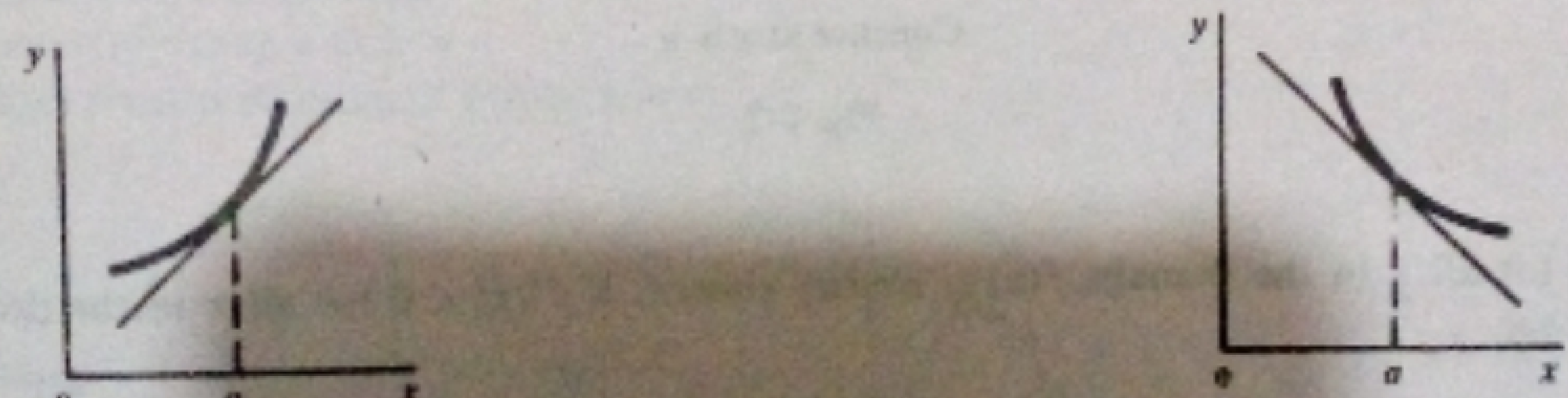
Uses of the Derivative in Mathematics and Economics

4.1 INCREASING AND DECREASING FUNCTIONS

A function $f(x)$ is said to be *increasing* (*decreasing*) at $x = a$ if in the immediate vicinity of the point $[a, f(a)]$ the graph of the function rises (falls) as it moves from left to right. Since the first derivative measures the rate of change and slope of a function, a positive first derivative at $x = a$ indicates the function is increasing at a; a negative first derivative indicates it is decreasing. In short, as seen in Fig. 4-1,

$$f'(a) > 0: \quad \text{increasing function at } x = a$$
$$f'(a) < 0: \quad \text{decreasing function at } x = a$$

A function that increases (or decreases) over its entire domain is called a *monotonic function*. It is said to increase (decrease) *monotonically*. See Problems 4.1 to 4.3.



Slope > 0
Increasing function at $x = a$

(*a*)

Slope < 0
Decreasing function at $x = a$

(*b*)

Fig. 4-1

4.2 CONCAVITY AND CONVEXITY

A function $f(x)$ is *concave* at $x = a$ if in some small region close to the point $[a, f(a)]$ the graph of the function lies completely below its tangent line. A function is *convex* at $x = a$ if in the area very close to $[a, f(a)]$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$; a negative second derivative at $x = a$ denotes the function is concave at a. The slope of the first derivative is irrelevant for concavity. In brief, as seen in Fig. 4-2 and Problems 4.1 to 4.4,

$$f''(a) > 0: \quad f(x) \text{ is convex at } x = a$$
$$f''(a) < 0: \quad f(x) \text{ is concave at } x = a$$

