1 The Integers

In the most general sense, number theory deals with the properties of different sets of numbers. In this chapter, we will discuss some particularly important sets of numbers, including the integers, the rational numbers, and the algebraic numbers. We will briefly introduce the notion of approximating real numbers by rational numbers. We will also introduce the concept of a sequence, and particular sequences of integers, including some figurate numbers studied in ancient Greece. A common problem is the identification of a particular integer sequence from its initial terms; we will briefly discuss how to attack such problems.

Using the concept of a sequence, we will define countable sets and show that the set of rational numbers is countable. We will also introduce notations for sums and products, and establish some useful summation formulas.

One of the most important proof techniques in number theory (and in much of mathematics) is mathematical induction. We will discuss the two forms of mathematical induction, illustrate how they can be used to prove various results, and explain why mathematical induction is a valid proof technique.

Continuing, we will introduce the intriguing sequence of Fibonacci numbers, and describe the original problem from which they arose. We will establish some identities and inequalities involving the Fibonacci numbers, using mathematical induction for some of our proofs.

The final section of this chapter deals with a fundamental notion in number theory, that of divisibility. We will establish some of the basic properties of division of integers, including the "division algorithm." We will show how the quotient and remainder of a division of one integer by another can be expressed using values of the greatest integer function (we will describe a few of the many useful properties of this function, as well).

1.1 Numbers and Sequences

In this section, we introduce basic material that will be used throughout the text. In particular, we cover the important sets of numbers studied in number theory, the concept of integer sequences, and summations and products.

Numbers

To begin, we will introduce several different types of numbers. The *integers* are the numbers in the set

$$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

The integers play center stage in the study of number theory. One property of the positive integers deserves special mention.

The Well-Ordering Property Every nonempty set of positive integers has a least element.

The well-ordering property may seem obvious, but it is the basic principle that allows us to prove many results about sets of integers, as we will see in Section 1.3.

The well-ordering property can be taken as one of the axioms defining the set of positive integers or it may be derived from a set of axioms in which it is not included. (See Appendix A for axioms for the set of integers.) We say that the set of positive integers is *well ordered.* However, the set of all integers (positive, negative, and zero) is not well ordered, as there are sets of integers without a smallest element, such as the set of negative integers, the set of even integers less than 100, and the set of all integers itself.

Another important class of numbers in the study of number theory is the set of numbers that can be written as a ratio of integers.

Definition. The real number r is *rational* if there are integers p and q, with $q \neq 0$, such that $r = p/q$. If r is not rational, it is said to be *irrational.*

Example 1.1. The numbers $-22/7$, $0 = 0/1$, $2/17$, and $1111/41$ are rational numbers. ◀

Note that every integer n is a rational number, because $n = n/1$. Examples of irrational numbers are $\sqrt{2}$, π, and e. We can use the well-ordering property of the set of positive integers to show that $\sqrt{2}$ is irrational. The proof that we provide, although quite clever, is not the simplest proof that $\sqrt{2}$ is irrational. You may prefer the proof that we will give in Chapter 4, which depends on concepts developed in that chapter. (The proof that e is irrational is left as Exercise 44. We refer the reader to [HaWr08] for a proof that π is irrational. It is not easy.)

Theorem 1.1. $\sqrt{2}$ is irrational.

Proof. Suppose that $\sqrt{2}$ were rational. Then there would exist positive integers a and b such that $\sqrt{2} = a/b$. Consequently, the set $S = \{k\sqrt{2} \mid k \text{ and } k\sqrt{2} \text{ are positive integers}\}$ is a nonempty set of positive integers (it is nonempty because $a = b\sqrt{2}$ is a member of S). Therefore, by the well-ordering property, S has a smallest element, say, $s = t\sqrt{2}$.

We have $s\sqrt{2} - s = s\sqrt{2} - t\sqrt{2} = (s - t)\sqrt{2}$. Because $s\sqrt{2} = 2t$ and s are both integers, $s\sqrt{2} - s = s\sqrt{2} - t\sqrt{2} = (s - t)\sqrt{2}$ must also be an integer. Furthermore, it is positive, because $s\sqrt{2} - s = s(\sqrt{2} - 1)$ and $\sqrt{2} > 1$. It is less than s, because $\sqrt{2} < 2$ so that $\sqrt{2} - 1 < 1$. This contradicts the choice of s as the smallest positive integer in S. It follows that $\sqrt{2}$ is irrational. ■

The sets of integers, positive integers, rational numbers, and real numbers are traditionally denoted by $\mathbf{Z}$, $\mathbf{Z}^+$, $\mathbf{Q}$, and $\mathbf{R}$, respectively. Also, we write $x \in S$ to indicate that x belongs to the set S. Such notation will be used occasionally in this book.

