MATRIX ALGEBRA

Matrices are one of the most important tools which is used not only to mention the coefficient in the system of linear equations but also used to solve the problems of input-output model in economics.

A matrix is simply a set of numbers arranged in rows and columns in a rectangular table.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{bmatrix}$$

This is an example of a 3×3 matrix. The number of rows and columns that a matrix has is called its dimension or its order. Thus, we can say that the dimension (or order) of the above matrix is 3×3. We usually write matrices inside brackets [ ].

Numbers that appear in the rows and columns of a matrix are called elements of the matrix. In the above matrix, the element in the first column of the first row is 1; the element in the second column of the first row is 2; and so on.

TYPES OF MATRICES

1- Column matrix

A matrix is said to be a column matrix if it has only one column. For example, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is a column matrix of order 3×1.

2- Row matrix

A matrix is said to be a row matrix if it has only one row. For example, $\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$ is a row matrix of order 1×3.

3- Square matrix

A matrix in which the number of rows are equal to the number of columns, is said to be a square matrix. Thus an $m \times n$ matrix is said to be a square matrix if $m = n$ and is known as a square matrix of order 'n'. For example,

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 1 & 0 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Matrix A is a square matrix of order 2 while Matrix B is a square matrix of order 3.

4- Equal Metrices:

Two matrices are equal if their corresponding entries are equal. If two metrices fulfill the following conditions, they are said to be equal:

Each matrix has the same number of rows.

Each matrix has the same number of columns.

Corresponding elements within each matrix are equal.

Consider the three matrices shown below.

$$A = \begin{bmatrix} 2 & x \\ y & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 4 \end{bmatrix}$$

If A = B, we know that x = 3 and y = 0; since corresponding elements of equal matrices are also equal. And we know that matrix C is not equal to A or B, because C has more columns than A or B.

5- Diagonal matrix

A square matrix is said to be a diagonal matrix if all its non diagonal elements are zero. For example,

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ are diagonal matrices of order 2, 3, 3, respectively.

6- Scalar matrix

A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal. For example,

$$\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

are scalar matrices of order 2 and 3, respectively.

7- Identity matrix

A square matrix in which elements in the diagonal are all 1 and rest are all zero is called an identity matrix or unit matrix. For example,

For example,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ are identity matrices of order 2 and 3, respectively.

Note: A scalar matrix is an identity matrix when K = 1. But every identity matrix is clearly a scalar matrix. The identity matrix has a unique feature that if a matrix that is multiplied by identity matrix remains the same; that is: AI = IA = A

8- Null matrix

A matrix is said to be zero matrix or null matrix if all its elements are zero. For example,

$$\begin{bmatrix} 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

are all zero matrices of orders 1×2, 2×2, 3×1 and 2×3

TRANSPOSE OF A MATRIX:

Transpose of a matrix is obtained by interchanging rows and columns of a matrix. For example if a matrix is:

$\begin{bmatrix} 3 & 2 \\ 5 & 4 \\ 1 & 2 \end{bmatrix}$ then transpose of above matrix will be $\begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 2 \end{bmatrix}$

When we transpose a matrix then the order of matrix changes, but for a square matrix order remains same.

INVERSE MATRICES

Now we come to Inverse Matrices. We know that the inverse matrix can be found out if the matrix is a square matrix, it is non–singular and it meets the following relationship: $AA^{-1} = I = A^{-1}A$. It means that if we multiply the matrix with its inverse we get Identity Matrix. Now we present a general method to find inverse of square matrix; (1) The $|A|$ is calculated. (2) The cofactors of all the elements of matrix are found out. (3) The transposes of C matrix ('C) be made to get AdjA. (4) The AdjA be divided by $|A|$. As a result of such operation A^{-1} (inverse matrix) will be obtained.

Example 1: If $A = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$

If any matrix is of 2 × 2 it can directly yield C′ or Adj*A*. For this purpose the elements in principal diagonal are interchanged while the elements in off principal diagonal are changed as well as their signs.

$$|A| = 3(0) - 2(1) = \boxed{-2} \neq 0$$

$$C = \begin{bmatrix} |c_{11}| & |c_{12}| \\ |c_{21}| & |c_{22}| \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & 3 \end{bmatrix} \Rightarrow C' = \text{Adj}A = \begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj}A = \frac{1}{-2}\begin{bmatrix} 0 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1/2 & -3/2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{3}{2} \end{bmatrix}$$

CRAMER'S RULE FOR MATRIX SOLUTIONS

According to Cramer Rule $x_i = \frac{|A_i|}{|A|}$. where x_i is unknown variable while $|A|$ is the determinant of matrix of coefficients. While $|A_i|$ is the determinant of that special matrix which comes into being by placing the column vectors against its columns in the matrix i.e. find The value of x_1 and x_2.