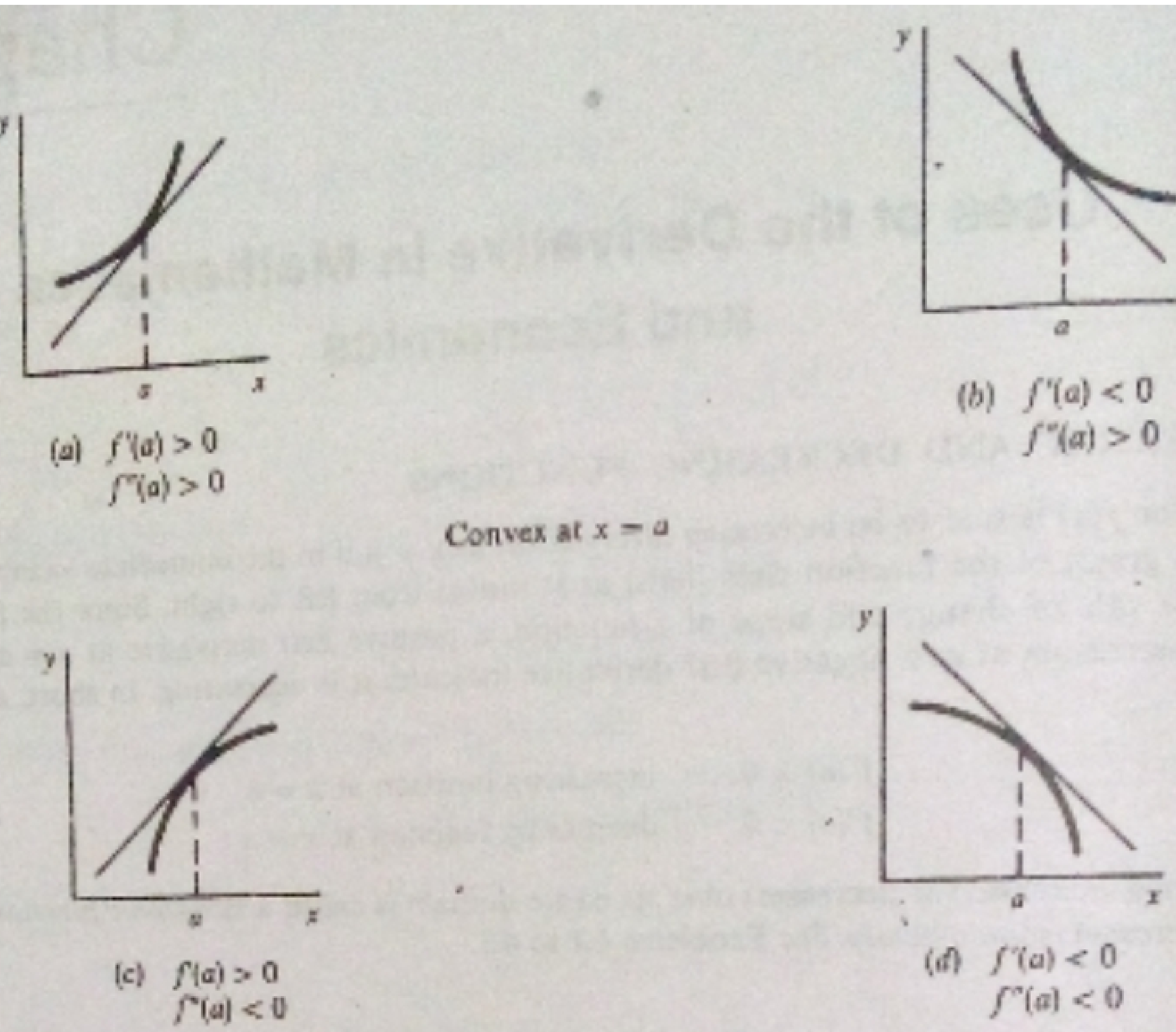


Fig. 4-2

If $f''(x) > 0$ for all x in the domain, $f(x)$ is *strictly convex*. If $f''(x) < 0$ for all x in the domain, $f(x)$ is *strictly concave*.

4.3 RELATIVE EXTREMA

A *relative extremum* is a point at which a function is at a relative maximum or minimum. To be at a relative maximum or minimum at a point a, the function must be at a relative *plateau*, i.e., neither increasing nor decreasing at a. If the function is neither increasing nor decreasing at a, the first derivative of the function at a must equal zero or be undefined. A point in the domain of a function where the derivative equals zero or is undefined is called a *critical point or value*.

