sigma\, a_n$ is also convergent.

(b) Use part (a) to show that the series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{\sqrt{n}\, e^n}$

41. (a) Suppose that $\Sigma\, a_n$ and $\Sigma\, b_n$ are series with positive terms and $\Sigma\, b_n$ is divergent. Prove that if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

then $\Sigma\, a_n$ is also divergent.

(b) Use part (a) to show that the series diverges.

(i) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

42. Give an example of a pair of series $\Sigma\, a_n$ and $\Sigma\, b_n$ with positive terms where $\lim_{n\to\infty}(a_n/b_n) = 0$ and $\Sigma\, b_n$ diverges, but $\Sigma\, a_n$ converges. (Compare with Exercise 40.)

43. Show that if $a_n > 0$ and $\lim_{n\to\infty} n a_n \neq 0$, then $\Sigma\, a_n$ is divergent.

44. Show that if $a_n > 0$ and $\Sigma\, a_n$ is convergent, then $\Sigma \ln(1 + a_n)$ is convergent.

45. If $\Sigma\, a_n$ is a convergent series with positive terms, is it true that $\Sigma \sin(a_n)$ is also convergent?

46. If $\Sigma\, a_n$ and $\Sigma\, b_n$ are both convergent series with positive terms, is it true that $\Sigma\, a_n b_n$ is also convergent?

11.5 Alternating Series

The convergence tests that we have looked at so far apply only to series with positive terms. In this section and the next we learn how to deal with series whose terms are not necessarily positive. Of particular importance are *alternating series*, whose terms alternate in sign.

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$$

We see from these examples that the nth term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^n b_n$$

where b_n is a positive number. (In fact, $b_n = |a_n|$.)

The following test says that if the terms of an alternating series decrease toward 0 in absolute value, then the series converges.

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

$$\text{(i)} \quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)} \quad \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

Before giving the proof let's look at Figure 1, which gives a picture of the idea behind the proof. We first plot $s_1 = b_1$ on a number line. To find s_2 we subtract b_2, so s_2 is to the left of s_1. Then to find s_3 we add b_3, so s_3 is to the right of s_2. But, since $b_3 < b_2$, s_3 is to the left of s_1. Continuing in this manner, we see that the partial sums oscillate back and forth. Since $b_n \to 0$, the successive steps are becoming smaller and smaller. The even partial sums $s_2, s_4, s_6, \ldots$ are increasing and the odd partial sums $s_1, s_3, s_5, \ldots$ are decreasing. Thus it seems plausible that both are converging to some number s, which is the sum of the series. Therefore we consider the even and odd partial sums separately in the following proof.

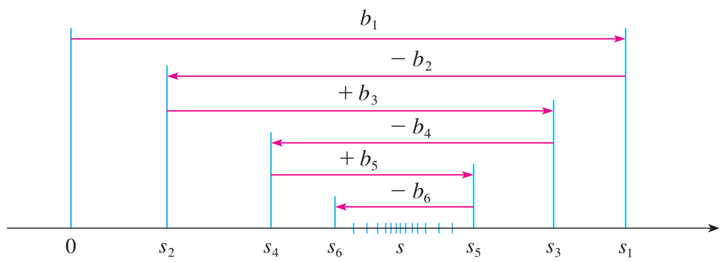


FIGURE 1

PROOF OF THE ALTERNATING SERIES TEST We first consider the even partial sums:

$$s_2 = b_1 - b_2 \geqslant 0 \qquad \text{since } b_2 \leqslant b_1$$

$$s_4 = s_2 + (b_3 - b_4) \geqslant s_2 \qquad \text{since } b_4 \leqslant b_3$$

In general $$s_{2n} = s_{2n-2} + (b_{2n-1} - b_{2n}) \geqslant s_{2n-2} \qquad \text{since } b_{2n} \leqslant b_{2n-1}$$

Thus $$0 \leqslant s_2 \leqslant s_4 \leqslant s_6 \leqslant \cdots \leqslant s_{2n} \leqslant \cdots$$

But we can also write

$$s_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n}$$

Every term in brackets is positive, so $s_{2n} \leqslant b_1$ for all n. Therefore the sequence $\{s_{2n}\}$ of even partial sums is increasing and bounded above. It is therefore convergent by the Monotonic Sequence Theorem. Let's call its limit s, that is,

$$\lim_{n\to\infty} s_{2n} = s$$

Now we compute the limit of the odd partial sums:

$$\begin{aligned} \lim_{n\to\infty} s_{2n+1} &= \lim_{n\to\infty} (s_{2n} + b_{2n+1}) \\ &= \lim_{n\to\infty} s_{2n} + \lim_{n\to\infty} b_{2n+1} \\ &= s + 0 \qquad \text{[by condition (ii)]} \\ &= s \end{aligned}$$

Since both the even and odd partial sums converge to s, we have $\lim_{n\to\infty} s_n = s$ [see Exercise 92(a) in Section 11.1] and so the series is convergent. ■

Figure 2 illustrates Example 1 by showing the graphs of the terms $a_n = (-1)^{n-1}/n$ and the partial sums s_n. Notice how the values of s_n zigzag across the limiting value, which appears to be about 0.7. In fact, it can be proved that the exact sum of the series is $\ln 2 \approx 0.693$ (see Exercise 36).

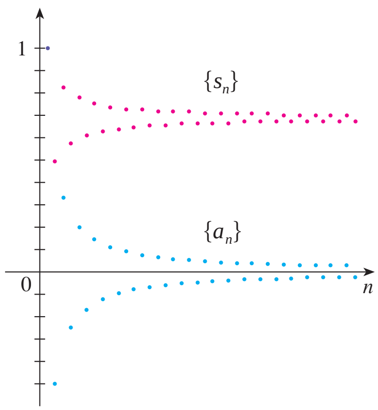


FIGURE 2

V EXAMPLE 1 The alternating harmonic series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

satisfies

$$\text{(i)}\quad b_{n+1} < b_n \qquad \text{because} \qquad \frac{1}{n+1} < \frac{1}{n}$$

$$\text{(ii)}\quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the series is convergent by the Alternating Series Test.

V EXAMPLE 2 The series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ is alternating, but

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{3}{4 - \dfrac{1}{n}} = \frac{3}{4}$$

so condition (ii) is not satisfied. Instead, we look at the limit of the nth term of the series:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(-1)^n 3n}{4n-1}$$

This limit does not exist, so the series diverges by the Test for Divergence.

EXAMPLE 3 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ for convergence or divergence.

SOLUTION The given series is alternating so we try to verify conditions (i) and (ii) of the Alternating Series Test.

Unlike the situation in Example 1, it is not obvious that the sequence given by $b_n = n^2/(n^3+1)$ is decreasing. However, if we consider the related function $f(x) = x^2/(x^3+1)$, we find that

$$f'(x) = \frac{x(2 - x^3)}{(x^3+1)^2}$$

Since we are considering only positive x, we see that $f'(x) < 0$ if $2 - x^3 < 0$, that is, $x > \sqrt[3]{2}$. Thus f is decreasing on the interval $(\sqrt[3]{2}, \infty)$. This means that $f(n+1) < f(n)$ and therefore $b_{n+1} < b_n$ when $n \geqslant 2$. (The inequality $b_2 < b_1$ can be verified directly but all that really matters is that the sequence $\{b_n\}$ is eventually decreasing.)