We briefly mention several other types of numbers here, though we do not return to them until Chapter 12.

Definition. A number α is *algebraic* if it is the root of a polynomial with integer coefficients; that is, α is algebraic if there exist integers $a_0, a_1, \ldots, a_n$ such that $a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0$. The number α is called *transcendental* if it is not algebraic.

Example 1.2. The irrational number $\sqrt{2}$ is algebraic, because it is a root of the polynomial $x^2 - 2$. ◄

Note that every rational number is algebraic. This follows from the fact that the number a/b, where a and b are integers and $b \neq 0$, is the root of $bx - a$. In Chapter 12, we will give an example of a transcendental number. The numbers e and π are also transcendental, but the proofs of these facts (which can be found in [HaWr08]) are beyond the scope of this book.

The Greatest Integer Function

In number theory, a special notation is used for the largest integer that is less than or equal to a particular real number.

Definition. The *greatest integer* in a real number x, denoted by $[x]$, is the largest integer less than or equal to x. That is, $[x]$ is the integer satisfying

$$[x] \leq x < [x] + 1.$$

Example 1.3. We have $[5/2] = 2$, $[-5/2] = -3$, $[\pi] = 3$, $[-2] = -2$, and $[0] = 0$. ◄

Remark. The greatest integer function is also known as the *floor function.* Instead of using the notation $[x]$ for this function, computer scientists usually use the notation $\lfloor x \rfloor$. The *ceiling function* is a related function often used by computer scientists. The ceiling function of a real number x, denoted by $\lceil x \rceil$, is the smallest integer greater than or equal to x. For example, $\lceil 5/2 \rceil = 3$ and $\lceil -5/2 \rceil = -2$.

The greatest integer function arises in many contexts. Besides being important in number theory, as we will see throughout this book, it plays an important role in the analysis of algorithms, a branch of computer science. The following example establishes a useful property of this function. Additional properties of the greatest integer function are found in the exercises at the end of this section and in [GrKnPa94].

Example 1.4. Show that if n is an integer, then $[x+n]=[x]+n$ whenever x is a real number. To show that this property holds, let $[x]=m$, so that m is an integer. This implies that $m \leq x < m+1$. We can add n to this inequality to obtain $m+n \leq x+n < m+n+1$. This shows that $m+n=[x]+n$ is the greatest integer less than or equal to $x+n$. Hence, $[x+n]=[x]+n$. ◀

Definition. The *fractional part* of a real number x, denoted by $\{x\}$, is the difference between x and the largest integer less than or equal to x, namely, $[x]$. That is, $\{x\}=x-[x]$.

Because $[x] \leq x < [x]+1$, it follows that $0 \leq \{x\} = x-[x] < 1$ for every real number x. The greatest integer in x is also called the *integral part* of x because $x=[x]+\{x\}$.

Example 1.5. We have $\{5/4\}=5/4-[5/4]=5/4-1=1/4$ and $\{-2/3\}=-2/3-[-2/3]=-2/3-(-1)=1/3$. ◀

Diophantine Approximation

We know that the distance of a real number to the integer closest to it is at most $1/2$. But can we show that one of the first k multiples of a real number must be much closer to an integer? An important part of number theory called *diophantine approximation* studies questions such as this. In particular, it concentrates on questions that involve the approximation of real numbers by rational numbers. (The adjective *diophantine* comes from the Greek mathematician Diophantus, whose biography can be found in Section 13.1.)

Here we will show that among the first n multiples of a real number α, there must be at least one at a distance less than $1/n$ from the integer nearest it. The proof will depend on the famous *pigeonhole principle*, introduced by the German mathematician Dirichlet.[1] Informally, this principle tells us if we have more objects than boxes, when these objects are placed in the boxes, at least two must end up in the same box. Although this seems like a particularly simple idea, it turns out to be extremely useful in number theory and combinatorics. We now state and prove this important fact, which is known as the pigeonhole principle, because if you have more pigeons than roosts, two pigeons must end up in the same roost.

Theorem 1.2. ***The Pigeonhole Principle.*** If $k+1$ or more objects are placed into k boxes, then at least one box contains two or more of the objects.

---

[1] Instead of calling Theorem 1.2 the pigeonhole principle, Dirichlet called it the *Schubfachprinzip* in German, which translates to the *drawer principle* in English. A biography of Dirichlet can be found in Section 3.1.

