6

The Feynman Calculus

In this chapter, we begin the quantitative formulation of elementary particle dynamics, which amounts, in practice, to the calculation of decay rates (Γ) *and scattering cross sections* (σ). *The procedure involves two distinct parts: (1) evaluation of the relevant Feynman diagrams to determine the 'amplitude'* ($\mathscr{M}$) *for the process in question and (2) insertion of* $\mathscr{M}$ *into Fermi's 'Golden Rule' to compute* Γ *or* σ, *as the case may be. To avoid distracting algebraic complications, I introduce here a simplified model. Realistic theories – QED, QCD, and GWS – are developed in succeeding chapters. If you like, Chapter 6 can be read immediately after Chapter 3. Study it with scrupulous care, or what follows will be unintelligible.*

6.1
Decays and Scattering

As I mentioned in the Introduction, we have three experimental probes of elementary particle interactions: bound states, decays, and scattering. Nonrelativistic quantum mechanics (in Schrödinger's formulation) is particularly well adapted to handle bound states, which is why we used it, as far as possible, in Chapter 5. By contrast, the relativistic theory (in Feynman's formulation) is especially well suited to describe decays and scattering. In this chapter I'll introduce the basic ideas and strategies of the Feynman 'calculus'; in subsequent chapters we will use it to develop the theories of strong, electromagnetic, and weak interactions.

6.1.1
Decay Rates

To begin with, we must decide what physical quantities we would like to calculate. In the case of decays, the item of greatest interest is the *lifetime* of the particle in question. What precisely do we mean by the lifetime of, say, the muon? We have in mind, of course, a muon at *rest*; a moving muon lasts longer (from our perspective) because of time dilation. But even stationary muons don't all last the same amount of time, for there is an intrinsically random element in the decay

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process. We cannot hope to calculate the lifetime of any *particular* muon; rather, what we are after is the *average* (or 'mean') lifetime, τ, of the muons in any large sample.

Now, elementary particles have no memories, so the probability of a given muon decaying in the next microsecond is independent of how long ago that muon was created. (It's quite different in biological systems: an 80-year-old man is much more likely to die in the next year than is a 20-year-old, and his body shows the signs of eight decades of wear and tear. But all muons are identical, regardless of when they were produced; from an actuarial point of view they're all on an equal footing.) The critical parameter, then, is the *decay rate*, Γ, the *probability per unit time* that any given muon will disintegrate. If we had a large collection of muons, say, $N(t)$, at time t, then $N\Gamma \mathrm{d}t$ of them would decay in the next instant $\mathrm{d}t$. This would, of course, *decrease* the number remaining:

$$\mathrm{d}N = -\Gamma N\,\mathrm{d}t \tag{6.1}$$

It follows that

$$N(t) = N(0)e^{-\Gamma t} \tag{6.2}$$

Evidently, the number of particles left decreases exponentially with time. As you can check for yourself (Problem 6.1), the mean lifetime is simply the reciprocal of the decay rate:

$$\tau = \frac{1}{\Gamma} \tag{6.3}$$

Actually, most particles can decay by several different routes. The π^+, for instance, usually decays to $\mu^+ + \nu_\mu$, but sometimes one goes to $e^+ + \nu_e$; occasionally, a π^+ decays to $\mu^+ + \nu_\mu + \gamma$, and they have even been known to go to $e^+ + \nu_e + \pi^0$. In such circumstances, the *total* decay rate is the sum of the individual decay rates:

$$\Gamma_{\mathrm{tot}} = \sum_{i=1}^{n} \Gamma_i \tag{6.4}$$

and the lifetime of the particle is the reciprocal of Γ_{tot}:

$$\tau = \frac{1}{\Gamma_{\mathrm{tot}}} \tag{6.5}$$

In addition to τ, we want to calculate the various *branching ratios*, that is, the fraction of all particles of the given type that decay by each mode. Branching ratios are determined by the decay rates:

$$\text{Branching ratio for ith decay mode} = \Gamma_i / \Gamma_{\mathrm{tot}} \tag{6.6}$$

For decays, then, the essential problem is to calculate the decay rate Γ_i for each mode; from there it is an easy matter to obtain the lifetime and branching ratios.

6.1.2
Cross Sections

How about scattering? What quantity should the experimentalist measure and the theorist calculate? If we were talking about an *archer* aiming at a 'bull's-eye', the parameter of interest would be the *size of the target* or, more precisely, the cross-sectional area it presents to a stream of incoming arrows. In a crude sense, the same goes for elementary particle scattering: if you fire a stream of electrons into a tank of hydrogen (which is essentially a collection of protons), the parameter of interest is the size of the proton – the cross-sectional area σ it presents to the incident beam. The situation is more complicated than in archery, however, for several reasons. First of all the target is 'soft'; it's not a simple case of 'hit-or-miss', but rather 'the closer you come the greater the deflection'. Nevertheless, it is still possible to define an 'effective' cross section; I'll show you how in a moment. Secondly, the cross section depends on the nature of the 'arrow' as well as the structure of the 'target'. Electrons scatter off hydrogen more sharply than neutrinos and less so than pions, because different interactions are involved. It depends, too, on the *outgoing* particles; if the energy is high enough we can have not only *elastic* scattering ($e + p \to e + p$), but also a variety of *inelastic* processes, such as $e + p \to e + p + \gamma$, or $e + p + \pi^0$, or even, in principle, $\nu_e + \Lambda$. Each one of these has its own ('exclusive') scattering cross section, σ_i (for process i). In some experiments, however, the final products are not examined, and we are interested only in the *total* ('inclusive') cross section:

$$\sigma_{\text{tot}} = \sum_{i=1}^{n} \sigma_i \tag{6.7}$$

Finally, each cross section typically depends on the *velocity* of the incident particle. At the most naive level we might expect the cross section to be proportional to the amount of time the incident particle spends in the vicinity of the target, which is to say that σ should be inversely proportional to v. But this behavior is dramatically altered in the neighborhood of a 'resonance' – a special energy at which the particles involved 'like' to interact, forming a short-lived semibound state before breaking apart. Such 'bumps' in the graph of σ versus v (or, as it is more commonly plotted, σ versus E) are in fact the principal means by which short-lived particles are discovered (see Figure 4.6). So, unlike the archer's target, there's a lot of physics in an elementary particle cross section.

Let's go back, now, to the question of what we mean by a 'cross section' when the target is soft. Suppose a particle (maybe an electron) comes along, encounters some kind of potential (perhaps the Coulomb potential of a stationary proton), and scatters off at an angle θ. This *scattering angle* is a function of the *impact parameter* b, the distance by which the incident particle would have missed the scattering center, had it continued on its original trajectory (Figure 6.1). Ordinarily, the smaller the impact parameter, the larger the deflection, but the actual functional form of $\theta(b)$ depends on the particular potential involved.

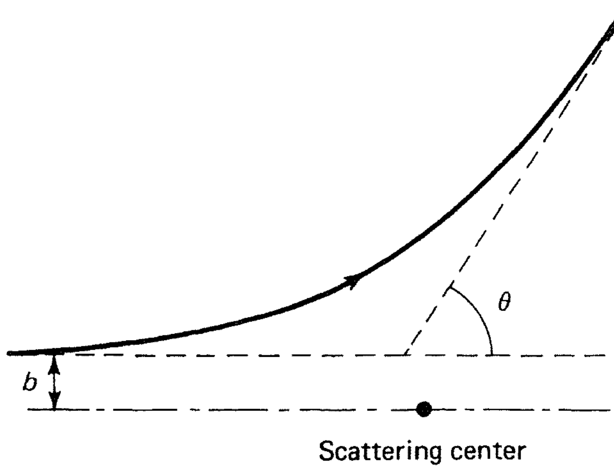


Fig. 6.1 Scattering from a fixed potential: θ is the scattering angle and b is the impact parameter.