Instead of verifying condition (i) of the Alternating Series Test by computing a derivative, we could verify that $b_{n+1} < b_n$ directly by using the technique of Solution 1 of Example 13 in Section 11.1.

Condition (ii) is readily verified:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} = \lim_{n\to\infty} \frac{\dfrac{1}{n}}{1 + \dfrac{1}{n^3}} = 0$$

Thus the given series is convergent by the Alternating Series Test.

Estimating Sums

A partial sum s_n of any convergent series can be used as an approximation to the total sum s, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$. The next theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

You can see geometrically why the Alternating Series Estimation Theorem is true by looking at Figure 1 (on page 752). Notice that $s - s_4 < b_5$, $|s - s_5| < b_6$, and so on. Notice also that s lies between any two consecutive partial sums.

Alternating Series Estimation Theorem If $s = \sum (-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

$$\text{(i)}\ b_{n+1} \leqslant b_n \qquad \text{and} \qquad \text{(ii)}\ \lim_{n\to\infty} b_n = 0$$

then $$|R_n| = |s - s_n| \leqslant b_{n+1}$$

PROOF We know from the proof of the Alternating Series Test that s lies between any two consecutive partial sums s_n and s_{n+1}. (There we showed that s is larger than all even partial sums. A similar argument shows that s is smaller than all the odd sums.) It follows that

$$|s - s_n| \leqslant |s_{n+1} - s_n| = b_{n+1}$$

By definition, $0! = 1$.

V EXAMPLE 4 Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

SOLUTION We first observe that the series is convergent by the Alternating Series Test because

$$\text{(i)} \quad \frac{1}{(n+1)!} = \frac{1}{n!\,(n+1)} < \frac{1}{n!}$$

$$\text{(ii)} \quad 0 < \frac{1}{n!} < \frac{1}{n} \to 0 \quad \text{so} \quad \frac{1}{n!} \to 0 \ \text{as}\ n \to \infty$$

To get a feel for how many terms we need to use in our approximation, let's write out the first few terms of the series:

$$\begin{aligned} s &= \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdots \\ &= 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} - \tfrac{1}{5040} + \cdots \end{aligned}$$

Notice that $$b_7 = \tfrac{1}{5040} < \tfrac{1}{5000} = 0.0002$$

and $$s_6 = 1 - 1 + \tfrac{1}{2} - \tfrac{1}{6} + \tfrac{1}{24} - \tfrac{1}{120} + \tfrac{1}{720} \approx 0.368056$$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leqslant b_7 < 0.0002$$

In Section 11.10 we will prove that $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all x, so what we have obtained in Example 4 is actually an approximation to the number e^{-1}.

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

NOTE The rule that the error (in using s_n to approximate s) is smaller than the first neglected term is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. The rule does not apply to other types of series.

11.5 Exercises

1. (a) What is an alternating series?
(b) Under what conditions does an alternating series converge?
(c) If these conditions are satisfied, what can you say about the remainder after n terms?

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

4. $\dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{3}} + \dfrac{1}{\sqrt{4}} - \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{6}} - \cdots$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n+1}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)}$

7. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

8. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^3+2}}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^n e^{-n}$

10. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$

11. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

12. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

13. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

14. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

15. $\displaystyle\sum_{n=0}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{1+\sqrt{n}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

17. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{\pi}{n}\right)$

18. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

20. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\sqrt{n+1} - \sqrt{n}\right)$

21–22 Graph both the sequence of terms and the sequence of partial sums on the same screen. Use the graph to make a rough estimate of the sum of the series. Then use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.

21. $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

22. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

23–26 Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?

23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6} \quad (|\text{error}| < 0.00005)$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,5^n} \quad (|\text{error}| < 0.0001)$

25. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!} \quad (|\text{error}| < 0.000005)$

26. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n} \quad (|\text{error}| < 0.01)$

27–30 Approximate the sum of the series correct to four decimal places.

27. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n n!}$

31. Is the 50th partial sum s_{50} of the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1}/n$ an overestimate or an underestimate of the total sum? Explain.

32–34 For what values of p is each series convergent?

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

33. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+p}$

34. $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$

35. Show that the series $\sum (-1)^{n-1} b_n$, where $b_n = 1/n$ if n is odd and $b_n = 1/n^2$ if n is even, is divergent. Why does the Alternating Series Test not apply?

36. Use the following steps to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$$

Let h_n and s_n be the partial sums of the harmonic and alternating harmonic series.

(a) Show that $s_{2n} = h_{2n} - h_n$.

(b) From Exercise 44 in Section 11.3 we have

$$h_n - \ln n \to \gamma \qquad \text{as } n \to \infty$$

and therefore

$$h_{2n} - \ln(2n) \to \gamma \qquad \text{as } n \to \infty$$

Use these facts together with part (a) to show that $s_{2n} \to \ln 2$ as $n \to \infty$.

11.6 Absolute Convergence and the Ratio and Root Tests

Given any series $\Sigma\, a_n$, we can consider the corresponding series

$$\sum_{n=1}^{\infty} |a_n| = |a_1| + |a_2| + |a_3| + \cdots$$

whose terms are the absolute values of the terms of the original series.

We have convergence tests for series with positive terms and for alternating series. But what if the signs of the terms switch back and forth irregularly? We will see in Example 3 that the idea of absolute convergence sometimes helps in such cases.

1 Definition A series $\Sigma\, a_n$ is called **absolutely convergent** if the series of absolute values $\Sigma\, |a_n|$ is convergent.

Notice that if $\Sigma\, a_n$ is a series with positive terms, then $|a_n| = a_n$ and so absolute convergence is the same as convergence in this case.

EXAMPLE 1 The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

is absolutely convergent because

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

is a convergent p-series ($p = 2$).

EXAMPLE 2 We know that the alternating harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is convergent (see Example 1 in Section 11.5), but it is not absolutely convergent because the corresponding series of absolute values is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^{n-1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

which is the harmonic series (p-series with $p = 1$) and is therefore divergent.

2 Definition A series $\Sigma\, a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Example 2 shows that the alternating harmonic series is conditionally convergent. Thus it is possible for a series to be convergent but not absolutely convergent. However, the next theorem shows that absolute convergence implies convergence.

3 Theorem If a series $\Sigma\, a_n$ is absolutely convergent, then it is convergent.

PROOF Observe that the inequality

$$0 \leqslant a_n + |a_n| \leqslant 2|a_n|$$

is true because $|a_n|$ is either a_n or $-a_n$. If $\Sigma\, a_n$ is absolutely convergent, then $\Sigma\, |a_n|$ is convergent, so $\Sigma\, 2|a_n|$ is convergent. Therefore, by the Comparison Test, $\Sigma\, (a_n + |a_n|)$ is convergent. Then

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|$$

is the difference of two convergent series and is therefore convergent.

V EXAMPLE 3 Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2} = \frac{\cos 1}{1^2} + \frac{\cos 2}{2^2} + \frac{\cos 3}{3^2} + \cdots$$

is convergent or divergent.

Figure 1 shows the graphs of the terms a_n and partial sums s_n of the series in Example 3. Notice that the series is not alternating but has positive and negative terms.