Proof. If none of the k boxes contains more than one object, then the total number of objects would be at most k. This contradiction shows that one of the boxes contains at least two or more of the objects. ∎

We now state and prove the approximation theorem, which guarantees that one of the first n multiples of a real number must be within $1/n$ of an integer. The proof we give illustrates the utility of the pigeonhole principle. (See [Ro07] for more applications of the pigeonhole principle.) (Note that in the proof we make use of the *absolute value function.* Recall that $|x|$, the absolute value of x, equals x if $x \geq 0$ and $-x$ if $x < 0$. Also recall that $|x - y|$ gives the distance between x and y.)

Theorem 1.3. ***Dirichlet's Approximation Theorem.*** *If α is a real number and n is a positive integer, then there exist integers a and b with $1 \leq a \leq n$ such that $|a\alpha - b| < 1/n$.*

Proof. Consider the $n + 1$ numbers $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$. These $n + 1$ numbers are the fractional parts of the numbers $j\alpha$, $j = 0, 1, \ldots, n$, so that $0 \leq \{j\alpha\} < 1$ for $j = 0, 1, \ldots, n$. Each of these $n + 1$ numbers lies in one of the n disjoint intervals $0 \leq x < 1/n$, $1/n \leq x < 2/n, \ldots, (j - 1)/n \leq x < j/n, \ldots, (n - 1)/n \leq x < 1$. Because there are $n + 1$ numbers under consideration, but only n intervals, the pigeonhole principle tells us that at least two of these numbers lie in the same interval. Because each of these intervals has length $1/n$ and does not include its right endpoint, we know that the distance between two numbers that lie in the same interval is less than $1/n$. It follows that there exist integers j and k with $0 \leq j < k \leq n$ such that $|\{k\alpha\} - \{j\alpha\}| < 1/n$. We will now show that when $a = k - j$, the product $a\alpha$ is within $1/n$ of an integer, namely, the integer $b = [k\alpha] - [j\alpha]$. To see this, note that

$$\begin{aligned} |a\alpha - b| &= |(k - j)\alpha - ([k\alpha] - [j\alpha])| \\ &= |(k\alpha - [k\alpha]) - (j\alpha - [j\alpha])| \\ &= |\{k\alpha\} - \{j\alpha\}| < 1/n. \end{aligned}$$

Furthermore, note that because $0 \leq j < k \leq n$, we have $1 \leq a = k - j \leq n$. Consequently, we have found integers a and b with $1 \leq a \leq n$ and $|a\alpha - b| < 1/n$, as desired. ∎

Example 1.6. Suppose that $\alpha = \sqrt{2}$ and $n = 6$. We find that $1 \cdot \sqrt{2} \approx 1.414$, $2 \cdot \sqrt{2} \approx 2.828$, $3 \cdot \sqrt{2} \approx 4.243$, $4 \cdot \sqrt{2} \approx 5.657$, $5 \cdot \sqrt{2} \approx 7.071$, and $6 \cdot \sqrt{2} \approx 8.485$. Among these numbers $5 \cdot \sqrt{2}$ has the smallest fractional part. We see that $|5 \cdot \sqrt{2} - 7| \approx |7.071 - 7| = 0.071 \leq 1/6$. It follows that when $\alpha = \sqrt{2}$ and $n = 6$, we can take $a = 5$ and $b = 7$ to make $|a\alpha - b| < 1/n$. ◀

Our proof of Theorem 1.3 follows Dirichlet's original 1834 proof. Proving a stronger version of Theorem 1.3 with $1/(n + 1)$ replacing $1/n$ in the approximation is not difficult (see Exercise 32). Furthermore, in Exercise 34 we show how to use the Dirichlet approximation theorem to show that, given an irrational number α, there are infinitely many different rational numbers p/q such that $|\alpha - p/q| < 1/q^2$, an important result in the theory of diophantine approximation. We will return to this topic in Chapter 12.

Sequences

A *sequence* $\{a_n\}$ is a list of numbers $a_1, a_2, a_3, \ldots$. We will consider many particular integer sequences in our study of number theory. We introduce several useful sequences in the following examples.

Example 1.7. The sequence $\{a_n\}$, where $a_n = n^2$, begins with the terms 1, 4, 9, 16, 25, 36, 49, 64, ... This is the sequence of the squares of integers. The sequence $\{b_n\}$, where $b_n = 2^n$, begins with the terms 2, 4, 8, 16, 32, 64, 128, 256, ... This is the sequence of powers of 2. The sequence $\{c_n\}$, where $c_n = 0$ if n is odd and $c_n = 1$ if n is even, begins with the terms 0, 1, 0, 1, 0, 1, 0, 1, ... ◀

There are many sequences in which each successive term is obtained from the previous term by multiplying by a common factor. For example, each term in the sequence of powers of 2 is 2 times the previous term. This leads to the following definition.