Example 6.1 Hard-sphere Scattering Suppose the particle bounces elastically off a sphere of radius R. From Figure 6.2, we have

$$b = R\sin\alpha, \qquad 2\alpha + \theta = \pi$$

Thus,

$$\sin\alpha = \sin(\pi/2 - \theta/2) = \cos(\theta/2)$$

and hence

$$b = R\cos(\theta/2) \quad \text{or} \quad \theta = 2\cos^{-1}(b/R)$$

This is the relation between θ and b for classical hard-sphere scattering.

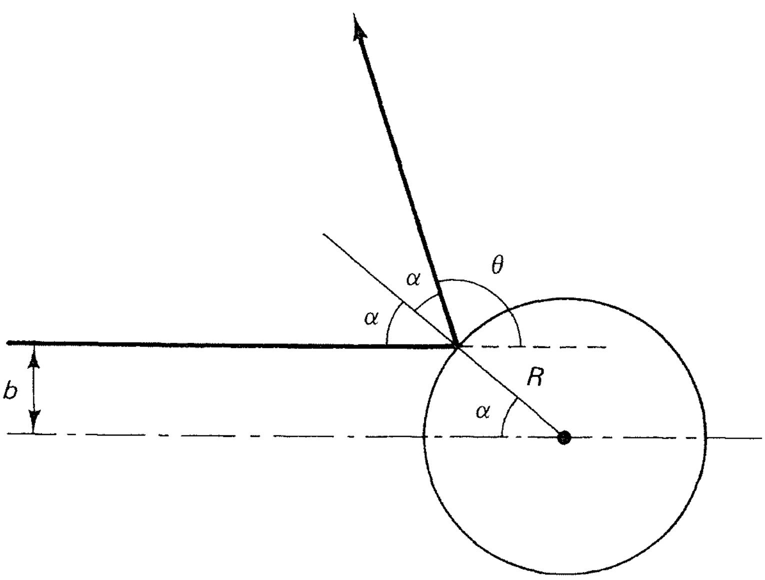


Fig. 6.2 Hard-sphere scattering.

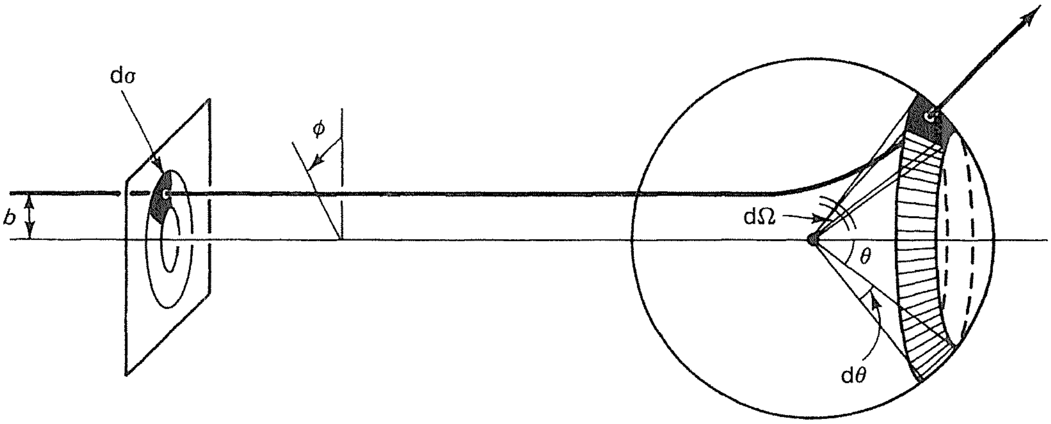


Fig. 6.3 Particle incident in area $d\sigma$ scatters into solid angle $d\Omega$.

If the particle comes in with an impact parameter between b and $b + db$, it will emerge with a scattering angle between θ and $\theta + d\theta$. More generally, if it passes through an infinitesimal *area* $d\sigma$, it will scatter into a corresponding *solid angle* $d\Omega$ (Figure 6.3). Naturally, the larger we make $d\sigma$, the larger $d\Omega$ will be. The proportionality factor is called the *differential (scattering) cross section, D:**

$$d\sigma = D(\theta)\, d\Omega \tag{6.8}$$

The name is poorly chosen; it's not a differential, or even a derivative, in the mathematical sense. The words would apply more naturally to $d\sigma$ than to $d\sigma/d\Omega$ . . . but I'm afraid we're stuck with it.

Now, from Figure 6.3 we see that

$$d\sigma = |b\, db\, d\phi|, \quad d\Omega = |\sin\theta\, d\theta\, d\phi| \tag{6.9}$$

(Areas and solid angles are intrinsically positive, hence the absolute value signs.) Accordingly,

$$D(\theta) = \frac{d\sigma}{d\Omega} = \left|\frac{b}{\sin\theta}\left(\frac{db}{d\theta}\right)\right| \tag{6.10}$$

Example 6.2 In the case of hard-sphere scattering, Example 6.1, we find

$$\frac{db}{d\theta} = -\frac{R}{2}\sin\left(\frac{\theta}{2}\right)$$

* In principle D can depend on the azimuthal angle ϕ; however, most potentials of interest are spherically symmetrical, in which case the differential cross section depends only on θ (or, if you prefer, on b). By the way, the notation (D) is my own; most people just write $d\sigma/d\Omega$, and in the rest of the book I'll do the same.

and hence

$$D(\theta) = \frac{Rb\sin(\theta/2)}{2\sin\theta} = \frac{R^2}{2}\frac{\cos(\theta/2)\sin(\theta/2)}{\sin\theta} = \frac{R^2}{4} \quad \blacksquare$$

Finally, the *total* cross section is the integral of $\mathrm{d}\sigma$ over all solid angles:

$$\sigma = \int \mathrm{d}\sigma = \int D(\theta)\,\mathrm{d}\Omega \tag{6.11}$$

Example 6.3 For hard-sphere scattering,

$$\sigma = \int \frac{R^2}{4}\,\mathrm{d}\Omega = \pi R^2$$

which is, of course, the total cross section the sphere presents to an incoming beam: any particles *within* this area will scatter, and any *outside* will pass by unaffected. $\blacksquare$

As Example 6.3 indicates, the formalism developed here is consistent with our naive sense of the term 'cross section', in the case of a 'hard' target; its virtue is that it applies as well to 'soft' targets, which do not have sharp edges.

Example 6.4 Rutherford Scattering A particle of charge q_1 scatters off a stationary particle of charge q_2. In classical mechanics, the formula relating the impact parameter to the scattering angle is [1]

$$b = \frac{q_1 q_2}{2E}\cot(\theta/2)$$

where E is the initial kinetic energy of the incident charge. The differential cross section is therefore

$$D(\theta) = \left(\frac{q_1 q_2}{4E\sin^2(\theta/2)}\right)^2$$

In this case, the total cross section is actually *infinite*:*

$$\sigma = 2\pi\left(\frac{q_1 q_2}{4E}\right)^2 \int_0^{\pi} \frac{1}{\sin^4(\theta/2)}\sin\theta\,\mathrm{d}\theta = \infty \quad \blacksquare$$

Suppose we have a *beam* of incoming particles, with uniform *luminosity* $\mathcal{L}$ ($\mathcal{L}$ is the number of particles passing down the line per unit time, per unit area). Then

* This is related to the fact that the Coulomb potential has infinite range (see footnote in Section 1.3).

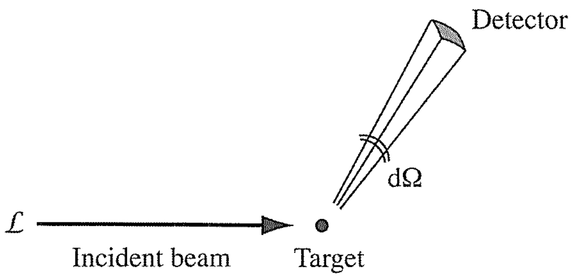


Fig. 6.4 Scattering of a beam with luminosity $\mathcal{L}$.