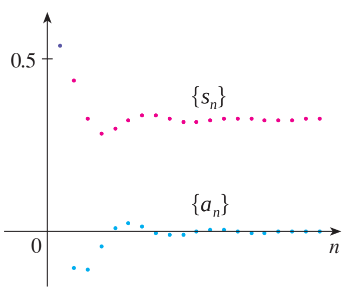


FIGURE 1

SOLUTION This series has both positive and negative terms, but it is not alternating. (The first term is positive, the next three are negative, and the following three are positive: The signs change irregularly.) We can apply the Comparison Test to the series of absolute values

$$\sum_{n=1}^{\infty} \left| \frac{\cos n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$$

Since $|\cos n| \leqslant 1$ for all n, we have

$$\frac{|\cos n|}{n^2} \leqslant \frac{1}{n^2}$$

We know that $\Sigma\, 1/n^2$ is convergent (p-series with $p = 2$) and therefore $\Sigma\, |\cos n|/n^2$ is convergent by the Comparison Test. Thus the given series $\Sigma\, (\cos n)/n^2$ is absolutely convergent and therefore convergent by Theorem 3.

The following test is very useful in determining whether a given series is absolutely convergent.

The Ratio Test

(i) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\Sigma\, a_n$.

PROOF

(i) The idea is to compare the given series with a convergent geometric series. Since $L < 1$, we can choose a number r such that $L < r < 1$. Since

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \qquad \text{and} \qquad L < r$$

the ratio $|a_{n+1}/a_n|$ will eventually be less than r; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < r \qquad \text{whenever } n \geqslant N$$

or, equivalently,

$$\boxed{4} \qquad |a_{n+1}| < |a_n|\, r \qquad \text{whenever } n \geqslant N$$

Putting n successively equal to N, $N + 1$, $N + 2$, . . . in $\boxed{4}$, we obtain

$$|a_{N+1}| < |a_N|\, r$$

$$|a_{N+2}| < |a_{N+1}|\, r < |a_N|\, r^2$$

$$|a_{N+3}| < |a_{N+2}|\, r < |a_N|\, r^3$$

and, in general,

$$\boxed{5} \qquad |a_{N+k}| < |a_N|\, r^k \qquad \text{for all } k \geqslant 1$$

Now the series

$$\sum_{k=1}^{\infty} |a_N|\, r^k = |a_N|\, r + |a_N|\, r^2 + |a_N|\, r^3 + \cdots$$

is convergent because it is a geometric series with $0 < r < 1$. So the inequality $\boxed{5}$ together with the Comparison Test, shows that the series

$$\sum_{n=N+1}^{\infty} |a_n| = \sum_{k=1}^{\infty} |a_{N+k}| = |a_{N+1}| + |a_{N+2}| + |a_{N+3}| + \cdots$$

is also convergent. It follows that the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. (Recall that a finite number of terms doesn't affect convergence.) Therefore $\sum a_n$ is absolutely convergent.

(ii) If $|a_{n+1}/a_n| \to L > 1$ or $|a_{n+1}/a_n| \to \infty$, then the ratio $|a_{n+1}/a_n|$ will eventually be greater than 1; that is, there exists an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \qquad \text{whenever } n \geqslant N$$

This means that $|a_{n+1}| > |a_n|$ whenever $n \geqslant N$ and so

$$\lim_{n\to\infty} a_n \neq 0$$

Therefore $\sum a_n$ diverges by the Test for Divergence.

NOTE Part (iii) of the Ratio Test says that if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the test gives no information. For instance, for the convergent series $\sum 1/n^2$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{(n+1)^2}}{\dfrac{1}{n^2}} = \frac{n^2}{(n+1)^2} = \frac{1}{\left(1+\dfrac{1}{n}\right)^2} \to 1 \qquad \text{as } n \to \infty$$

whereas for the divergent series $\sum 1/n$ we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\dfrac{1}{n+1}}{\dfrac{1}{n}} = \frac{n}{n+1} = \frac{1}{1+\dfrac{1}{n}} \to 1 \qquad \text{as } n \to \infty$$

The Ratio Test is usually conclusive if the nth term of the series contains an exponential or a factorial, as we will see in Examples 4 and 5.

Therefore, if $\lim_{n\to\infty} |a_{n+1}/a_n| = 1$, the series $\sum a_n$ might converge or it might diverge. In this case the Ratio Test fails and we must use some other test.

EXAMPLE 4 Test the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{3^n}$ for absolute convergence.

SOLUTION We use the Ratio Test with $a_n = (-1)^n n^3/3^n$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\dfrac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\dfrac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$$

$$= \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the Ratio Test, the given series is absolutely convergent and therefore convergent.

Estimating Sums

In the last three sections we used various methods for estimating the sum of a series—the method depended on which test was used to prove convergence. What about series for which the Ratio Test works? There are two possibilities: If the series happens to be an alternating series, as in Example 4, then it is best to use the methods of Section 11.5. If the terms are all positive, then use the special methods explained in Exercise 38.

V EXAMPLE 5 Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

SOLUTION Since the terms $a_n = n^n/n!$ are positive, we don't need the absolute value signs.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \\ &= \frac{(n+1)(n+1)^n}{(n+1)n!} \cdot \frac{n!}{n^n} \\ &= \left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n \to e \qquad \text{as } n \to \infty\end{aligned}$$

(see Equation 6.4.9 or 6.4*.9). Since $e > 1$, the given series is divergent by the Ratio Test.

NOTE Although the Ratio Test works in Example 5, an easier method is to use the Test for Divergence. Since

$$a_n = \frac{n^n}{n!} = \frac{n \cdot n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} \geqslant n$$

it follows that a_n does not approach 0 as $n \to \infty$. Therefore the given series is divergent by the Test for Divergence.

The following test is convenient to apply when nth powers occur. Its proof is similar to the proof of the Ratio Test and is left as Exercise 41.

The Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, then part (iii) of the Root Test says that the test gives no information. The series $\Sigma\, a_n$ could converge or diverge. (If $L = 1$ in the Ratio Test, don't try the Root Test because L will again be 1. And if $L = 1$ in the Root Test, don't try the Ratio Test because it will fail too.)

V EXAMPLE 6 Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

SOLUTION

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2 + \frac{3}{n}}{3 + \frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the given series converges by the Root Test.

Rearrangements

The question of whether a given convergent series is absolutely convergent or conditionally convergent has a bearing on the question of whether infinite sums behave like finite sums.

If we rearrange the order of the terms in a finite sum, then of course the value of the sum remains unchanged. But this is not always the case for an infinite series. By a **rearrangement** of an infinite series $\Sigma\, a_n$ we mean a series obtained by simply changing the order of the terms. For instance, a rearrangement of $\Sigma\, a_n$ could start as follows:

$$a_1 + a_2 + a_5 + a_3 + a_4 + a_{15} + a_6 + a_7 + a_{20} + \cdots$$

It turns out that

> if $\Sigma\, a_n$ is an absolutely convergent series with sum s, then any rearrangement of $\Sigma\, a_n$ has the same sum s.