Definition. A *geometric progression* is a sequence of the form $a, ar, ar^2, ar^3, \ldots, ar^k, \ldots$, where a, the *initial term*, and r, the *common ratio*, are real numbers.

Example 1.8. The sequence $\{a_n\}$, where $a_n = 3 \cdot 5^n$, $n = 0, 1, 2, \ldots$, is a geometric sequence with initial term 3 and common ratio 5. (Note that we have started the sequence with the term a_0. We can start the index of the terms of a sequence with 0 or any other integer that we choose.) ◀

A common problem in number theory is finding a formula or rule for constructing the terms of a sequence, even when only a few terms are known (such as trying to find a formula for the nth triangular number $1 + 2 + 3 + \cdots + n$). Even though the initial terms of a sequence do not determine the sequence, knowing the first few terms can lead to a conjecture for a formula or rule for the terms. Consider the following examples.

Example 1.9. Conjecture a formula for a_n, where the first eight terms of $\{a_n\}$ are 4, 11, 18, 25, 32, 39, 46, 53. We note that each term, starting with the second, is obtained by adding 7 to the previous term. Consequently, the nth term could be the initial term plus $7(n-1)$. A reasonable conjecture is that $a_n = 4 + 7(n-1) = 7n - 3$. ◀

The sequence proposed in Example 1.9 is an *arithmetic progression*, that is, a sequence of the form $a, a+d, a+2d, \ldots, a+nd, \ldots$. The particular sequence in Example 1.9 has $a = 4$ and $d = 7$.

Example 1.10. Conjecture a formula for a_n, where the first eight terms of the sequence $\{a_n\}$ are 5, 11, 29, 83, 245, 731, 2189, 6563. We note that each term is approximately 3 times the previous term, suggesting a formula for a_n in terms of 3^n. The integers 3^n for $n = 1, 2, 3, \ldots$ are 3, 9, 27, 81, 243, 729, 2187, 6561. Looking at these two sequences together, we find that the formula $a_n = 3^n + 2$ produces these terms. ◀

Example 1.11. Conjecture a formula for a_n, where the first ten terms of the sequence $\{a_n\}$ are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. After examining this sequence from different perspectives, we notice that each term of this sequence, after the first two terms, is the sum of the two preceding terms. That is, we see that $a_n = a_{n-1} + a_{n-2}$ for $3 \leq n \leq 10$. This is an example of a recursive definition of a sequence, discussed in Section 1.3. The terms listed in this example are the initial terms of the Fibonacci sequence, which is discussed in Section 1.4. ◀

Integer sequences arise in many contexts in number theory. Among the sequences we will study are the Fibonacci numbers, the prime numbers (covered in Chapter 3), and the perfect numbers (introduced in Section 7.3). Integer sequences appear in an amazing range of subjects besides number theory. Neil Sloane has amassed a fantastically diverse collection of more than 170,000 integer sequences (as of early 2010) in his *On-Line Encyclopedia of Integer Sequences*. This collection is available on the Web. (Note that in early 2010, the OEIS Foundation took over maintenance of this collection.) (The book [SlPl95] is an earlier printed version containing only a small percentage of the current contents of the encyclopdia.) This site provides a program for finding sequences that match initial terms provided as input. You may find this a valuable resource as you continue your study of number theory (as well as other subjects).

We now define what it means for a set to be countable, and show that a set is countable if and only if its elements can be listed as the terms of a sequence.

Definition. A set is *countable* if it is finite or it is infinite and there exists a one-to-one correspondence between the set of positive integers and the set. A set that is not countable is called *uncountable*.

An infinite set is countable if and only if its elements can be listed as the terms of a sequence indexed by the set of positive integers. To see this, simply note that a one-to-one correspondence f from the set of positive integers to a set S is exactly the same as a listing of the elements of the set in a sequence $a_1, a_2, \ldots, a_n, \ldots$, where $a_i = f(i)$.

Example 1.12. The set of integers is countable, because the integers can be listed starting with 0, followed by 1 and -1, followed by 2 and -2, and so on. This produces the sequence $0, 1, -1, 2, -2, 3, -3, \ldots$, where $a_1 = 0$, $a_{2n} = n$, and $a_{2n+1} = -n$ for $n = 1, 2, \ldots$. ◀

Is the set of rational numbers countable? At first glance, it may seem unlikely that there would be a one-to-one correspondence between the set of positive integers and the set of all rational numbers. However, there is such a correspondence, as the following theorem shows.