$\mathrm{d}N = \mathcal{L}\,\mathrm{d}\sigma$ is the number of particles per unit time passing through area $\mathrm{d}\sigma$, and hence also the number per unit time scattered into solid angle $\mathrm{d}\Omega$:

$$\mathrm{d}N = \mathcal{L}\,\mathrm{d}\sigma = \mathcal{L}D(\theta)\,\mathrm{d}\Omega \tag{6.12}$$

Suppose I set up a detector that subtends a solid angle $\mathrm{d}\Omega$ with respect to the collision point (Figure 6.4). I count the number of particles per unit time ($\mathrm{d}N$) reaching my detector – what an experimentalist would call the *event rate*. Equation 6.12 says that the event rate is equal to the luminosity times the differential cross section times the solid angle. Whoever is operating the accelerator controls the luminosity; whoever set up the detector determined the solid angle. With these parameters established, the differential cross section can be measured by simply counting the number of particles entering the detector:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}N}{\mathcal{L}\,\mathrm{d}\Omega} \tag{6.13}$$

If the detector completely surrounds the target, then $N = \sigma\mathcal{L}$; as accelerator physicists like to say, 'the event rate is the cross section times the luminosity'.*

6.2 The Golden Rule

In Section 6.1 I introduced the physical quantities we need to calculate: decay rates and cross sections. In both cases there are two ingredients in the recipe: (i) the *amplitude* ($\mathcal{M}$) for the process and (ii) the *phase space* available.† The amplitude contains all the *dynamical* information; we calculate it by evaluating the relevant Feynman diagrams, using the *Feynman rules* appropriate to the interaction in

* In this discussion, I have assumed that the target itself is *stationary* and that the incident particle is simply *deflected* as it passes through the scattering potential. My purpose was to introduce the essential ideas in the simplest possible context. But in Section 6.2 the formalism is completely general; it includes the recoil of the target, and allows for a change in the identity of the participants during the scattering process (in the reaction $\pi^- + p^+ \to K^+ + \Sigma^-$, for example, $\mathrm{d}\Omega$ might represent the solid angle into which the K^+ scatters).

† The *amplitude* is also called the *matrix element*; the *phase space* is sometimes called the *density of final states*.

question. The phase space factor is purely *kinematic*; it depends on the masses, energies, and momenta of the participants, and reflects the fact that a given process is more likely to occur the more 'room to maneuver' there is in the final state. For example, the decay of a heavy particle into light secondaries involves a large phase space factor, for there are many different ways to apportion the available energy. By contrast, the decay of the neutron ($n \to p + e + \overline{\nu}_e$), in which there is almost no extra mass to spare, is tightly constrained and the phase space factor is very small.*

The ritual for calculating reaction rates was dubbed the *Golden Rule* by Enrico Fermi. In essence, Fermi's Golden Rule says that a transition rate is given by the *product* of the phase space and the (absolute) square of the amplitude. You may have encountered the nonrelativistic version, in the context of time-dependent perturbation theory [2]. We need the relativistic version, which comes from quantum field theory [3]. I can't *derive* it here; what I will do is *state* the Golden Rule and try to make it plausible. Actually, I'll do it twice: once in a form appropriate to decays and again in a form suitable for scattering.

6.2.1 Golden Rule for Decays

Suppose particle 1 (at rest)† decays into several other particles 2, 3, 4, ..., n:

$$1 \to 2 + 3 + 4 + \cdots + n \tag{6.14}$$

The decay rate is given by the formula

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4 \left(p_1 - p_2 - p_3 \cdots - p_n\right) \times \prod_{j=2}^{n} 2\pi \delta\left(p_j^2 - m_j^2 c^2\right) \theta\left(p_j^0\right) \frac{\mathrm{d}^4 p_j}{(2\pi)^4} \tag{6.15}$$

where m_i is the mass of the ith particle and p_i is its four-momentum. S is a statistical factor that corrects for double-counting when there are identical particles in the final state: for each such group of s particles, S gets a factor of $(1/s!)$. For instance, if $a \to b + b + c + c + c$, then $S = (1/2!)(1/3!) = 1/12$. If there are *no* identical particles in the final state (the most common circumstance), then $S = 1$.

Remember: The *dynamics* of the process is contained in the *amplitude*, $\mathcal{M}(p_1, p_2, \ldots, p_n)$, which is a function of the various momenta; we'll calculate it (later)

* For a more extreme case, consider the (kinematically forbidden) decay $\Omega^- \to \Xi^- + \overline{K}^0$. Since the final products weigh more than the Ω, there is no phase space available at all and the decay rate is zero.

† There is no loss of generality in assuming particle 1 is at rest; this is simply an astute choice of reference frame.

by evaluating the appropriate Feynman diagrams. The rest is *phase space*; it tells us to *integrate over all outgoing four-momenta*, subject to three kinematical constraints:

1. Each outgoing particle lies on its mass shell: $p_j^2 = m_j^2 c^2$ (which is to say, $E_j^2 - \mathbf{p}_j^2 c^2 = m_j^2 c^4$). This is enforced by the delta function $\delta\left(p_j^2 - m_j^2 c^2\right)$, which is zero unless its argument vanishes.*
2. Each outgoing energy is positive: $p_j^0 = E_j/c > 0$. Hence the θ function.†
3. Energy and momentum must be conserved: $p_1 = p_2 + p_3 \cdots + p_n$. This is ensured by the factor $\delta^4(p_1 - p_2 - p_3 \cdots - p_n)$.

The Golden Rule (Equation 6.15) may *look* forbidding, but what it actually *says* could hardly be simpler: all outcomes consistent with the three natural kinematic constraints are a priori equally likely. To be sure, the *dynamics* (contained in $\mathscr{M}$) may favor some combinations of momenta over others, but with that modulation you just *add up all the possibilities*. How about all those factors of 2π? These are easy to keep track of if you adhere scrupulously to the following rule:‡

$$\text{Every } \delta \text{ gets } (2\pi); \text{ every d gets } 1/(2\pi). \tag{6.16}$$

Four-dimensional 'volume' elements can be split into spatial and temporal parts:

$$d^4 p = dp^0\, d^3\mathbf{p} \tag{6.17}$$

(I'll drop the subscript j, for simplicity – this argument applies to *each* of the outgoing momenta). The p^0 integrals§ can be performed immediately, by exploiting the delta function

$$\delta\left(p^2 - m^2c^2\right) = \delta\left[(p^0)^2 - \mathbf{p}^2 - m^2c^2\right] \tag{6.18}$$

Now

$$\delta(x^2 - a^2) = \frac{1}{2a}\left[\delta(x-a) + \delta(x+a)\right] \quad (a > 0) \tag{6.19}$$

* If you are unfamiliar with the Dirac delta function, you *must* study Appendix A carefully before proceeding.

† $\theta(x)$ is the (Heaviside) step function: 0 if $x < 0$ and 1 if $x > 0$ (see Appendix A).

‡ Some of these factors eventually cancel out, and you might wonder if there is a more efficient way to manage them. I don't think so. Feynman is supposed to have shouted in exasperation (at a graduate student who 'couldn't be bothered with such trivial matters') 'If you can't get the 2π's right, you don't know *nothing!*'

§ The integral sign in Equation 6.15 actually stands for $4(n-1)$ integrations – one for each component of the $n-1$ outgoing momenta.