However, any conditionally convergent series can be rearranged to give a different sum. To illustrate this fact let's consider the alternating harmonic series

$$\boxed{6} \qquad 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \tfrac{1}{7} - \tfrac{1}{8} + \cdots = \ln 2$$

(See Exercise 36 in Section 11.5.) If we multiply this series by $\tfrac{1}{2}$, we get

$$\tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{6} - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Inserting zeros between the terms of this series, we have

Adding these zeros does not affect the sum of the series; each term in the sequence of partial sums is repeated, but the limit is the same.

$$\boxed{7} \qquad 0 + \tfrac{1}{2} + 0 - \tfrac{1}{4} + 0 + \tfrac{1}{6} + 0 - \tfrac{1}{8} + \cdots = \tfrac{1}{2}\ln 2$$

Now we add the series in Equations 6 and 7 using Theorem 11.2.8:

$$\boxed{8} \qquad 1 + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \cdots = \tfrac{3}{2}\ln 2$$

Notice that the series in $\boxed{8}$ contains the same terms as in $\boxed{6}$, but rearranged so that one negative term occurs after each pair of positive terms. The sums of these series, however, are different. In fact, Riemann proved that

> if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number whatsoever, then there is a rearrangement of $\Sigma\, a_n$ that has a sum equal to r.

A proof of this fact is outlined in Exercise 44.

11.6 Exercises

1. What can you say about the series $\Sigma\, a_n$ in each of the following cases?

(a) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 8$ (b) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0.8$

(c) $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

2–30 Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

2. $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n^2}$

3. $\displaystyle\sum_{n=1}^{\infty}\frac{n}{5^n}$ **4.** $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n^2+4}$

5. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{5n+1}$ **6.** $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^n}{(2n+1)!}$

7. $\displaystyle\sum_{k=1}^{\infty}k\left(\tfrac{2}{3}\right)^k$ **8.** $\displaystyle\sum_{n=1}^{\infty}\frac{n!}{100^n}$

9. $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(1.1)^n}{n^4}$ **10.** $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n}{\sqrt{n^3+2}}$

11. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n e^{1/n}}{n^3}$ **12.** $\displaystyle\sum_{n=1}^{\infty}\frac{\sin 4n}{4^n}$

13. $\displaystyle\sum_{n=1}^{\infty}\frac{10^n}{(n+1)4^{2n+1}}$ **14.** $\displaystyle\sum_{n=1}^{\infty}\frac{n^{10}}{(-10)^{n+1}}$

1. Homework Hints available at stewartcalculus.com

15. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^2}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{3 - \cos n}{n^{2/3} - 2}$

17. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

21. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2 + 1}{2n^2 + 1}\right)^n$

22. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-2n}{n + 1}\right)^{5n}$

23. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100} 100^n}{n!}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n^2}}{n!}$

27. $\displaystyle 1 - \frac{1 \cdot 3}{3!} + \frac{1 \cdot 3 \cdot 5}{5!} - \frac{1 \cdot 3 \cdot 5 \cdot 7}{7!} + \cdots + (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{(2n - 1)!} + \cdots$

28. $\displaystyle \frac{2}{5} + \frac{2 \cdot 6}{5 \cdot 8} + \frac{2 \cdot 6 \cdot 10}{5 \cdot 8 \cdot 11} + \frac{2 \cdot 6 \cdot 10 \cdot 14}{5 \cdot 8 \cdot 11 \cdot 14} + \cdots$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!}$

30. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n + 2)}$

31. The terms of a series are defined recursively by the equations

$$a_1 = 2 \qquad a_{n+1} = \frac{5n + 1}{4n + 3} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

32. A series $\Sigma\, a_n$ is defined by the equations

$$a_1 = 1 \qquad a_{n+1} = \frac{2 + \cos n}{\sqrt{n}} a_n$$

Determine whether $\Sigma\, a_n$ converges or diverges.

33–34 Let $\{b_n\}$ be a sequence of positive numbers that converges to $\frac{1}{2}$. Determine whether the given series is absolutely convergent.

33. $\displaystyle\sum_{n=1}^{\infty} \frac{b_n^n \cos n\pi}{n}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{n^n b_1 b_2 b_3 \cdots b_n}$

35. For which of the following series is the Ratio Test inconclusive (that is, it fails to give a definite answer)?

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1 + n^2}$

36. For which positive integers k is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$$

37. (a) Show that $\Sigma_{n=0}^{\infty}\, x^n/n!$ converges for all x.
(b) Deduce that $\lim_{n\to\infty} x^n/n! = 0$ for all x.

38. Let $\Sigma\, a_n$ be a series with positive terms and let $r_n = a_{n+1}/a_n$. Suppose that $\lim_{n\to\infty} r_n = L < 1$, so $\Sigma\, a_n$ converges by the Ratio Test. As usual, we let R_n be the remainder after n terms, that is,

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots$$

(a) If $\{r_n\}$ is a decreasing sequence and $r_{n+1} < 1$, show, by summing a geometric series, that

$$R_n \leqslant \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) If $\{r_n\}$ is an increasing sequence, show that

$$R_n \leqslant \frac{a_{n+1}}{1 - L}$$

39. (a) Find the partial sum s_5 of the series $\Sigma_{n=1}^{\infty}\, 1/(n2^n)$. Use Exercise 38 to estimate the error in using s_5 as an approximation to the sum of the series.
(b) Find a value of n so that s_n is within 0.00005 of the sum. Use this value of n to approximate the sum of the series.

40. Use the sum of the first 10 terms to approximate the sum of the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Use Exercise 38 to estimate the error.

41. Prove the Root Test. [*Hint for part (i):* Take any number r such that $L < r < 1$ and use the fact that there is an integer N such that $\sqrt[n]{|a_n|} < r$ whenever $n \geqslant N$.]

42. Around 1910, the Indian mathematician Srinivasa Ramanujan discovered the formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!(1103 + 26390n)}{(n!)^4 396^{4n}}$$

William Gosper used this series in 1985 to compute the first 17 million digits of π.
(a) Verify that the series is convergent.
(b) How many correct decimal places of π do you get if you use just the first term of the series? What if you use two terms?

43. Given any series $\Sigma\, a_n$, we define a series $\Sigma\, a_n^+$ whose terms are all the positive terms of $\Sigma\, a_n$ and a series $\Sigma\, a_n^-$ whose terms are all the negative terms of $\Sigma\, a_n$. To be specific, we let

$$a_n^+ = \frac{a_n + |a_n|}{2} \qquad a_n^- = \frac{a_n - |a_n|}{2}$$

Notice that if $a_n > 0$, then $a_n^+ = a_n$ and $a_n^- = 0$, whereas if $a_n < 0$, then $a_n^- = a_n$ and $a_n^+ = 0$.
(a) If $\Sigma\, a_n$ is absolutely convergent, show that both of the series $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ are convergent.
(b) If $\Sigma\, a_n$ is conditionally convergent, show that both of the series $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ are divergent.

44. Prove that if $\Sigma\, a_n$ is a conditionally convergent series and r is any real number, then there is a rearrangement of $\Sigma\, a_n$ whose sum is r. [*Hints:* Use the notation of Exercise 43. Take just enough positive terms a_n^+ so that their sum is greater than r. Then add just enough negative terms a_n^- so that the cumulative sum is less than r. Continue in this manner and use Theorem 11.2.6.]

45. Suppose the series $\Sigma\, a_n$ is conditionally convergent.
(a) Prove that the series $\Sigma\, n^2 a_n$ is divergent.
(b) Conditional convergence of $\Sigma\, a_n$ is not enough to determine whether $\Sigma\, na_n$ is convergent. Show this by giving an example of a conditionally convergent series such that $\Sigma\, na_n$ converges and an example where $\Sigma\, na_n$ diverges.

11.7 Strategy for Testing Series

We now have several ways of testing a series for convergence or divergence; the problem is to decide which test to use on which series. In this respect, testing series is similar to integrating functions. Again there are no hard and fast rules about which test to apply to a given series, but you may find the following advice of some use.