Theorem 1.4. The set of rational numbers is countable.

Proof. We can list the rational numbers as the terms of a sequence, as follows. First, we arrange all the rational numbers in a two-dimensional array, as shown in Figure 1.1. We put all fractions with a denominator of 1 in the first row. We arrange these by placing the fraction with a particular numerator in the position this numerator occupies in the list of

4. Find as many terms as you can of the spectrum sequence of π. (See the preamble to Exercise 38 for the definition of spectrum.)
5. Find the first 1000 Ulam numbers.
6. How many pairs of consecutive integers can you find where both are Ulam numbers?
7. Can the sum of any two consecutive Ulam numbers, other than 1 and 2, be another Ulam number? If so, how many examples can you find?
8. How large are the gaps between consecutive Ulam numbers? Do you think that these gaps can be arbitrarily long?
9. What conjectures can you make about the number of Ulam numbers less than an integer n? Do your computations support these conjectures?

Programming Projects

1. Given a number α, find rational numbers p/q such that $|\alpha - p/q| \leq 1/q^2$.
2. Given a number α, find its spectrum sequence.
3. Find the first n Ulam numbers, where n is a positive integer.

1.2 Sums and Products

Because summations and products arise so often in the study of number theory, we now introduce notation for summations and products. The following notation represents the sum of the numbers $a_1, a_2, \ldots, a_n$:

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The letter k, the *index of summation,* is a "dummy variable" and can be replaced by any letter. For instance,

$$\sum_{k=1}^{n} a_k = \sum_{j=1}^{n} a_j = \sum_{i=1}^{n} a_i, \text{ and so forth.}$$

Example 1.13. We see that $\sum_{j=1}^{5} j = 1+2+3+4+5 = 15$, $\sum_{j=1}^{5} 2 = 2+2+2+2+2 = 10$, and $\sum_{j=1}^{5} 2^j = 2 + 2^2 + 2^3 + 2^4 + 2^5 = 62$.

We also note that, in summation notation, the index of summation may range between any two integers, as long as the lower limit does not exceed the upper limit. If m and n are integers such that $m \leq n$, then $\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$. For instance, we have $\sum_{k=3}^{5} k^2 = 3^2 + 4^2 + 5^2 = 50$, $\sum_{k=0}^{2} 3^k = 3^0 + 3^1 + 3^2 = 13$, and $\sum_{k=-2}^{1} k^3 = (-2)^3 + (-1)^3 + 0^3 + 1^3 = -8$. ◀

We will often need to consider sums in which the index of summation ranges over all those integers that possess a particular property. We can use summation notation to specify the particular property or properties the index must have for a term with that index to be included in the sum. This use of notation is illustrated in the following example.

Example 1.14. We see that

$$\sum_{\substack{j \le 10 \\ j \in \{n^2 \mid n \in \mathbf{Z}\}}} 1/(j+1) = 1/1 + 1/2 + 1/5 + 1/10 = 9/5,$$

because the terms in the sum are all those for which j is an integer not exceeding 10 that is a perfect square. ◀

The following three properties for summations are often useful. We leave their proofs to the reader.

(1.1) $$\sum_{j=m}^{n} ca_j = c\sum_{j=m}^{n} a_j$$

(1.2) $$\sum_{j=m}^{n}(a_j + b_j) = \sum_{j=m}^{n} a_j + \sum_{j=m}^{n} b_j$$

(1.3) $$\sum_{i=m}^{n}\sum_{j=p}^{q} a_i b_j = \left(\sum_{i=m}^{n} a_i\right)\left(\sum_{j=p}^{q} b_j\right) = \sum_{j=p}^{q}\sum_{i=m}^{n} a_i b_j$$

Next, we develop several useful summation formulas. We often need to evaluate sums of consecutive terms of a geometric series. The following example shows how a formula for such sums can be derived.

