exceed 50 units i.e., $x + y \le 50$, find units of x and y where π are maximized.

BORDERED HESSIAN DETERMINANT (UOP: 2007-A, 2010-S) (UOS: 2010)

The Lagrangian method is used to present first order conditions of constraint maxima and minima. Here the constraint is added or subtracted in objective function where constraint is multiplied with λ, as : $F(x, y, \lambda) = f(x, y) + \lambda g(x, y)$

Now we present constraint maxima or minima with second order condition. For this purpose, the Bordered Hessians are used. They are as :

$$|\overline{H}| = \begin{vmatrix} F_{xx} & F_{xy} & g_x \\ F_{yx} & F_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix} \quad \text{OR} \quad \begin{vmatrix} 0 & g_x & g_y \\ g_x & F_{xx} & F_{xy} \\ g_y & F_{yx} & F_{yy} \end{vmatrix}$$

It is told that the Bordered Hessian comes into being by attaching first partial derivatives of constraint function and zeroes in principal diagonal of the matrix. The order of Bordered principal minor depends upon principal minors with whom the border has been imposed. Thus, the previous $|\overline{H}|$ is of $|\overline{H}_2|$. As, here principal minor has order of 2×2. If there are n variables in a function like $f(x_1, x_2, x_3 \ldots\ldots x_n)$ and constraint function is $g(x_1, x_2, x_3 \ldots\ldots x_n)$ the Hessian Bordered determinant will be as :

$$|\overline{H}| = \begin{vmatrix} F_{11} & F_{12} & \ldots\ldots & F_{1n} & g_1 \\ F_{21} & F_{22} & \ldots\ldots & F_{2n} & g_2 \\ \ldots & \ldots & \ldots\ldots & \ldots & \\ F_{n1} & F_{n2} & \ldots\ldots & F_{nn} & g_n \\ g_1 & g_2 & \ldots\ldots & g_n & 0 \end{vmatrix} = \begin{vmatrix} 0 & g_1 & g_2 & \ldots\ldots & g_n \\ g_1 & F_{11} & F_{12} & \ldots\ldots & F_{1n} \\ g_2 & F_{21} & F_{22} & \ldots\ldots & F_{2n} \\ \ldots & \ldots & \ldots & \ldots\ldots & \ldots \\ g_n & F_{n1} & F_{n2} & \ldots\ldots & F_{nn} \end{vmatrix}$$

Here $|\overline{H}| = |\overline{H}_n|$ as the principal minor has been bordered $n \times n$. If $|\overline{H}_2|, |\overline{H}_3| \ldots |\overline{H}_n| < 0$, the Hessian Bordered determinant will be positive definite. This is the sufficient condition of minima. Here test starts from $|\overline{H}_2|$ instead of $|\overline{H}_1|$. If $|\overline{H}_2| > 0$, $|\overline{H}_3| < 0$ and $|\overline{H}_4| > 0$ the Hessian Bordered determinant will be negative definite which is sufficient condition of maxima.

Example. If $z_y = 1$, $z_x = 1$ while $z_{xy} = z_{yx} = 3$, $z_{xx} = 8$, $z_{yy} = 12$, state whether the constraint function is maximized or minimized .

Solution.

$$|\overline{H}| = \begin{vmatrix} Z_{xx} & Z_{xy} & g_x \\ Z_{yx} & Z_{yy} & g_y \\ g_x & g_y & 0 \end{vmatrix} \Rightarrow |\overline{H}| = \begin{vmatrix} 8 & 3 & 1 \\ 3 & 12 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$|\overline{H}| = |\overline{H}_2| = 8(-1) - 3(-1) + 1(3 - 12) = -14$$

Hence $|\overline{H}_2| < 0$ —— positive definite. The z is minima here.

FURTHER EXAMPLES OF CONSTRAINED OPTIMIZATION

In this chapter we have discussed earlier the concept of constrained optimization. In this respect, we followed the method of Lagrangian Multiplier. The necessary condition of Maxima required $F_x = F_y = F_\lambda = 0$. While according to sufficient conditions:

$$2f_{xy}(g_x)(g_y) - f_{xx}(g_y)^2 - f_{yy}(g_x)^2 > 0.$$

This second order condition has actually been derived from Bordered Hassian Determinant which states that at maxima $|\overline{H}| = |\overline{H}_1| > 0$.

The necessary condition required for minima is : $F_x = F_y = F_\lambda = 0$, while according to sufficient condition : $2F_{xy}(g_x)(g_y) - F_{xx}(g_y)^2 - F_{yy}(g_x)^2 < 0$.

The second order condition has actually been derived from Bordered Hessian determinant which states that at minimum : $|\overline{H}| = |\overline{H}_2| < 0$.

Example 1: If the function is $z = f(x, y) = xy$ while constraint function is

(UOP: 2012-A) (GCUF: 2014)

$$g(x, y) = 6 - x - y \text{ or } x + y = 6.$$

Constructing Lagrangian function. Find extremum, $z = f(x, y) + \lambda g(x, y) = xy + \lambda(6 - x - y)$

To find stationary values, z_x, z_y and z_λ are found and kept equal to zero.

$z_x = y - \lambda = 0$ ...... (1)

$z_y = x - \lambda = 0$ ...... (2)

$z_\lambda = 6 - x - y = 0$ ...... (3)

By solving (1), (2) and (3), we get $\overline{\lambda} = 3$, $\overline{x} = 3$, $\overline{y} = 3$.