It is not wise to apply a list of the tests in a specific order until one finally works. That would be a waste of time and effort. Instead, as with integration, the main strategy is to classify the series according to its *form.*

1. If the series is of the form $\Sigma\, 1/n^p$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.

2. If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply the Comparison Test to $\Sigma\, |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\Sigma\, (-1)^{n-1} b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

In the following examples we don't work out all the details but simply indicate which tests should be used.

V EXAMPLE 1 $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

Since $a_n \to \frac{1}{2} \neq 0$ as $n \to \infty$, we should use the Test for Divergence.

EXAMPLE 2 $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\Sigma\, b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{3/2}}{3n^3} = \frac{1}{3n^{3/2}}$$

V EXAMPLE 3 $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

Since the integral $\int_1^{\infty} xe^{-x^2}\,dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

EXAMPLE 4 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$

Since the series is alternating, we use the Alternating Series Test.

V EXAMPLE 5 $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

Since the series involves $k!$, we use the Ratio Test.

EXAMPLE 6 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+3^n}$

Since the series is closely related to the geometric series $\Sigma\, 1/3^n$, we use the Comparison Test.

11.7 Exercises

1–38 Test the series for convergence or divergence.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n+3^n}$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n+1)^n}{n^{2n}}$

3. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$

4. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+2}$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2\, 2^{n-1}}{(-5)^n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$

7. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{2^k k!}{(k+2)!}$

9. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

10. $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$

12. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2+1}}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

15. $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k-1}3^{k+1}}{k^k}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+1}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}$

18. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}-1}$

19. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)}$

21. $\displaystyle\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

22. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2 + \sin k}$

23. $\displaystyle\sum_{n=1}^{\infty} \tan(1/n)$

24. $\displaystyle\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$

27. $\displaystyle\sum_{k=1}^{\infty} \frac{k \ln k}{(k + 1)^3}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j + 5}$

31. $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n + 1}\right)^{n^2}$

34. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

35. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

36. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$

38. $\displaystyle\sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)$

11.8 Power Series

A **power series** is a series of the form

$$\boxed{1} \qquad \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series. For each fixed x, the series $\boxed{1}$ is a series of constants that we can test for convergence or divergence. A power series may converge for some values of x and diverge for other values of x. The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

whose domain is the set of all x for which the series converges. Notice that f resembles a polynomial. The only difference is that f has infinitely many terms.

For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geqslant 1$. (See Equation 11.2.5.)

More generally, a series of the form

$$\boxed{2} \qquad \sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots$$

is called a **power series in $(x - a)$** or a **power series centered at a** or a **power series about a**. Notice that in writing out the term corresponding to $n = 0$ in Equations 1 and 2 we have adopted the convention that $(x - a)^0 = 1$ even when $x = a$. Notice also that when $x = a$ all of the terms are 0 for $n \geqslant 1$ and so the power series $\boxed{2}$ always converges when $x = a$.

Trigonometric Series

A power series is a series in which each term is a power function. A **trigonometric series**

$$\sum_{n=0}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is a series whose terms are trigonometric functions. This type of series is discussed on the website

www.stewartcalculus.com

Click on *Additional Topics* and then on *Fourier Series*.

V EXAMPLE 1 For what values of x is the series $\displaystyle\sum_{n=0}^{\infty} n!\,x^n$ convergent?

SOLUTION We use the Ratio Test. If we let a_n, as usual, denote the nth term of the series, then $a_n = n!\,x^n$. If $x \neq 0$, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n + 1)!\,x^{n+1}}{n!\,x^n}\right| = \lim_{n\to\infty} (n + 1)|x| = \infty$$

Notice that

$$(n + 1)! = (n + 1)n(n - 1) \cdot \cdots \cdot 3 \cdot 2 \cdot 1 = (n + 1)n!$$

By the Ratio Test, the series diverges when $x \neq 0$. Thus the given series converges only when $x = 0$.

V EXAMPLE 2 For what values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

SOLUTION Let $a_n = (x-3)^n/n$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n}\right|$$

$$= \frac{1}{1+\dfrac{1}{n}}|x-3| \to |x-3| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series is absolutely convergent, and therefore convergent, when $|x-3| < 1$ and divergent when $|x-3| > 1$. Now

$$|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$$

so the series converges when $2 < x < 4$ and diverges when $x < 2$ or $x > 4$.

The Ratio Test gives no information when $|x-3| = 1$ so we must consider $x = 2$ and $x = 4$ separately. If we put $x = 4$ in the series, it becomes $\Sigma\, 1/n$, the harmonic series, which is divergent. If $x = 2$, the series is $\Sigma\, (-1)^n/n$, which converges by the Alternating Series Test. Thus the given power series converges for $2 \leqslant x < 4$.

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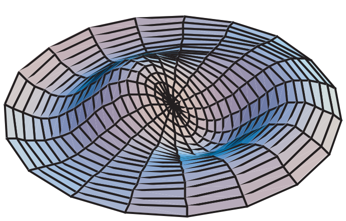

Notice how closely the computer-generated model (which involves Bessel functions and cosine functions) matches the photograph of a vibrating rubber membrane.

We will see that the main use of a power series is that it provides a way to represent some of the most important functions that arise in mathematics, physics, and chemistry. In particular, the sum of the power series in the next example is called a **Bessel function**, after the German astronomer Friedrich Bessel (1784–1846), and the function given in Exercise 35 is another example of a Bessel function. In fact, these functions first arose when Bessel solved Kepler's equation for describing planetary motion. Since that time, these functions have been applied in many different physical situations, including the temperature distribution in a circular plate and the shape of a vibrating drumhead.

EXAMPLE 3 Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

SOLUTION Let $a_n = (-1)^n x^{2n}/[2^{2n}(n!)^2]$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1}x^{2(n+1)}}{2^{2(n+1)}[(n+1)!]^2} \cdot \frac{2^{2n}(n!)^2}{(-1)^n x^{2n}}\right|$$

$$= \frac{x^{2n+2}}{2^{2n+2}(n+1)^2(n!)^2} \cdot \frac{2^{2n}(n!)^2}{x^{2n}}$$

$$= \frac{x^2}{4(n+1)^2} \to 0 < 1 \qquad \text{for all } x$$

Thus, by the Ratio Test, the given series converges for all values of x. In other words, the domain of the Bessel function J_0 is $(-\infty, \infty) = \mathbb{R}$.

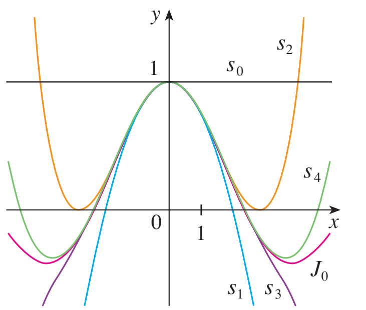

FIGURE 1
Partial sums of the Bessel function J_0

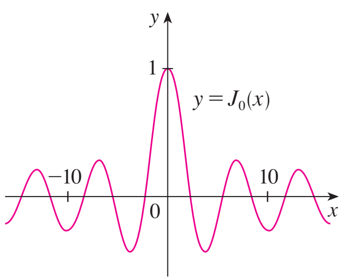

FIGURE 2

Recall that the sum of a series is equal to the limit of the sequence of partial sums. So when we define the Bessel function in Example 3 as the sum of a series we mean that, for every real number x,

$$J_0(x) = \lim_{n\to\infty} s_n(x) \qquad \text{where} \qquad s_n(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{2^{2i}(i!)^2}$$

The first few partial sums are

$$s_0(x) = 1 \qquad s_1(x) = 1 - \frac{x^2}{4} \qquad s_2(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64}$$

$$s_3(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} \qquad s_4(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \frac{x^8}{147{,}456}$$

Figure 1 shows the graphs of these partial sums, which are polynomials. They are all approximations to the function J_0, but notice that the approximations become better when more terms are included. Figure 2 shows a more complete graph of the Bessel function.