Example 1.15. To evaluate

$$S = \sum_{j=0}^{n} ar^j,$$

the sum of the first $n+1$ terms of the geometric series $a, ar, \ldots, ar^k, \ldots$, we multiply both sides by r and manipulate the resulting sum to find:

$$\begin{aligned}
rS &= r\sum_{j=0}^{n} ar^j \\
&= \sum_{j=0}^{n} ar^{j+1} \\
&= \sum_{k=1}^{n+1} ar^k && \textit{(shifting the index of summation, taking } k = j+1\text{)} \\
&= \sum_{k=0}^{n} ar^k + (ar^{n+1} - a) && \textit{(removing the term with } k = n+1 \\
& && \textit{from the set and adding the term with } k = 0\text{)} \\
&= S + (ar^{n+1} - a).
\end{aligned}$$

It follows that

$$rS - S = (ar^{n+1} - a).$$

Solving for S shows that when $r \neq 1$,

$$S = \frac{ar^{n+1} - a}{r - 1}.$$

Note that when $r = 1$, we have $\sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$. ◀

Example 1.16. Taking $a = 3$, $r = -5$, and $n = 6$ in the formula found in Example 1.15, we see that $\sum_{j=0}^{6} 3(-5)^j = \frac{3(-5)^7 - 3}{-5-1} = 39{,}063$. ◀

The following example shows that the sum of the first n consecutive powers of 2 is 1 less than the next power of 2.

Example 1.17. Let n be a positive integer. To find the sum

$$\sum_{k=0}^{n} 2^k = 1 + 2 + 2^2 + \cdots + 2^n,$$

we use Example 1.15, with $a = 1$ and $r = 2$, to obtain

$$1 + 2 + 2^2 + \cdots + 2^n = \frac{2^{n+1} - 1}{2 - 1} = 2^{n+1} - 1.$$

◀

A summation of the form $\sum_{j=1}^{n}(a_j - a_{j-1})$, where $a_0, a_1, a_2, \ldots, a_n$ is a sequence of numbers, is said to be *telescoping*. Telescoping sums are easily evaluated because

$$\begin{aligned}
\sum_{j=1}^{n} a_j - a_{j-1} &= (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) \\
&= a_n - a_0.
\end{aligned}$$

The ancient Greeks were interested in sequences of numbers that can be represented by regular arrangements of equally spaced points. The following example illustrates one such sequence of numbers.

Example 1.18. The *triangular numbers* $t_1, t_2, t_3, \ldots, t_k, \ldots$ is the sequence where t_k is the number of dots in the triangular array of k rows with j dots in the jth row. ◀

Figure 1.2 illustrates that t_k counts the dots in successively larger regular triangles for $k = 1, 2, 3, 4$, and 5.

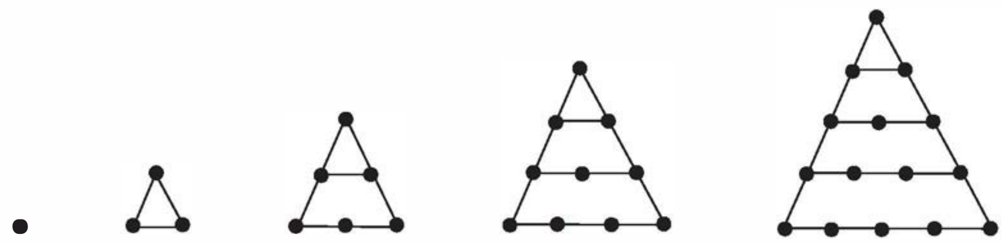


Figure 1.2 *The Triangular Numbers.*

Next, we will determine an explicit formula for the nth triangular number t_n.

Example 1.19. How can we find a formula for the nth triangular number? One approach is to use the identity $(k+1)^2 - k^2 = 2k + 1$. When we isolate the factor k, we find that $k = ((k+1)^2 - k^2)/2 - 1/2$. When we sum this expression for k over the values $k = 1, 2, \ldots, n$, we obtain

$$\begin{aligned}
t_n &= \sum_{k=1}^{n} k \\
&= \left(\sum_{k=1}^{n}((k+1)^2 - k^2)/2\right) - \sum_{k=1}^{n} 1/2 \quad \text{(\textit{replacing k with} } (((k+1)^2 - k^2)/2) - 1/2) \\
&= ((n+1)^2/2 - 1/2) - n/2 \qquad \text{(\textit{simplifying a telescoping sum})} \\
&= (n^2 + 2n)/2 - n/2 \\
&= (n^2 + n)/2 \\
&= n(n+1)/2.
\end{aligned}$$

The second equality here follows by the formula for the sum of a telescoping series with $a_k = (k+1)^2 - k^2$. We conclude that the nth triangular number $t_n = n(n+1)/2$. (See Exercise 7 for another way to find t_n.) ◀

We also define a notation for products, analogous to that for summations. The product of the numbers $a_1, a_2, \ldots, a_n$ is denoted by

$$\prod_{j=1}^{n} a_j = a_1 a_2 \cdots a_n.$$

The letter j above is a "dummy variable," and can be replaced arbitrarily.