Example 1: Cramer rule is applied to solve the system of equations.

$6x_1 + 5x_2 = 49, \ 3x_1 + 4x_2 = 32$ Writing the equation in matrix.

$$\begin{bmatrix} 6 & 5 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 49 \\ 32 \end{bmatrix} \qquad |A| = \begin{vmatrix} 6 & 5 \\ 3 & 4 \end{vmatrix} 6(4) - 3(5) = 9$$

Replacing the column 1 of matrix A by constants we construct A_1 and then find $|A|$.

$$A_1 = \begin{bmatrix} 49 & 5 \\ 32 & 4 \end{bmatrix}, \ |A_1| = 49(4) - 32(5) = 36$$

According to Cramer rule. $x_1 = \frac{|A_1|}{|A|} = \frac{36}{9} = 4$

To find the value of x_2 the column vectors are entered in the column 2 of matrix. It is given the name A_2. It is found as:

$$A_2 = \begin{bmatrix} 6 & 49 \\ 3 & 32 \end{bmatrix}, \ |A_2| = \begin{vmatrix} 6 & 49 \\ 3 & 32 \end{vmatrix} = 6(32) - 3(49) = 45$$

Value of x_2 is found with Cramer rule. $x_2 = \frac{|A_2|}{|A|} = \frac{45}{9} = 5$ Thus we get $x_1 = 4$ and $x_2 = 5$.

Properties of Inverse Matrices

If A and B are two non–singular matrices having 2 × 2 orders, they are furnished with following properties.

$(A^{-1})^{-1} = A$ ———————— (1)

$(AB)^{-1} = B^{-1}A^{-1}$ ———————— (2)

$(A')^{-1} = (A^{-1})'$ ———————— (3)

[1 2]

The Gaussian Method of Inverting a Matrix

Under this method the augmented matrix is constructed having identity matrix on the right side. Then the row operation be made in such a way that the matrix of coefficients on the left side went on decreasing in the form of identity matrix. Then the matrix on the right side is the inverse matrix. Now we present in other words: We start with augmented matrix (A/I) and it is multiplied with A^{-1}, then as a result of row operations it will eventually be $\frac{1}{A^{-1}}$ where the identity matrix is on left side and inverse matrix is on right side.

Example1: Augmented matrix is constructed.

$$A = \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \qquad \begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Under row operation, the row 1 is multiplied with 1/4.

$$\begin{bmatrix} 1 & 1/4 & -5/4 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The double of row 1 be added in row 2 and three time of row 1 be subtracted from row 3.

$$\begin{bmatrix} 1 & 1/4 & -5/4 \\ 0 & 7/2 & -3/2 \\ 0 & -7/4 & 31/4 \end{bmatrix}\begin{bmatrix} 1/4 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/4 & 0 & 1 \end{bmatrix}$$

Multiplying the row 2 by 2/7.

$$\begin{bmatrix} 1 & 1/4 & -5/4 \\ 0 & 1 & -3/7 \\ 0 & -7/4 & 31/4 \end{bmatrix}\begin{bmatrix} 1/4 & 0 & 0 \\ 1/7 & 2/7 & 0 \\ -3/4 & 0 & 1 \end{bmatrix}$$

Subtracting 1/4 of row 2 and adding $\frac{7}{4}$ of row 2 in row 3.

$$\begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & -3/7 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 3/14 & -1/14 & 0 \\ 1/7 & 2/7 & 0 \\ -1/2 & 1/2 & 1 \end{bmatrix}$$

Multiplying row 2 by 1/7.

$$\begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & -3/7 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3/14 & -1/14 & 0 \\ 1/7 & 2/7 & 0 \\ -1/14 & 1/14 & 1/7 \end{bmatrix}$$

Adding 8/7 of row 3 in row 1 and adding 3/7 of row 3 in row 2.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 13/98 & 1/98 & 8/49 \\ 11/98 & 31/98 & 3/49 \\ -1/14 & 1/14 & 1/7 \end{bmatrix} \Rightarrow \qquad A^{-1} = \begin{bmatrix} 13/98 & 1/98 & 8/49 \\ 11/98 & 31/98 & 3/49 \\ -1/14 & 1/14 & 1/7 \end{bmatrix}$$