(see Problem A.7), so

$$\theta\left(p^0\right)\delta\left[\left(p^0\right)^2-\mathbf{p}^2-m^2c^2\right]=\frac{1}{2\sqrt{\mathbf{p}^2+m^2c^2}}\delta\left(p^0-\sqrt{\mathbf{p}^2+m^2c^2}\right) \tag{6.20}$$

(the theta function kills the spike at $p^0=-\sqrt{\mathbf{p}^2+m^2c^2}$, and it's 1 at $p^0=\sqrt{\mathbf{p}^2+m^2c^2}$). Thus Equation 6.15 reduces to

$$\Gamma=\frac{S}{2\hbar m_1}\int|\mathcal{M}|^2(2\pi)^4\delta^4\left(p_1-p_2-p_3\cdots-p_n\right)$$
$$\times\prod_{j=2}^{n}\frac{1}{2\sqrt{\mathbf{p}_j^2+m_j^2c^2}}\frac{\mathrm{d}^3\mathbf{p}_j}{(2\pi)^3} \tag{6.21}$$

with

$$p_j^0\to\sqrt{\mathbf{p}_j^2+m_j^2c^2} \tag{6.22}$$

wherever it appears (in $\mathcal{M}$ and in the remaining delta function). This is a more useful way to express the Golden Rule, though it obscures the physical content.*

6.2.1.1 Two-particle Decays

In particular, if there are only two particles in the final state

$$\Gamma=\frac{S}{32\pi^2\hbar m_1}\int|\mathcal{M}|^2\frac{\delta^4(p_1-p_2-p_3)}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_3^2+m_3^2c^2}}\,\mathrm{d}^3\mathbf{p}_2 x\,\mathrm{d}^3\mathbf{p}_3 \tag{6.23}$$

The four-dimensional delta function is a product of temporal and spatial parts:

$$\delta^4(p_1-p_2-p_3)=\delta\left(p_1^0-p_2^0-p_3^0\right)\delta^3\left(\mathbf{p}_1-\mathbf{p}_2-\mathbf{p}_3\right) \tag{6.24}$$

But particle 1 is at rest, so $\mathbf{p}_1=\mathbf{0}$ and $p_1^0=m_1c$. Meanwhile, p_2^0 and p_3^0 have been replaced (Equation 6.22), so†

$$\Gamma=\frac{S}{32\pi^2\hbar m_1}\int|\mathcal{M}|^2\frac{\delta\left(m_1c-\sqrt{\mathbf{p}_2^2+m_2^2c^2}-\sqrt{\mathbf{p}_3^2+m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2+m_2^2c^2}\sqrt{\mathbf{p}_3^2+m_3^2c^2}}$$
$$\times\delta^3\left(\mathbf{p}_2+\mathbf{p}_3\right)\mathrm{d}^3\mathbf{p}_2\,\mathrm{d}^3\mathbf{p}_3 \tag{6.25}$$

* You might recognize the quantity $\sqrt{\mathbf{p}_j^2+m_j^2c^2}$ as E_j/c, and many books write it this way. It's dangerous notation: $\mathbf{p}_j$ is an integration variable, so E_j is not some constant you can take outside the integral. Use it as shorthand, if you like, but remember that E_j is a function of $\mathbf{p}_j$, not an independent variable.

† We can drop the minus sign in the final delta function, since $\delta(-x)=\delta(x)$.

The $\mathbf{p}_3$ integral is now trivial: in view of the final delta function it simply makes the replacement

$$\mathbf{p}_3 \to -\mathbf{p}_2 \tag{6.26}$$

leaving

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int |\mathcal{M}|^2 \frac{\delta\left(m_1c - \sqrt{\mathbf{p}_2^2 + m_2^2c^2} - \sqrt{\mathbf{p}_2^2 + m_3^2c^2}\right)}{\sqrt{\mathbf{p}_2^2 + m_2^2c^2}\sqrt{\mathbf{p}_2^2 + m_3^2c^2}}\,\mathrm{d}^3\mathbf{p}_2 \tag{6.27}$$

For the remaining integral we adopt spherical coordinates, $\mathbf{p}_2 \to (r,\ \theta,\ \phi)$, $\mathrm{d}^3\mathbf{p}_2 \to r^2\sin\theta\mathrm{d}r\,\mathrm{d}\theta\ \mathrm{d}\phi$ (this is *momentum* space, of course: $r = |\mathbf{p}_2|$).

$$\Gamma = \frac{S}{32\pi^2\hbar m_1}\int |\mathcal{M}|^2 \frac{\delta\left(m_1c - \sqrt{r^2 + m_2^2c^2} - \sqrt{r^2 + m_3^2c^2}\right)}{\sqrt{r^2 + m_2^2c^2}\sqrt{r^2 + m_3^2c^2}} \times r^2\ \sin\theta\ \mathrm{d}r\,\mathrm{d}\theta\ \mathrm{d}\phi \tag{6.28}$$

Now, $\mathcal{M}$ was originally a function of the four-momenta p_1, p_2, and p_3, but $p_1 = (m_1c, \mathbf{0})$ is a constant (as far as the integration is concerned), and the integrals already performed have made the replacements $p_2^0 \to \sqrt{\mathbf{p}_2^2 + m_2^2c^2}$, $p_3^0 \to \sqrt{\mathbf{p}_3^2 + m_3^2c^2}$, and $\mathbf{p}_3 \to -\mathbf{p}_2$, so by now $\mathcal{M}$ depends only on $\mathbf{p}_2$. As we shall see, however, amplitudes must be *scalars*, and the only scalar you can make out of a vector is the dot product with itself:* $\mathbf{p}_2 \cdot \mathbf{p}_2 = r^2$. At this stage, then, $\mathcal{M}$ is a function only of r (not of θ or ϕ). That being the case we can do the angular integrals

$$\int_0^{\pi} \sin\theta\ \mathrm{d}\theta = 2, \qquad \int_0^{2\pi} \mathrm{d}\phi = 2\pi \tag{6.29}$$

and there remains only the r integral:

$$\Gamma = \frac{S}{8\pi\hbar m_1}\int_0^{\infty} |\mathcal{M}(r)|^2 \frac{\delta\left(m_1c - \sqrt{r^2 + m_2^2c^2} - \sqrt{r^2 + m_3^2c^2}\right)}{\sqrt{r^2 + m_2^2c^2}\sqrt{r^2 + m_3^2c^2}}\,r^2\,\mathrm{d}r \tag{6.30}$$

To simplify the argument of the delta function, let

$$u \equiv \sqrt{r^2 + m_2^2c^2} + \sqrt{r^2 + m_3^2c^2} \tag{6.31}$$

* If the particles carry spin, then $\mathcal{M}$ might depend also on $(\mathbf{p}_i \cdot \mathbf{S}_j)$ and $(\mathbf{S}_i \cdot \mathbf{S}_j)$. However, since experiments rarely measure the spin orientation, we almost always work with the spin-averaged amplitude. In that case, and of course in the case of spin 0, the only vector in sight is $\mathbf{p}_2$ and the only scalar variable is $(\mathbf{p}_2)^2$.

so

$$\frac{du}{dr} = \frac{ur}{\sqrt{r^2 + m_2^2 c^2}\sqrt{r^2 + m_3^2 c^2}} \tag{6.32}$$

Then

$$\Gamma = \frac{S}{8\pi \hbar m_1} \int_{(m_2+m_3)c}^{\infty} |\mathcal{M}(r)|^2 \delta(m_1 c - u) \frac{r}{u} \, du \tag{6.33}$$

The last integral sends* u to $m_1 c$, and hence r to

$$r_0 = \frac{c}{2m_1}\sqrt{m_1^4 + m_2^4 + m_3^4 - 2m_1^2 m_2^2 - 2m_1^2 m_3^2 - 2m_2^2 m_3^2} \tag{6.34}$$

(Problem 6.5). Remember that r was short for the variable $|\mathbf{p}_2|$; r_0 is the *particular* value of $|\mathbf{p}_2|$ that is consistent with conservation of energy, and Equation 6.25 simply reproduces the result we obtained back in Chapter 3 (Problem 3.19). In more comprehensible notation, then,

$$\Gamma = \frac{S|\mathbf{p}|}{8\pi \hbar m_1^2 c} |\mathcal{M}|^2 \tag{6.35}$$

where $|\mathbf{p}|$ is the magnitude of either outgoing momentum, given in terms of the three masses by Equation 6.34, and $\mathcal{M}$ is evaluated at the momenta dictated by the conservation laws. The various substitutions (Equations 6.22, 6.26, and 6.34) have systematically enforced these conservation laws – hardly a surprise, since they were built into the Golden Rule.

The two-body decay formula (Equation 6.35) is surprisingly simple; we were able to carry out all the integrals *without ever knowing the functional form of* $\mathcal{M}$! Mathematically, there were just enough delta functions to cover all the variables; physically, two-body decays are *kinematically determined*: the particles have to come out back-to-back with opposite three-momenta – the *direction* of this axis is not fixed, but since the initial state was symmetric, it doesn't matter. We will use Equation 6.35 frequently. Unfortunately, when there are three or more particles in the final state, the integrals cannot be done until we know the specific functional form of $\mathcal{M}$. In such cases (of which we shall encounter mercifully few), you have to go back to the Golden Rule and work it out from scratch.