Example 1.20. To illustrate the notation for products, we have

$$\prod_{j=1}^{5} j = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120,$$
$$\prod_{j=1}^{5} 2 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32, \text{ and}$$
$$\prod_{j=1}^{5} 2^j = 2 \cdot 2^2 \cdot 2^3 \cdot 2^4 \cdot 2^5 = 2^{15}.$$

◀

The *factorial function* arises throughout number theory.

Definition. Let n be a positive integer. Then $n!$ (read as "n factorial") is the product of the integers 1, 2, . . . , n. We also specify that $0! = 1$. In terms of product notation, we have $n! = \prod_{j=1}^{n} j$.

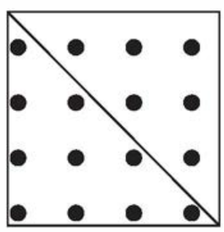
Example 1.21. We have $1! = 1$, $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$, and $12! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 = 479{,}001{,}600$. ◀

1.2 Exercises

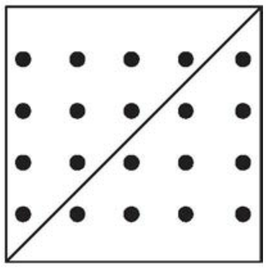
1. Find each of the following sums.
   a) $\sum_{j=1}^{5} j^2$ b) $\sum_{j=1}^{5}(-3)$ c) $\sum_{j=1}^{5} 1/(j+1)$
2. Find each of the following sums.
   a) $\sum_{j=0}^{4} 3$ b) $\sum_{j=0}^{4}(j-3)$ c) $\sum_{j=0}^{4}(j+1)/(j+2)$
3. Find each of the following sums.
   a) $\sum_{j=1}^{8} 2^j$ b) $\sum_{j=1}^{8} 5(-3)^j$ c) $\sum_{j=1}^{8} 3(-1/2)^j$
4. Find each of the following sums.
   a) $\sum_{j=0}^{10} 8 \cdot 3^j$ b) $\sum_{j=0}^{10}(-2)^{j+1}$ c) $\sum_{j=0}^{10}(1/3)^j$

* 5. Find and prove a formula for $\sum_{k=1}^{n}[\sqrt{k}]$ in terms of n and $[\sqrt{n}]$. (*Hint:* Use the formula $\sum_{k=1}^{t} k^2 = t(t+1)(2t+1)/6$.)

6. By putting together two triangular arrays, one with n rows and one with $n-1$ rows, to form a square (as illustrated for $n = 4$), show that $t_{n-1} + t_n = n^2$, where t_n is the nth triangular number.



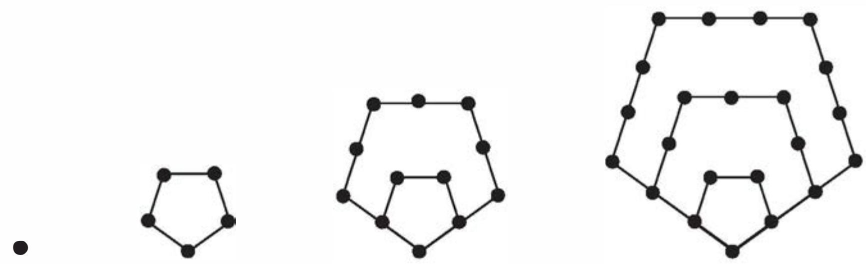
7. By putting together two triangular arrays, each with n rows, to form a rectangular array of dots of size n by $n+1$ (as illustrated for $n=4$), show that $2t_n = n(n+1)$. From this, conclude that $t_n = n(n+1)/2$.



8. Show that $3t_n + t_{n-1} = t_{2n}$, where t_n is the nth triangular number.

9. Show that $t_{n+1}^2 - t_n^2 = (n+1)^3$, where t_n is the nth triangular number.

The *pentagonal numbers* $p_1, p_2, p_3, \ldots, p_k, \ldots$, are the integers that count the number of dots in k nested pentagons, as shown in the following figure.



1 5 12 22

➢ 10. Show that $p_1 = 1$ and $p_k = p_{k-1} + (3k-2)$ for $k \geq 2$. Conclude that $p_n = \sum_{k=1}^{n}(3k-2)$ and evaluate this sum to find a simple formula for p_n.

➢ 11. Prove that the sum of the $(n-1)$st triangular number and the nth square number is the nth pentagonal number.