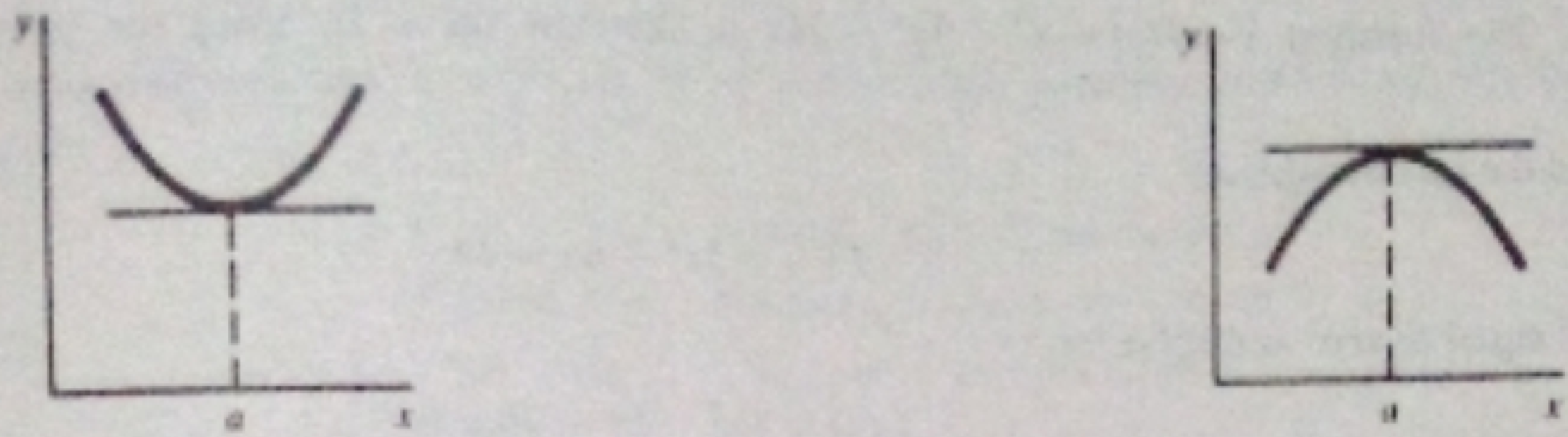
To distinguish mathematically between a relative maximum and minimum, the *second-derivative test* is used. Assuming $f'(a) = 0$,

1. If $f''(a) > 0$, indicating that the function is convex and the graph of the function lies completely above its tangent line at $x = a$, the function is at a relative minimum at $x = a$.
2. If $f''(a) < 0$, denoting that the function is concave and the graph of the function lies completely below its tangent line at $x = a$, the function is at a relative maximum at $x = a$.
3. If $f''(a) = 0$, the test is inconclusive.

For functions which are differentiable at all values of x, called *differentiable or smooth functions*, one need only consider cases where $f'(x) = 0$ in looking for critical points. To summarize,

$$f'(a) = 0 \quad f''(a) > 0: \quad \text{relative minimum at } x = a$$
$$f'(a) = 0 \quad f''(a) < 0: \quad \text{relative maximum at } x = a$$

See Fig. 4-3 and Problems 4.5 and 4.6.



$f'(a) = 0$
$f''(a) > 0$
Relative Minimum at $x = a$

(a)

$f'(a) = 0$
$f''(a) < 0$
Relative Maximum at $x = a$

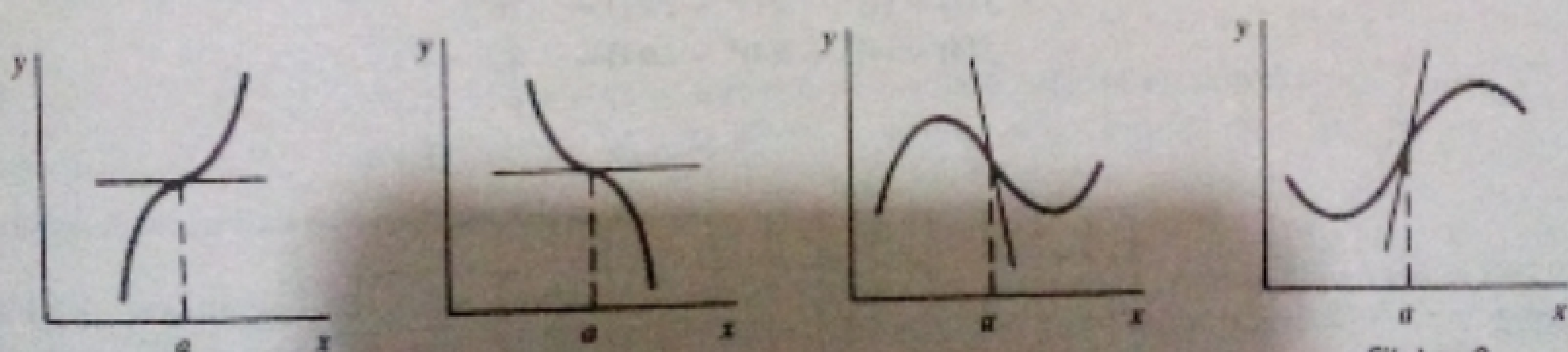
(b)

Fig. 4-3

4.4 INFLECTION POINTS

An *inflection point* is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa. Inflection points occur only where the *second* derivative equals zero or is undefined. The sign of the first derivative is immaterial. In sum, for an inflection point at a, as seen in Fig. 4-4, Example 1, and Problems 4.6 and 4.8(*c*),

1. $f''(a) = 0$ or is undefined.
2. Concavity changes at $x = a$.
3. Graph crosses its tangent line at $x = a$.