6.2.2 Golden Rule for Scattering

Suppose particles 1 and 2 collide, producing particles 3, 4, ..., n:

$$1 + 2 \to 3 + 4 + \cdots + n \tag{6.36}$$

* This assumes $m_1 > (m_2 + m_3)$; otherwise the delta function spike is outside the domain of integration and we get $\Gamma = 0$, recording the fact that a particle cannot decay into heavier secondaries.

The scattering cross section is given by the formula

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n) \times \prod_{j=3}^{n} 2\pi\delta\left(p_j^2 - m_j^2 c^2\right)\theta\left(p_j^0\right)\frac{d^4 p_j}{(2\pi)^4} \tag{6.37}$$

where p_i is the four-momentum of particle i (mass m_i) and the statistical factor (S) is the same as before (Equation 6.15). The phase space is essentially the same as before: integrate over all outgoing momenta, subject to the three kinematical constraints (every outgoing particle is on its mass shell, every outgoing energy is positive, and energy and momentum are conserved), which are enforced by the delta and theta functions. Once again, we can simplify matters by performing the p_j^0 integrals:

$$\sigma = \frac{S\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 \cdots - p_n) \times \prod_{j=3}^{n} \frac{1}{2\sqrt{\mathbf{p}_j^2 + m_j^2 c^2}} \frac{d^3 \mathbf{p}_j}{(2\pi)^3} \tag{6.38}$$

with

$$p_j^0 = \sqrt{\mathbf{p}_j^2 + m_j^2 c^2} \tag{6.39}$$

wherever it occurs in $\mathcal{M}$ and the delta function.

6.2.2.1 Two-body Scattering in the CM Frame

Consider the process

$$1 + 2 \rightarrow 3 + 4 \tag{6.40}$$

in the CM frame, $\mathbf{p}_2 = -\mathbf{p}_1$ (Figure 6.5), where

$$\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2} = (E_1 + E_2)|\mathbf{p}_1|/c \tag{6.41}$$

(Problem 6.7). In this case, Equation 6.38 reduces to

$$\sigma = \frac{S\hbar^2 c}{64\pi^2 (E_1 + E_2)|\mathbf{p}_1|} \int |\mathcal{M}|^2 \frac{\delta^4(p_1 + p_2 - p_3 - p_4)}{\sqrt{\mathbf{p}_3^2 + m_3^2 c^2}\sqrt{\mathbf{p}_4^2 + m_4^2 c^2}} \, d^3\mathbf{p}_3 \, d^3\mathbf{p}_4 \tag{6.42}$$

Fig. 6.5 Two-body scattering in the *CM* frame.

As before, we begin by rewriting the delta function:*

$$\delta^4(p_1 + p_2 - p_3 - p_4) = \delta\left(\frac{E_1 + E_2}{c} - p_3^0 - p_4^0\right)\delta^3(\mathbf{p}_3 + \mathbf{p}_4) \tag{6.43}$$

Next we insert Equation 6.39 and carry out the $\mathbf{p}_4$ integral (which sends $\mathbf{p}_4 \to -\mathbf{p}_3$):

$$\sigma = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|} \int |\mathscr{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{\mathbf{p}_3^2 + m_3^2c^2} - \sqrt{\mathbf{p}_3^2 + m_4^2c^2}\right]}{\sqrt{\mathbf{p}_3^2 + m_3^2c^2}\sqrt{\mathbf{p}_3^2 + m_4^2c^2}}\, \mathrm{d}^3\mathbf{p}_3 \tag{6.44}$$

This time, however, $|\mathscr{M}|^2$ depends on the *direction* of $\mathbf{p}_3$ as well as its magnitude,† so we cannot carry out the angular integration. But that's all right – we didn't really want σ in the first place; what we're after is $d\sigma/d\Omega$. Adopting spherical coordinates, as before,

$$\mathrm{d}^3\mathbf{p}_3 = r^2\, \mathrm{d}r\, \mathrm{d}\Omega \tag{6.45}$$

(where r is shorthand for $|\mathbf{p}_3|$ and $\mathrm{d}\Omega = \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$), we obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{\hbar}{8\pi}\right)^2 \frac{Sc}{(E_1 + E_2)|\mathbf{p}_1|} \int_0^\infty |\mathscr{M}|^2 \times \frac{\delta\left[(E_1 + E_2)/c - \sqrt{r^2 + m_3^2c^2} - \sqrt{r^2 + m_4^2c^2}\right]}{\sqrt{r^2 + m_3^2c^2}\sqrt{r^2 + m_4^2c^2}}\, r^2\, \mathrm{d}r \tag{6.46}$$

* Observe that $\mathbf{p}_1$ and $\mathbf{p}_2$ are *fixed* vectors (related by our choice of reference frame: $\mathbf{p}_2 = -\mathbf{p}_1$), but at this stage $\mathbf{p}_3$ and $\mathbf{p}_4$ are integration variables. It is only *after* the $\mathbf{p}_4$ integration that they are restricted ($\mathbf{p}_4 = -\mathbf{p}_3$), and after the $|\mathbf{p}_3|$ integration that they are determined by the scattering angle θ.

† In general, $|\mathscr{M}|^2$ depends on all four-momenta. However, in this case $\mathbf{p}_2 = -\mathbf{p}_1$ and $\mathbf{p}_4 = -\mathbf{p}_3$, so it remains a function only of $\mathbf{p}_1$ and $\mathbf{p}_3$ (assuming again that spin does not come into it). From these vectors we can construct three scalars: $\mathbf{p}_1\cdot\mathbf{p}_1 = |\mathbf{p}_1|^2$, $\mathbf{p}_3\cdot\mathbf{p}_3 = |\mathbf{p}_3|^2$, and $\mathbf{p}_1\cdot\mathbf{p}_3 = |\mathbf{p}_1||\mathbf{p}_3|\cos\theta$. But $\mathbf{p}_1$ is fixed, so the only *integration* variables on which $|\mathscr{M}|^2$ can depend are $|\mathbf{p}_3|$ and θ.

The integral over r is the same as in Equation 6.30, with $m_2 \to m_4$ and $m_1 \to (E_1 + E_2)/c^2$. Quoting our previous result (Equation 6.35), I conclude that

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|\mathbf{p}_f|}{|\mathbf{p}_i|} \tag{6.47}$$

where $|\mathbf{p}_f|$ is the magnitude of either outgoing momentum and $|\mathbf{p}_i|$ is the magnitude of either incoming momentum.

As in the case of decays, the two-body final state is peculiarly simple, in the sense that we are able to carry the calculation through to the end *without knowing the explicit functional form of* $\mathcal{M}$. We will be using Equation 6.47 frequently in later chapters.

By the way, lifetimes obviously carry the dimensions of *time* (seconds); decay rates ($\Gamma = 1/\tau$), therefore, are measured in inverse seconds. Cross sections have dimensions of *area* – cm^2, or, more conveniently, 'barns':

$$1\ \text{b} = 10^{-24}\ \text{cm}^2 \tag{6.48}$$

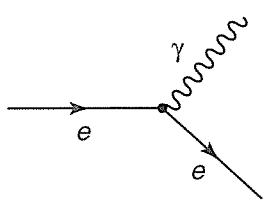
Differential cross sections, $d\sigma/d\Omega$, are given in barns per steradian or simply barns (steradians, like radians, being dimensionless). The amplitude, $\mathcal{M}$, has units that depend on the number of particles involved: if there are n external lines (incoming plus outgoing), the dimensions of $\mathcal{M}$ are those of momentum raised to the power $4 - n$:

$$\text{Dimensions of } \mathcal{M} = (mc)^{4-n} \tag{6.49}$$

For example, in a three-body process $(A \to B + C)$, $\mathcal{M}$ has dimensions of momentum; in a four-body process $(A \to B + C + D$ or $A + B \to C + D)$, $\mathcal{M}$ is dimensionless. You can check for yourself that the two Golden Rules then yield the correct units for Γ and σ.