> Solve this question with the help of Example 1.
> Q. Find the extremum of the following.
> $Z = x^2 + 2y^2$ subject to $x + y = 8$. (BZU: 2010)

Putting these values in z we get stationary values ($\overline{z}$). $\overline{Z} = \overline{z} = xy = 3(3) = 9$

The second partial derivatives of objective function and first partial derivatives of constraint function are taken.

$$z_x = y - \lambda \Rightarrow z_{xx} = 0, \quad z_{xy} = 1$$

$$z_y = x - \lambda \Rightarrow z_{yy} = 0, \quad z_{yx} = 1$$

$$g(x, y) = 6 - x - y \Rightarrow g_x = -1, \quad g_y = -1$$

The bordered Hessian determinant is constructed and it is expanded.

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & z_{xx} & z_{xy} \\ g_y & z_{yx} & z_{yy} \end{vmatrix} = \begin{vmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= 0[(z_{xx})(z_{yy}) - (z_{xy})(z_{yx})] - g_x[(g_x)(z_{yy}) - g_y(z_{yx})] + g_y[(g_x)(z_{xy}) - g_y(z_{xx})]$$

$$= 0 - g_x[(g_x)(z_{yy}) - g_y(z_{yx})] + g_y[(g_x)(z_{xy}) - g_y(z_{xx})]$$

$$= -g_x^2 z_{yy} + g_x g_y z_{yx} + g_x g_y z_{xy} - g_y^2 z_{xx}$$

$$= 2 z_{yx} g_x g_y - z_{yy} g_x^2 - z_{xx} g_y^2$$

$$= 2(1)(-1)(-1) - 0(-1)^2 - 0(-1)^2 = 2 - 0 - 0 = 2 > 0$$

OR Putting the above values :

$$|\overline{H}| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = [(0)(0) - (1)(1)] - (-1)[(-1)(0) - (-1)(0)] + (-1)[(-1)(1) - (-1)(0)]$$

$$= 0 + [0 + 1] - 1\,[-1 - 0] = 0 + 1 + 1 = 2 > 0 \Rightarrow |\overline{H}_2| > 0$$

Thus at $\overline{\lambda} = \overline{x} = \overline{y} = 3$ the function is maximized both at necessary and sufficient conditions. Moreover, at these values the constraint is satisfied. $x + y = 6 \Rightarrow 3 + 3 = 6$

Example 2: If $Z = x_1^2 + x_2^2$, find extremum values while the constraint is:

(BZU: 2011, 2013, 2014) (UOH: 2013)

$$g(x, y) = 2 - x_1 - 4x_2 \qquad \Rightarrow \qquad x_1 + 4x_2 = 2$$

Adding it in objective function : $Z = x_1^2 + x_2^2 + \lambda\,(2 - x_1 - 4x_2)$

To find stationary values $\longrightarrow$ $z_{x_1} = z_{x_2} = z\lambda = 0$

$$z_{x_1} = 2_{x_1} - \lambda = 0 \quad \ldots\ldots (1)$$

$$z_{x_2} = 2_{x_2} - 4\lambda = 0 \quad \ldots\ldots (2)$$

$$z\lambda = 2 - x_1 - 4x_2 = 0 \quad \ldots\ldots (3)$$

Solving we get : $\overline{\lambda} = \frac{4}{17}$, $\overline{x}_1 = \frac{2}{17}$, $\overline{x}_2 = \frac{8}{17}$. Putting them in Z we get ($\overline{z}$)

$$Z = \overline{z} = x_1^2 + x_2^2 = \left(\frac{2}{17}\right)^2 + \left(\frac{8}{17}\right)^2 = \frac{4}{198} + \frac{64}{189} = \frac{68}{189} = \frac{4}{17}$$

The second order partial derivatives of objective function and first partial derivatives of constraint function are taken.

$$z_1 = 2x_1 - \lambda \;\Rightarrow\; z_{11} = 2,\; z_{12} = 0,\; z_2 = 2x_2 - 4\lambda \;\Rightarrow\; z_{22} = 2,\; z_{21} = 0$$

$g(x, y) = 2 - x_1 - 4x_2 \;\Rightarrow\; g_1 = -1,\; g_2 = -4$. Bordered Hessian determinant is taken.

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & z_{11} & z_{21} \\ g_2 & z_{12} & z_{22} \end{vmatrix} = \begin{vmatrix} 0 & -1 & -4 \\ -1 & 2 & 0 \\ -4 & 0 & 2 \end{vmatrix}$$

$$= 2\, z_{21}\, g_1 g_2 - z_{22}\, g_1^2 - z_{11}\, g_2^2$$

$$= 2(0)(-1)(-4) - 2(-1)^2 - 2(-4)^2 = -32 - 2 = -34 < 0$$

$$\text{OR} \quad |\overline{H}_2| = \begin{vmatrix} 0 & -1 & -4 \\ -1 & 2 & 0 \\ -4 & 0 & 2 \end{vmatrix} = 0\;[(2)(2) - (0)(0)] - (-1)\;[(-1)(2) - (-4)(0)] + (-4)\;[(-1)(0) - (-4)(2)]$$

$$= 0 - 2 - 32 = -34 < 0$$

Both the methods show that $|\overline{H}_2| < 0$. As a result the sufficient condition is met. Thus we find that at such values $\overline{\lambda} = \frac{4}{17}, \overline{x}_1 = \frac{2}{17}, \overline{x}_2 = \frac{8}{17}$ the function is minimized both on necessary and sufficient conditions. Moreover, here the constraint equation is met.

$$Z = \overline{z} = x_1^2 + x_2^2 = \left(\frac{2}{17}\right)^2 + \left(\frac{8}{17}\right)^2 = \frac{4}{198} + \frac{64}{189} = \frac{68}{189} = \frac{4}{17}$$

... = 6 – x – y or x + y = 6, find extremum.