(a) $f'(a) = 0$, $f''(a) = 0$

(b) $f'(a) = 0$, $f''(a) = 0$

(c) $f'(a) < 0$, $f''(a) = 0$

(d) $f'(a) > 0$, $f''(a) = 0$

Inflection Points at $x = a$
Fig. 4-4

4.5 CURVE SKETCHING

The first and second derivatives provide useful information about the general shape of a curve and facilitate graphing. A relatively complicated function can be roughly sketched in a few easy steps. For a given $f(x)$:

1. Seek out any relative extremum by looking for points where the first derivative equals zero: $f'(x) = 0$.
2. Determine concavity at the critical point(s) by testing the sign of the second derivative to distinguish between a relative maximum [$f''(x) < 0$] and a relative minimum [$f''(x) > 0$].
3. Check for inflection points where $f''(x) = 0$ and concavity changes.

See Example 1 and Problems 4.7 and 4.8.

4.6 OPTIMIZATION OF FUNCTIONS

Optimization is the process of finding the relative maximum or minimum of a function. Without the aid of a graph, this is done with the techniques developed in Sections 4.3 through 4.5 and outlined below. Given the usual differentiable function,

1. Take the first derivative, set it equal to zero, and solve for the critical point(s). This step is known as the *first-order condition*.

2. Take the second derivative, evaluate it at the critical point(s), and check the sign(s). If at a critical point a,

$$f''(a) > 0: \quad \text{convex, relative minimum}$$
$$f''(a) < 0: \quad \text{concave, relative maximum}$$
$$f''(a) = 0: \quad \text{The test is inconclusive.}$$

This step, the *second-derivative test*, is also called the *second-order condition*. In sum,

Relative maximum	Relative minimum
$f'(a) = 0$	$f'(a) = 0$
$f''(a) < 0$	$f''(a) > 0$

See Examples 2 and 3 and Problems 4.9 and 4.10.

EXAMPLE 2. Optimize $f(x) = 2x^3 - 30x^2 + 126x + 59$.

(*a*) Find the critical points by taking the first derivative, setting it equal to zero, and solving for x.

$$f'(x) = 6x^2 - 60x + 126 = 0$$
$$6(x-3)(x-7) = 0$$
$$x = 3 \qquad x = 7 \qquad \text{critical points}$$

(*b*) Test for concavity by taking the second derivative, evaluating it at the critical points, and checking the signs to distinguish between a relative maximum and minimum.

$$f''(x) = 12x - 60$$
$$f''(3) = 12(3) - 60 = -24 < 0 \qquad \text{concave, relative maximum}$$
$$f''(7) = 12(7) - 60 = 24 > 0 \qquad \text{convex, relative minimum}$$

The function is maximized at $x = 3$ and minimized at $x = 7$.

EXAMPLE 3. The second-derivative test would be inconclusive for functions such as those illustrated in Fig. 4-4(*a*) and (*b*) where inflection points occur at the critical values. In the event that $f''(a) = 0$ and without a graph for guidance, apply the *successive-derivative test*:

1. If, when evaluated at a critical point, the first nonzero value of a higher-order derivative is an odd-numbered derivative (third, fifth, etc.), the function is at an inflection point. See Problem 4.8(*c*).
2. If, when evaluated at a critical value, the first nonzero value of a higher-order derivative is an even-numbered derivative, the function is at a relative extremum, with a positive value of the derivative signifying a relative minimum and a negative value indicating a relative maximum. See Problems 4.8(*d*) and 4.10(*c*) and (*d*).

4.7 MARGINAL CONCEPTS

Marginal cost in economics is defined as the change in total cost incurred from the production of an additional unit. *Marginal revenue* is defined as the change in total revenue brought about by the sale of an extra good. Since total cost (TC) and total revenue (TR) are both functions of the level of output (Q), marginal cost (MC) and marginal revenue (MR) can each be expressed mathematically as derivatives of their respective total functions. Thus,

if $\text{TC} = \text{TC}(Q)$, then $$\text{MC} = \frac{d\text{TC}}{dQ}$$

and if $\text{TR} = \text{TR}(Q)$, then $$\text{MR} = \frac{d\text{TR}}{dQ}$$