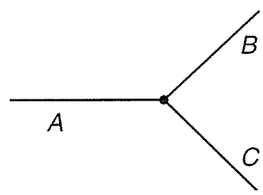
6.3
Feynman Rules for a Toy Theory

In Section 6.2, we learned how to calculate decay rates and scattering cross sections, in terms of the amplitude $\mathcal{M}$ for the process in question. Now I'll show you how to determine $\mathcal{M}$ itself, using the 'Feynman rules' to evaluate the relevant diagrams. We could go straight to a 'real-life' system, such as quantum electrodynamics, with electrons and photons interacting via the primitive vertex:

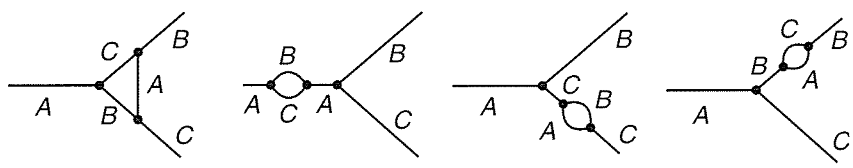


This is the original, the most important, and the best understood application of Feynman's technique. Unfortunately, it involves diverting complications (the electron has spin $\frac{1}{2}$, the photon is massless and carries spin 1), which have nothing to do with the Feynman calculus as such. In Chapter 7, I'll show you how to handle particles with spin, but for the moment I don't want to confuse the issue, so I'm going to introduce a 'toy' theory, which does not pretend to represent the real world, but will serve to illustrate the *method*, with a minimum of extraneous baggage [4].

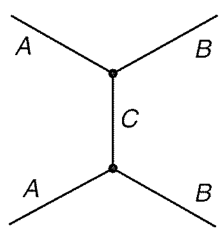
Imagine a world in which there are just three kinds of particles – call them A, B, and C – with masses m_A, m_B, and m_C. They all have spin 0 and each is its own antiparticle (so we don't need arrows on the lines). There is one primitive vertex, by which the three particles interact:



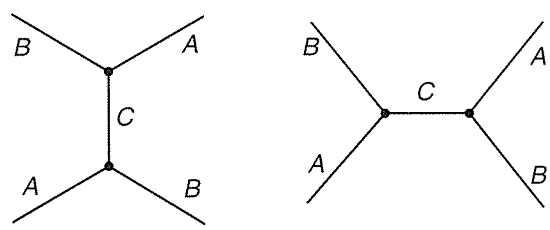
I shall assume that A is the heaviest of the three and in fact weighs more than B and C combined, so that it can decay into $B + C$. The lowest-order diagram describing this disintegration is the primitive vertex itself; to this there are (small) third-order corrections:



and even smaller ones of higher order. Our first project will be to calculate the lifetime of the A, to lowest order. After that, we'll look at various scattering processes, such as $A + A \rightarrow B + B$:



$A + B \rightarrow A + B$:



and so on.

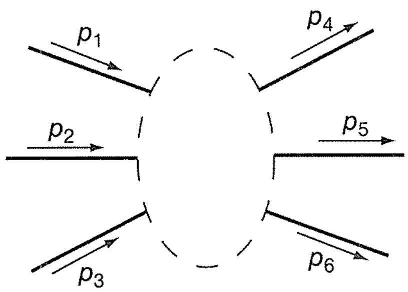


Fig. 6.6 A generic Feynman diagram, with external lines labeled (internal lines not shown).

Our problem is to find the amplitude $\mathscr{M}$ associated with a given Feynman diagram. The ritual is as follows [5]:

1. *Notation*: Label the incoming and outgoing four-momenta $p_1, p_2, \ldots, p_n$ (Figure 6.6). Label the internal momenta q_1, $q_2, \ldots$ Put an arrow beside each line, to keep track of the 'positive' direction (forward in time for external lines, arbitrary for internal lines).
2. *Vertex factors*: For each vertex, write down a factor

$$-ig$$

   g is called the *coupling constant*; it specifies the strength of the interaction between A, B, and C. In this toy theory, g has the dimensions of momentum; in the 'real-world' theories, we shall encounter later on, the coupling constant is always dimensionless.
3. *Propagators*: For each internal line, write a factor

$$\frac{i}{q_j^2 - m_j^2 c^2}$$

   where q_j is the four-momentum of the line and m_j is the mass of the particle the line describes. (Note that $q_j^2 \neq m_j^2 c^2$, because a virtual particle does not lie on its mass shell.)
4. *Conservation of energy and momentum*: For each vertex, write a delta function of the form

$$(2\pi)^4 \delta^4(k_1 + k_2 + k_3)$$

   where the k's are the three four-momenta coming *into* the vertex (if the arrow leads outward, then k is *minus* the four-momentum of that line). This factor imposes conservation of energy and momentum at each vertex, since the delta function is zero unless the sum of the incoming momenta equals the sum of the outgoing momenta.

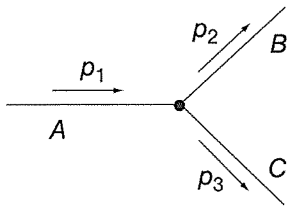


Fig. 6.7 Lowest-order contribution to $A \to B + C$.

5. *Integration over internal momenta*: For each internal line, write down a factor*

$$\frac{1}{(2\pi)^4} d^4 q_j$$

and integrate over all internal momenta.
6. *Cancel the delta function*: The result will include a delta function

$$(2\pi)^4 \delta^4 (p_1 + p_2 + \cdots - p_n)$$

reflecting overall conservation of energy and momentum. Erase this factor† and multiply by i. The result is $\mathscr{M}$.

6.3.1 Lifetime of the *A*

The simplest possible diagram, representing the lowest-order contribution to $A \to B + C$, has no internal lines at all (Figure 6.7). There is one vertex, at which we pick up a factor of $-ig$ (Rule 2) and a delta function

$$(2\pi)^4 \delta^4 (p_1 - p_2 - p_3)$$

(Rule 4), which we promptly discard (Rule 6). Multiplying by i, we get

$$\mathscr{M} = g \tag{6.50}$$

This is the *amplitude* (to lowest order); the decay rate is found by plugging $\mathscr{M}$ into Equation 6.35:

$$\Gamma = \frac{g^2 |\mathbf{p}|}{8\pi \hbar m_A^2 c} \tag{6.51}$$

* Notice (again) that every δ gets a factor of (2π) and every d gets a factor of $1/(2\pi)$.

† Of course, the Golden Rule immediately puts this factor back in Equations 6.15 and 6.37, and you might wonder why we don't just keep it in $\mathscr{M}$. The problem is that $|\mathscr{M}|^2$, not $\mathscr{M}$, comes into the Golden Rule and the *square* of a delta function is undefined. So you have to remove it here, even though you'll be putting it back at the next stage.

where $|\mathbf{p}|$ (the magnitude of either outgoing momentum) is

$$|\mathbf{p}| = \frac{c}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2} \tag{6.52}$$

The lifetime of the A, then, is

$$\tau = \frac{1}{\Gamma} = \frac{8\pi\hbar m_A^2 c}{g^2|\mathbf{p}|} \tag{6.53}$$

You should check for yourself that τ comes out with the correct units.