For the power series that we have looked at so far, the set of values of x for which the series is convergent has always turned out to be an interval [a finite interval for the geometric series and the series in Example 2, the infinite interval $(-\infty, \infty)$ in Example 3, and a collapsed interval $[0, 0] = \{0\}$ in Example 1]. The following theorem, proved in Appendix F, says that this is true in general.

3 Theorem For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$. When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints. Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a - R, a + R) \qquad (a - R, a + R] \qquad [a - R, a + R) \qquad [a - R, a + R]$$

The situation is illustrated in Figure 3.

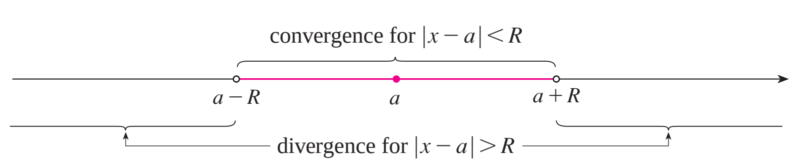

FIGURE 3

We summarize here the radius and interval of convergence for each of the examples already considered in this section.

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \dfrac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

In general, the Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

EXAMPLE 4 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

SOLUTION Let $a_n = (-3)^n x^n/\sqrt{n+1}$. Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n x^n}\right| = \left|-3x\sqrt{\frac{n+1}{n+2}}\right|$$

$$= 3\sqrt{\frac{1+(1/n)}{1+(2/n)}}\,|x| \to 3|x| \qquad \text{as } n \to \infty$$

By the Ratio Test, the given series converges if $3|x| < 1$ and diverges if $3|x| > 1$. Thus it converges if $|x| < \frac{1}{3}$ and diverges if $|x| > \frac{1}{3}$. This means that the radius of convergence is $R = \frac{1}{3}$.

We know the series converges in the interval $\left(-\frac{1}{3}, \frac{1}{3}\right)$, but we must now test for convergence at the endpoints of this interval. If $x = -\frac{1}{3}$, the series becomes

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(-\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \cdots$$

which diverges. (Use the Integral Test or simply observe that it is a p-series with $p = \frac{1}{2} < 1$.) If $x = \frac{1}{3}$, the series is

$$\sum_{n=0}^{\infty} \frac{(-3)^n\left(\frac{1}{3}\right)^n}{\sqrt{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

which converges by the Alternating Series Test. Therefore the given power series converges when $-\frac{1}{3} < x \leqslant \frac{1}{3}$, so the interval of convergence is $\left(-\frac{1}{3}, \frac{1}{3}\right]$.

V EXAMPLE 5 Find the radius of convergence and interval of convergence of the series

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

SOLUTION If $a_n = n(x+2)^n/3^{n+1}$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n}\right|$$

$$= \left(1 + \frac{1}{n}\right)\frac{|x+2|}{3} \to \frac{|x+2|}{3} \qquad \text{as } n \to \infty$$

Using the Ratio Test, we see that the series converges if $|x+2|/3 < 1$ and it diverges if $|x+2|/3 > 1$. So it converges if $|x+2| < 3$ and diverges if $|x+2| > 3$. Thus the radius of convergence is $R = 3$.

The inequality $|x+2| < 3$ can be written as $-5 < x < 1$, so we test the series at the endpoints -5 and 1. When $x = -5$, the series is

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} (-1)^n n$$

which diverges by the Test for Divergence [$(-1)^n n$ doesn't converge to 0]. When $x = 1$, the series is

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \tfrac{1}{3}\sum_{n=0}^{\infty} n$$

which also diverges by the Test for Divergence. Thus the series converges only when $-5 < x < 1$, so the interval of convergence is $(-5, 1)$.

11.8 Exercises

1. What is a power series?
2. (a) What is the radius of convergence of a power series? How do you find it?
(b) What is the interval of convergence of a power series? How do you find it?

3–28 Find the radius of convergence and interval of convergence of the series.

3. $\sum_{n=1}^{\infty} (-1)^n n x^n$
4. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{\sqrt[3]{n}}$
5. $\sum_{n=1}^{\infty} \frac{x^n}{2n-1}$
6. $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n^2}$
7. $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
8. $\sum_{n=1}^{\infty} n^n x^n$
9. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 x^n}{2^n}$
10. $\sum_{n=1}^{\infty} \frac{10^n x^n}{n^3}$
11. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n\sqrt{n}} x^n$
12. $\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$
13. $\sum_{n=2}^{\infty} (-1)^n \frac{x^n}{4^n \ln n}$
14. $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$
15. $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$
16. $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2n+1}$
17. $\sum_{n=1}^{\infty} \frac{3^n(x+4)^n}{\sqrt{n}}$
18. $\sum_{n=1}^{\infty} \frac{n}{4^n}(x+1)^n$
19. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$
20. $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{5^n\sqrt{n}}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b > 0$

22. $\displaystyle\sum_{n=2}^{\infty} \frac{b^n}{\ln n}(x-a)^n, \quad b > 0$

23. $\displaystyle\sum_{n=1}^{\infty} n!(2x-1)^n$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 x^n}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{x^{2n}}{n(\ln n)^2}$

27. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$

---

29. If $\sum_{n=0}^{\infty} c_n 4^n$ is convergent, does it follow that the following series are convergent?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n(-2)^n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n(-4)^n$

30. Suppose that $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

(a) $\displaystyle\sum_{n=0}^{\infty} c_n$ (b) $\displaystyle\sum_{n=0}^{\infty} c_n 8^n$

(c) $\displaystyle\sum_{n=0}^{\infty} c_n(-3)^n$ (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

31. If k is a positive integer, find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

32. Let p and q be real numbers with $p < q$. Find a power series whose interval of convergence is

(a) (p, q) (b) $(p, q]$

(c) $[p, q)$ (d) $[p, q]$

33. Is it possible to find a power series whose interval of convergence is $[0, \infty)$? Explain.

34. Graph the first several partial sums $s_n(x)$ of the series $\sum_{n=0}^{\infty} x^n$, together with the sum function $f(x) = 1/(1-x)$, on a common screen. On what interval do these partial sums appear to be converging to $f(x)$?

35. The function J_1 defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$$

is called the *Bessel function of order 1.*

(a) Find its domain.

(b) Graph the first several partial sums on a common screen.

CAS (c) If your CAS has built-in Bessel functions, graph J_1 on the same screen as the partial sums in part (b) and observe how the partial sums approximate J_1.

36. The function A defined by

$$A(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \cdots$$

is called an *Airy function* after the English mathematician and astronomer Sir George Airy (1801–1892).

(a) Find the domain of the Airy function.

(b) Graph the first several partial sums on a common screen.

CAS (c) If your CAS has built-in Airy functions, graph A on the same screen as the partial sums in part (b) and observe how the partial sums approximate A.