6.3.2
$A + A \to B + B$ Scattering

The lowest-order contribution to the process $A + A \to B + B$ is shown in Figure 6.8 In this case, there are two vertices (hence two factors of $-ig$), one internal line, with the propagator

$$\frac{i}{q^2 - m_C^2 c^2}$$

two delta functions:

$$(2\pi)^4\delta^4(p_1 - p_3 - q) \quad \textit{and} \quad (2\pi)^4\delta^4(p_2 + q - p_4)$$

and one integration:

$$\frac{1}{(2\pi)^4}\mathrm{d}^4 q$$

Rules 1–5, then, yield

$$-i(2\pi)^4 g^2 \int \frac{1}{q^2 - m_C^2 c^2}\,\delta^4(p_1 - p_3 - q)\,\delta^4(p_2 + q - p_4)\,\mathrm{d}^4 q$$

Doing the integral, the second delta function sends $q \to p_4 - p_2$, and we have

$$-ig^2\frac{1}{(p_4 - p_2)^2 - m_C^2 c^2}(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$$

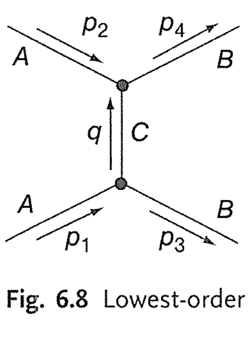


Fig. 6.8 Lowest-order contribution to $A + A \to B + B$.

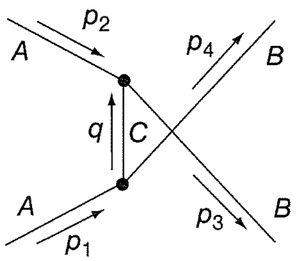


Fig. 6.9 Second diagram contributing in lowest order to $A + A \to B + B$.

As promised, there is one remaining delta function, reflecting overall conservation of energy and momentum. Erasing it and multiplying by i (Rule 6), we are left with

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2} \tag{6.54}$$

But that's not the whole story, for there is another diagram of order g^2, obtained by 'twisting' the B lines (Figure 6.9).* Since this differs from Figure 6.8 only by the interchange $p_3 \leftrightarrow p_4$, there is no need to compute it from scratch; quoting Equation 6.54, we can write down immediately the total amplitude (to order g^2) for the process $A + A \to B + B$:

$$\mathcal{M} = \frac{g^2}{(p_4 - p_2)^2 - m_C^2 c^2} + \frac{g^2}{(p_3 - p_2)^2 - m_C^2 c^2} \tag{6.55}$$

Notice, incidentally, that $\mathcal{M}$ is a Lorentz-invariant (scalar) quantity. This is *always* the case; it is built into the Feynman rules.

Suppose we are interested in the differential cross section $(d\sigma/d\Omega)$ for this process, in the CM system (Figure 6.10). Say, for simplicity, that $m_A = m_B = m$ and $m_C = 0$. Then

$$(p_4 - p_2)^2 - m_C^2 c^2 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 = -2\mathbf{p}^2(1 - \cos\theta) \tag{6.56}$$

$$(p_3 - p_2)^2 - m_C^2 c^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2\mathbf{p}^2(1 + \cos\theta) \tag{6.57}$$

(where $\mathbf{p}$ is the incident momentum of particle 1), and hence

$$\mathcal{M} = -\frac{g^2}{\mathbf{p}^2 \sin^2\theta} \tag{6.58}$$

According to Equation 6.47, then,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{\hbar c g^2}{16\pi E \mathbf{p}^2 \sin^2\theta}\right)^2 \tag{6.59}$$

(there are two identical particles in the final state, so $S = 1/2$). As in the case of Rutherford scattering (Example 6.4), the *total* cross section is infinite.

* You don't get yet another new diagram by twisting the A lines; the only choice here is whether p_3 connects to p_1 or to p_2.

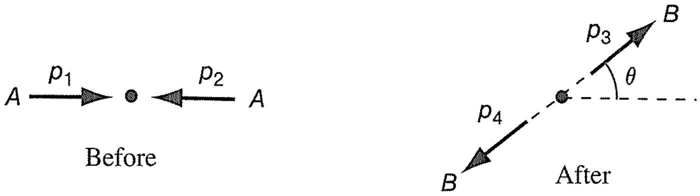
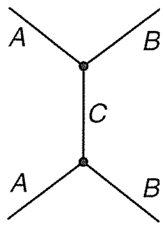


Fig. 6.10 $A + A \to B + B$ in the CM frame.

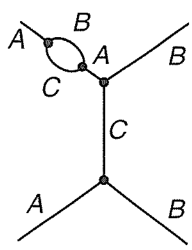
6.3.3
Higher-order Diagrams

So far we have looked only at lowest-order ('tree level') Feynman diagrams; in the case of $A + A \to B + B$, for instance, we considered the graph:

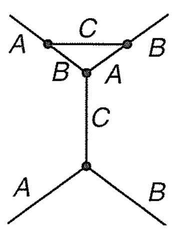


This diagram has two vertices, so $\mathscr{M}$ is proportional to g^2. But there are eight diagrams with *four* vertices (and eight more with the external B lines 'twisted'):

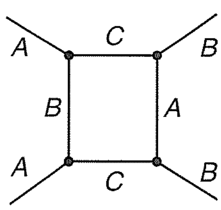
- five 'self-energy' diagrams, in which one of the lines sprouts a loop:



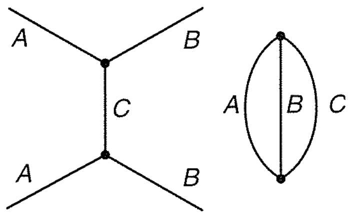
- two 'vertex corrections', in which a vertex becomes a triangle:



- and one 'box' diagram:

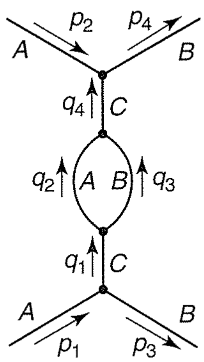


(Disconnected diagrams, such as



don't count.)

I am certainly not going to evaluate all these 'one-loop' diagrams (or even *think* about *two*-loop diagrams), but I would like to take a closer look at *one* of them – the one with a bubble on the virtual C line:



Applying Feynman rules 1–5, we obtain

$$g^4 \int \frac{\delta^4(p_1 - q_1 - p_3)\,\delta^4(q_1 - q_2 - q_3)\,\delta^4(q_2 + q_3 - q_4)\,\delta^4(q_4 + p_2 - p_4)}{(q_1^2 - m_C^2c^2)(q_2^2 - m_A^2c^2)(q_3^2 - m_B^2c^2)(q_4^2 - m_C^2c^2)} \times \mathrm{d}^4q_1\,\mathrm{d}^4q_2\,\mathrm{d}^4q_3\,\mathrm{d}^4q_4 \tag{6.60}$$

Integration over q_1, using the first delta function, replaces q_1 by $(p_1 - p_3)$; integration over q_4, using the last delta function, replaces q_4 by $(p_4 - p_2)$:

$$\frac{g^4}{[(p_1 - p_3)^2 - m_C^2c^2][(p_4 - p_2)^2 - m_C^2c^2]} \times \int \frac{\delta^4(p_1 - p_3 - q_2 - q_3)\,\delta^4(q_2 + q_3 - p_4 + p_2)}{(q_2^2 - m_A^2c^2)(q_3^2 - m_B^2c^2)}\,\mathrm{d}^4q_2\,\mathrm{d}^4q_3 \tag{6.61}$$

Here, the first delta function sends $q_2 \to p_1 - p_3 - q_3$, and the second delta function becomes

$$\delta^4(p_1 + p_2 - p_3 - p_4)$$

which, by Rule 6, we erase, leaving

$$\mathscr{M} = i\left(\frac{g}{2\pi}\right)^4 \frac{1}{[(p_1 - p_3)^2 - m_C^2c^2]^2} \int \frac{1}{[(p_1 - p_3 - q)^2 - m_A^2c^2](q^2 - m_B^2c^2)}\,\mathrm{d}^4q \tag{6.62}$$

(I drop the subscript on q_3 at this point.)