37. A function f is defined by

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + \cdots$$

that is, its coefficients are $c_{2n} = 1$ and $c_{2n+1} = 2$ for all $n \geqslant 0$. Find the interval of convergence of the series and find an explicit formula for $f(x)$.

38. If $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_{n+4} = c_n$ for all $n \geqslant 0$, find the interval of convergence of the series and a formula for $f(x)$.

39. Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

40. Suppose that the power series $\sum c_n(x-a)^n$ satisfies $c_n \neq 0$ for all n. Show that if $\lim_{n\to\infty} |c_n/c_{n+1}|$ exists, then it is equal to the radius of convergence of the power series.

41. Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n)x^n$?

42. Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?

11.9 Representations of Functions as Power Series

In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series. You might wonder why we would ever want to express a known function as a sum of infinitely many terms. We will see later that this strategy is useful for integrating functions that don't have elementary antiderivatives, for solving differential equations, and for approximating functions by polynomials. (Scientists do this to simplify the expressions they deal with; computer scientists do this to represent functions on calculators and computers.)

We start with an equation that we have seen before:

$$\boxed{1} \qquad \boxed{\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \qquad |x| < 1}$$

A geometric illustration of Equation 1 is shown in Figure 1. Because the sum of a series is the limit of the sequence of partial sums, we have

$$\frac{1}{1-x} = \lim_{n\to\infty} s_n(x)$$

where

$$s_n(x) = 1 + x + x^2 + \cdots + x^n$$

is the nth partial sum. Notice that as n increases, $s_n(x)$ becomes a better approximation to $f(x)$ for $-1 < x < 1$.

We first encountered this equation in Example 6 in Section 11.2, where we obtained it by observing that the series is a geometric series with $a = 1$ and $r = x$. But here our point of view is different. We now regard Equation 1 as expressing the function $f(x) = 1/(1 - x)$ as a sum of a power series.

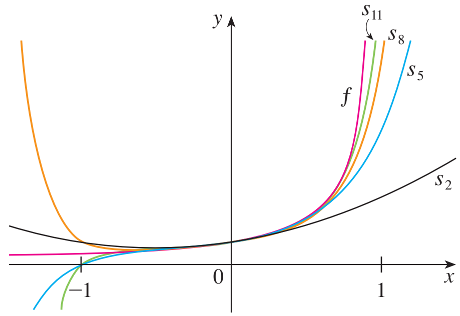

FIGURE 1

$f(x) = \dfrac{1}{1-x}$ and some partial sums

V EXAMPLE 1 Express $1/(1 + x^2)$ as the sum of a power series and find the interval of convergence.

SOLUTION Replacing x by $-x^2$ in Equation 1, we have

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

Because this is a geometric series, it converges when $|-x^2| < 1$, that is, $x^2 < 1$, or $|x| < 1$. Therefore the interval of convergence is $(-1, 1)$. (Of course, we could have determined the radius of convergence by applying the Ratio Test, but that much work is unnecessary here.)

EXAMPLE 2 Find a power series representation for $1/(x + 2)$.

SOLUTION In order to put this function in the form of the left side of Equation 1, we first factor a 2 from the denominator:

$$\frac{1}{2+x} = \frac{1}{2\left(1+\dfrac{x}{2}\right)} = \frac{1}{2\left[1-\left(-\dfrac{x}{2}\right)\right]}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n$$

This series converges when $|-x/2| < 1$, that is, $|x| < 2$. So the interval of convergence is $(-2, 2)$.

EXAMPLE 3 Find a power series representation of $x^3/(x+2)$.

SOLUTION Since this function is just x^3 times the function in Example 2, all we have to do is to multiply that series by x^3:

It's legitimate to move x^3 across the sigma sign because it doesn't depend on n. [Use Theorem 11.2.8(i) with $c = x^3$.]

$$\begin{aligned}\frac{x^3}{x+2} &= x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} \\ &= \tfrac{1}{2}x^3 - \tfrac{1}{4}x^4 + \tfrac{1}{8}x^5 - \tfrac{1}{16}x^6 + \cdots\end{aligned}$$

Another way of writing this series is as follows:

$$\frac{x^3}{x+2} = \sum_{n=3}^{\infty} \frac{(-1)^{n-1}}{2^{n-2}} x^n$$

As in Example 2, the interval of convergence is $(-2, 2)$.

Differentiation and Integration of Power Series

The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series. We would like to be able to differentiate and integrate such functions, and the following theorem (which we won't prove) says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial. This is called **term-by-term differentiation and integration**.

2 Theorem If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\begin{aligned}\int f(x)\,dx &= C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots \\ &= C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}\end{aligned}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

In part (ii), $\int c_0\,dx = c_0x + C_1$ is written as $c_0(x-a) + C$, where $C = C_1 + ac_0$, so all the terms of the series have the same form.

NOTE 1 Equations (i) and (ii) in Theorem 2 can be rewritten in the form

(iii) $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n(x-a)^n\right]$$

(iv) $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$$

We know that, for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. Equations (iii) and (iv) assert that the same is true for infinite sums, provided we are dealing with *power series*. (For other types of series of functions the situation is not as simple; see Exercise 38.)

NOTE 2 Although Theorem 2 says that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the *interval* of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there. (See Exercise 39.)

NOTE 3 The idea of differentiating a power series term by term is the basis for a powerful method for solving differential equations. We will discuss this method in Chapter 17.

EXAMPLE 4 In Example 3 in Section 11.8 we saw that the Bessel function

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

is defined for all x. Thus, by Theorem 2, J_0 is differentiable for all x and its derivative is found by term-by-term differentiation as follows:

$$J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$$

V EXAMPLE 5 Express $1/(1-x)^2$ as a power series by differentiating Equation 1. What is the radius of convergence?

SOLUTION Differentiating each side of the equation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{n=1}^{\infty} n x^{n-1}$$

If we wish, we can replace n by $n + 1$ and write the answer as

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1) x^n$$

According to Theorem 2, the radius of convergence of the differentiated series is the same as the radius of convergence of the original series, namely, $R = 1$.

EXAMPLE 6 Find a power series representation for $\ln(1 + x)$ and its radius of convergence.

SOLUTION We notice that the derivative of this function is $1/(1 + x)$. From Equation 1 we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + \cdots \qquad |x| < 1$$

Integrating both sides of this equation, we get

$$\begin{aligned}\ln(1+x) &= \int \frac{1}{1+x}\,dx = \int (1 - x + x^2 - x^3 + \cdots)\,dx \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + C \\ &= \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} + C \qquad |x| < 1\end{aligned}$$

To determine the value of C we put $x = 0$ in this equation and obtain $\ln(1 + 0) = C$. Thus $C = 0$ and

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n} \qquad |x| < 1$$

The radius of convergence is the same as for the original series: $R = 1$.

V EXAMPLE 7 Find a power series representation for $f(x) = \tan^{-1}x$.

SOLUTION We observe that $f'(x) = 1/(1 + x^2)$ and find the required series by integrating the power series for $1/(1 + x^2)$ found in Example 1.

$$\begin{aligned}\tan^{-1}x &= \int \frac{1}{1+x^2}\,dx = \int (1 - x^2 + x^4 - x^6 + \cdots)\,dx \\ &= C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\end{aligned}$$

The power series for $\tan^{-1}x$ obtained in Example 7 is called *Gregory's series* after the Scottish mathematician James Gregory (1638–1675), who had anticipated some of Newton's discoveries. We have shown that Gregory's series is valid when $-1 < x < 1$, but it turns out (although it isn't easy to prove) that it is also valid when $x = \pm 1$. Notice that when $x = 1$ the series becomes

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

This beautiful result is known as the Leibniz formula for π.