You can try calculating this integral, if you've got the energy, but I'll tell you right now you're going to hit a snag. The four-dimensional volume element could be written as $d^4q = q^3\, \mathrm{d}q\, \mathrm{d}\Omega'$ (where $\mathrm{d}\Omega'$ stands for the angular part), just as in two-dimensional polar coordinates the element of area is $r\, \mathrm{d}r\, \mathrm{d}\theta$ and in three-dimensional spherical coordinates the volume element is $r^2\, \mathrm{d}r \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$. At large q the integrand is essentially just $1/q^4$, so the q integral has the form

$$\int^{\infty} \frac{1}{q^4} q^3\, dq = \ln q|^{\infty} = \infty \tag{6.63}$$

The integral is logarithmically divergent at large q. This disaster, in one form or another, held up the development of quantum electrodynamics for nearly two decades, until, through the combined efforts of many great physicists – from Dirac, Pauli, Kramers, Weisskopf, and Bethe through Tomonaga, Schwinger, and Feynman – systematic methods were developed for 'sweeping the infinities under the rug'. The first step is to *regularize* the integral, using a suitable cutoff procedure that renders it finite without spoiling other desirable features (such as Lorentz invariance). In the case of Equation 6.62, this can be accomplished by introducing a factor

$$\frac{-M^2c^2}{(q^2 - M^2c^2)} \tag{6.64}$$

under the integral sign. The *cutoff* mass M is assumed to be very large, and will be taken to infinity at the end of the calculation (note that the 'fudge factor', Equation 6.64, goes to 1 as $M \to \infty$).* The integral can now be calculated [6] and separated into two parts: a finite term, independent of M, and a term involving (in this case) the logarithm of M, which blows up as $M \to \infty$.

At this point, a miraculous thing happens: all the divergent, M-dependent terms appear in the final answer in the form of *additions to the masses and the coupling constant.* If we take this seriously, it means that the *physical* masses and couplings

* No one would deny that this procedure is artificial. Still, it can be argued that the inclusion of Equation 6.64 merely confesses our ignorance of the high-energy (short distance) behavior of quantum field theory. Perhaps the Feynman propagators are not quite right in this regime, and M is simply a crude way of accounting for the unknown modification. (This would be the case, for example, if the 'particles' have substructure that becomes relevant at extremely close range.) Dirac said, of renormalization,

> *It's just a stop-gap procedure. There must be some fundamental change in our ideas, probably a change just as fundamental as the passage from Bohr's orbit theory to quantum mechanics. When you get a number turning out to be infinite which ought to be finite, you should admit that there is something wrong with your equations, and not hope that you can get a good theory just by doctoring up that number.*

P. Buckley and F. D. Peat, *A Question of Physics* (Toronto: University of Toronto Press, 1979), page 39.

are not the m's and g's that appeared in the original Feynman rules, but rather the 'renormalized' ones, containing these extra factors:

$$m_{\text{physical}} = m + \delta m; \quad g_{\text{physical}} = g + \delta g \tag{6.65}$$

The fact that δm and δg are infinite (in the limit $M \to \infty$) is disturbing, but not catastrophic, for we never measure them anyway; all we ever see in the laboratory are the *physical* values, and these are (obviously) finite (evidently the unmeasurable 'bare' masses and couplings, m and g, contain compensating infinities).* As a practical matter, we take account of the infinities by using the *physical* values of m and g in the Feynman rules, and then systematically ignoring the divergent contributions from higher-order diagrams.

Meanwhile, there remain the *finite* (M-*in*dependent) contributions from the loop diagrams. They, too, lead to modifications in m and g (perfectly calculable ones, in this case) – which, however, are functions of the four-momentum of the line in which the loop is inserted ($p_1 - p_3$ in the example). This means that the *effective* masses and coupling constants actually depend on the *energies* of the particles involved; we call them 'running' masses and 'running' coupling constants. The dependence is typically rather slight, at low energies, and can ordinarily be ignored, but it does have observable consequences, in the form of the Lamb shift (in QED) and asymptotic freedom (in QCD).†

* In case it is some comfort, essentially the same thing occurs in *classical* electrodynamics: the electrostatic energy of a point charge is infinite, and makes an infinite contribution (via $E = mc^2$) to the particle's mass. Perhaps this means that there are no true point charges, in classical electrodynamics; perhaps that's what it means in quantum field theory, too. In neither case, however, do we know how to avoid the point particle as a theoretical construct.

† A *physical* interpretation of the running coupling constant in *QED* and *QCD* was suggested in Chapter 2, Section 2.3. A nice explanation of mass renormalization is given by P. Nelson in *American Scientist*, 73, 66 (1985):

> *According to renormalization theory, not only the strengths of the various interactions but the masses of the participating particles appear to vary on differing length scales. To get a feel for this seemingly paradoxical statement, imagine firing a cannon underwater. Even neglecting friction, the trajectory will be very different from the corresponding one on land, since the cannonball must now drag with it a considerable amount of water, modifying its apparent, or "effective," mass. We can experimentally measure the cannonball's effective mass by shaking it to and fro at a rate ω, computing the mass from $F = ma$. (This is how astronauts "weigh" themselves in space.) Having found the effective mass, we can now replace the difficult problem of underwater ballistics by a simplified approximation: we ignore the water altogether, but in Newton's equations we simply replace the true cannonball mass by the effective mass. The complicated details of the interaction with the medium are*

The procedure I have sketched in the last three paragraphs is called *renormalization* [7]. If *all* the infinities arising from higher-order diagrams can be accommodated in this way, we say that the theory is *renormalizable. ABC* theory and quantum electrodynamics are renormalizable. In the early 1970s, 't Hooft showed that all *gauge* theories, including chromodynamics and the electroweak theory of Glashow, Weinberg, and Salam, are renormalizable. This was a profoundly important discovery, because, beyond lowest-order calculations, a nonrenormalizable theory yields answers that are cutoff-dependent and, therefore, really, quite meaningless.

References

1 See, for example, Taylor, J. R.(**2005**) *Classical Mechanics*, University Science Books, Sausalito, C.A. Sect.14.6, or; (a) Goldstein, H., Poole, C. and Safko, J. (**2002**) *Classical Mechanics*, 3rd edn, Addison-Wesley, San Francisco, C.A. Sect. 3-10.

2 See, for example, Park, D. (**1992**) *Introduction to the Quantum Theory*, 3rd edn, McGraw-Hill, New York. Sect. 7.9; (a) Townsend, J. S. (**2000**) *A Modern Approach to Quantum Mechanics*, University Science Books, Sausalito, C.A. Sect. 14.7; (b) Griffiths, D. J. (**2005**) *Introduction to Quantum Mechanics*, 2nd edn, Prentice Hall, Upper Saddle River, N.J. Sect. 9.2.3.

3 Convincing and accessible accounts are in fact difficult to find. Feynman, R. P. (**1961**) *Theory of Fundamental Processes*, Benjamin, Reading, M.A. Chapters 15 and 16, is a good place to start. See also; (a) Jauch, J. M. and Rohrlich, F. (**1976**) *Theory of Photons and Electrons*, 2nd edn, Springer-Verlag, Berlin. Sect. 8.6; (b) Hagedorn, R. (**1963**) *Relativistic Kinematics*, Benjamin, New York. Chap. 7; (c) Ryder, L. H. (**1985**) *Quantum Field Theory*, Cambridge University Press, Cambridge. Sect. 6.10; (d) Peskin, M. E. and Schroeder, D. V. (**1995**) *An Introduction to Quantum Field Theory*, Perseus, Cambridge. Chap. 4; (e) Weinberg, S. (**1995**) *The Quantum Theory of Fields*, vol. I, Cambridge University Press, Cambridge. Sect. 3.4.

4 This model was shown to me by Max Dresden. For a more sophisticated treatment see Aitchison, I. J. R. and Hey, A. J. G. (**2003**) *Gauge Theories in Particle Physics*, 3rd edn, vol. I, Institute of Physics, Bristol, UK. Section 6.3.

5 You may well ask where these rules *come* from, and I am not sure Feynman himself could have given you

thus reduced to determining one effective parameter. A key feature of this approach is that the effective mass so computed depends on ω, since as ω approaches zero, for example, the water has no effect whatever. In other words, the presence of a medium can introduce a scale-dependent effective mass. We say that the effective mass is "renormalized" by the medium. In quantum physics, every particle moves through a "medium" consisting of the quantum fluctuations of all particles present in the theory. We again take into account this medium by ignoring it but changing the values of our parameters to scale-dependent "effective" values.