To find C we put $x = 0$ and obtain $C = \tan^{-1}0 = 0$. Therefore

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

Since the radius of convergence of the series for $1/(1 + x^2)$ is 1, the radius of convergence of this series for $\tan^{-1}x$ is also 1.

EXAMPLE 8
(a) Evaluate $\int [1/(1 + x^7)]\,dx$ as a power series.
(b) Use part (a) to approximate $\int_0^{0.5} [1/(1 + x^7)]\,dx$ correct to within 10^{-7}.

SOLUTION
(a) The first step is to express the integrand, $1/(1 + x^7)$, as the sum of a power series. As in Example 1, we start with Equation 1 and replace x by $-x^7$:

$$\begin{aligned}\frac{1}{1+x^7} &= \frac{1}{1-(-x^7)} = \sum_{n=0}^{\infty} (-x^7)^n \\ &= \sum_{n=0}^{\infty} (-1)^n x^{7n} = 1 - x^7 + x^{14} - \cdots\end{aligned}$$

This example demonstrates one way in which power series representations are useful. Integrating $1/(1+x^7)$ by hand is incredibly difficult. Different computer algebra systems return different forms of the answer, but they are all extremely complicated. (If you have a CAS, try it yourself.) The infinite series answer that we obtain in Example 8(a) is actually much easier to deal with than the finite answer provided by a CAS.

Now we integrate term by term:

$$\int \frac{1}{1+x^7}\,dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1}$$

$$= C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots$$

This series converges for $|-x^7| < 1$, that is, for $|x| < 1$.

(b) In applying the Fundamental Theorem of Calculus, it doesn't matter which antiderivative we use, so let's use the antiderivative from part (a) with $C = 0$:

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx = \left[x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \cdots \right]_0^{1/2}$$

$$= \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \cdots + \frac{(-1)^n}{(7n+1)2^{7n+1}} + \cdots$$

This infinite series is the exact value of the definite integral, but since it is an alternating series, we can approximate the sum using the Alternating Series Estimation Theorem. If we stop adding after the term with $n = 3$, the error is smaller than the term with $n = 4$:

$$\frac{1}{29 \cdot 2^{29}} \approx 6.4 \times 10^{-11}$$

So we have

$$\int_0^{0.5} \frac{1}{1+x^7}\,dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374$$

11.9 Exercises

1. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is 10, what is the radius of convergence of the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$? Why?

2. Suppose you know that the series $\sum_{n=0}^{\infty} b_n x^n$ converges for $|x| < 2$. What can you say about the following series? Why?

$$\sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$$

3–10 Find a power series representation for the function and determine the interval of convergence.

3. $f(x) = \dfrac{1}{1+x}$

4. $f(x) = \dfrac{5}{1-4x^2}$

5. $f(x) = \dfrac{2}{3-x}$

6. $f(x) = \dfrac{1}{x+10}$

7. $f(x) = \dfrac{x}{9+x^2}$

8. $f(x) = \dfrac{x}{2x^2+1}$

9. $f(x) = \dfrac{1+x}{1-x}$

10. $f(x) = \dfrac{x^2}{a^3-x^3}$

11–12 Express the function as the sum of a power series by first using partial fractions. Find the interval of convergence.

11. $f(x) = \dfrac{3}{x^2-x-2}$

12. $f(x) = \dfrac{x+2}{2x^2-x-1}$

13. (a) Use differentiation to find a power series representation for

$$f(x) = \frac{1}{(1+x)^2}$$

What is the radius of convergence?

(b) Use part (a) to find a power series for

$$f(x) = \frac{1}{(1+x)^3}$$

(c) Use part (b) to find a power series for

$$f(x) = \frac{x^2}{(1+x)^3}$$

14. (a) Use Equation 1 to find a power series representation for $f(x) = \ln(1 - x)$. What is the radius of convergence?
(b) Use part (a) to find a power series for $f(x) = x \ln(1 - x)$.
(c) By putting $x = \frac{1}{2}$ in your result from part (a), express ln 2 as the sum of an infinite series.

15–20 Find a power series representation for the function and determine the radius of convergence.

15. $f(x) = \ln(5 - x)$

16. $f(x) = x^2 \tan^{-1}(x^3)$

17. $f(x) = \dfrac{x}{(1+4x)^2}$

18. $f(x) = \left(\dfrac{x}{2-x}\right)^3$

19. $f(x) = \dfrac{1+x}{(1-x)^2}$

20. $f(x) = \dfrac{x^2+x}{(1-x)^3}$

21–24 Find a power series representation for f, and graph f and several partial sums $s_n(x)$ on the same screen. What happens as n increases?

21. $f(x) = \dfrac{x}{x^2+16}$

22. $f(x) = \ln(x^2 + 4)$

23. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right)$

24. $f(x) = \tan^{-1}(2x)$

25–28 Evaluate the indefinite integral as a power series. What is the radius of convergence?

25. $\displaystyle\int \frac{t}{1-t^8}\,dt$

26. $\displaystyle\int \frac{t}{1+t^3}\,dt$

27. $\displaystyle\int x^2 \ln(1+x)\,dx$

28. $\displaystyle\int \frac{\tan^{-1}x}{x}\,dx$

29–32 Use a power series to approximate the definite integral to six decimal places.

29. $\displaystyle\int_0^{0.2} \frac{1}{1+x^5}\,dx$

30. $\displaystyle\int_0^{0.4} \ln(1+x^4)\,dx$

31. $\displaystyle\int_0^{0.1} x \arctan(3x)\,dx$

32. $\displaystyle\int_0^{0.3} \frac{x^2}{1+x^4}\,dx$

33. Use the result of Example 7 to compute arctan 0.2 correct to five decimal places.

34. Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

is a solution of the differential equation

$$f''(x) + f(x) = 0$$

35. (a) Show that J_0 (the Bessel function of order 0 given in Example 4) satisfies the differential equation

$$x^2 J_0''(x) + x J_0'(x) + x^2 J_0(x) = 0$$

(b) Evaluate $\int_0^1 J_0(x)\,dx$ correct to three decimal places.

36. The Bessel function of order 1 is defined by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

(a) Show that J_1 satisfies the differential equation

$$x^2 J_1''(x) + x J_1'(x) + (x^2 - 1) J_1(x) = 0$$

(b) Show that $J_0'(x) = -J_1(x)$.

37. (a) Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is a solution of the differential equation

$$f'(x) = f(x)$$

(b) Show that $f(x) = e^x$.

38. Let $f_n(x) = (\sin nx)/n^2$. Show that the series $\Sigma f_n(x)$ converges for all values of x but the series of derivatives $\Sigma f_n'(x)$ diverges when $x = 2n\pi$, n an integer. For what values of x does the series $\Sigma f_n''(x)$ converge?

39. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Find the intervals of convergence for f, f', and f''.

40. (a) Starting with the geometric series $\sum_{n=0}^{\infty} x^n$, find the sum of the series

$$\sum_{n=1}^{\infty} n x^{n-1} \qquad |x| < 1$$

(b) Find the sum of each of the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} n x^n, \quad |x| < 1$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$