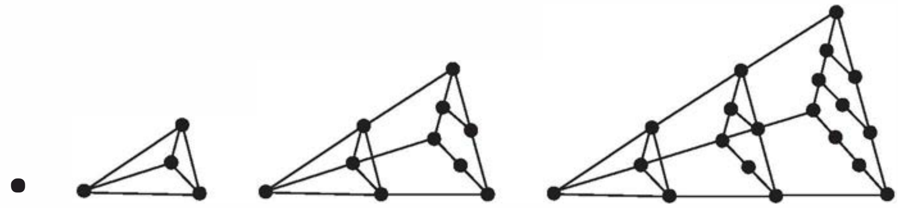
12. a) Define the hexagonal numbers h_n for $n = 1, 2, \ldots$ in a manner analogous to the definitions of triangular, square, and pentagonal numbers. (Recall that a hexagon is a six-sided polygon.)
    b) Find a closed formula for hexagonal numbers.

13. a) Define the heptagonal numbers in a manner analogous to the definitions of triangular, square, and pentagonal numbers. (Recall that a heptagon is a seven-sided polygon.)
    b) Find a closed formula for heptagonal numbers.

14. Show that $h_n = t_{2n-1}$ for all positive integers n where h_n is the nth hexagonal number, defined in Exercise 12, and t_{2n-1} is the $(2n-1)$st triangular number.

15. Show that $p_n = t_{3n-1}/3$ where p_n is the nth pentagonal number and t_{3n-1} is the $(3n-1)$st triangular number.

The *tetrahedral numbers* $T_1, T_2, T_3, \ldots, T_k, \ldots$, are the integers that count the number of dots on the faces of k nested tetrahedra, as shown in the following figure.



16. Show that the nth tetrahedral number is the sum of the first n triangular numbers.

17. Find and prove a closed formula for the nth tetrahedral number.

18. Find $n!$ for n equal to each of the first ten positive integers.

19. List the integers $100!$, 100^{100}, 2^{100}, and $(50!)^2$ in order of increasing size. Justify your answer.

20. Express each of the following products in terms of $\prod_{i=1}^{n} a_i$, where k is a constant.

a) $\prod_{i=1}^{n} ka_i$ b) $\prod_{i=1}^{n} ia_i$ c) $\prod_{i=1}^{n} a_i^k$

21. Use the identity $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$ to evaluate $\sum_{k=1}^{n} \frac{1}{k(k+1)}$.

22. Use the identity $\frac{1}{k^2-1} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right)$ to evaluate $\sum_{k=2}^{n} \frac{1}{k^2-1}$.

23. Find a formula for $\sum_{k=1}^{n} k^2$ using a technique analogous to that in Example 1.21 and the formula found there.

24. Find a formula for $\sum_{k=1}^{n} k^3$ using a technique analogous to that in Example 1.19, and the results of that example and Exercise 21.

25. Without multiplying all the terms, verify these equalities.

a) $10! = 6!\,7!$ b) $10! = 7!\,5!\,3!$ c) $16! = 14!\,5!\,2!$ d) $9! = 7!\,3!\,3!\,2!$

26. Let $a_1, a_2, \ldots, a_n$ be positive integers. Let $b = (a_1!\,a_2!\ldots a_n!) - 1$, and $c = a_1!\,a_2!\ldots a_n!$. Show that $c! = a_1!\,a_2!\cdots a_n!b!$.

27. Find all positive integers x, y, and z such that $x! + y! = z!$.

28. Find the values of the following products.

a) $\prod_{j=2}^{n}(1 - 1/j)$ b) $\prod_{j=2}^{n}(1 - 1/j^2)$

Computations and Explorations

1. What are the largest values of n for which $n!$ has fewer than 100 decimal digits, fewer than 1000 decimal digits, and fewer than 10,000 decimal digits?

2. Find as many triangular numbers that are perfect squares as you can. (We will study this question in the Exercises in Section 13.4.)

3. Find as many tetrahedral numbers that are perfect squares as you can.

Programming Projects

1. Given the terms of a sequence $a_1, a_2, \ldots, a_n$, compute $\sum_{j=1}^{n} a_j$ and $\prod_{j=1}^{n} a_j$.

2. Given the terms of a geometric progression, find the sum of its terms.

3. Given a positive integer n, find the nth triangular number, the nth perfect square, the nth pentagonal number, and the nth tetrahedral number.

1.3 Mathematical Induction

By examining the sums of the first n odd positive integers for small values of n, we can conjecture a formula for this sum. We have

$$\begin{aligned} 1 &= 1, \\ 1+3 &= 4, \\ 1+3+5 &= 9, \\ 1+3+5+7 &= 16, \\ 1+3+5+7+9 &= 25, \\ 1+3+5+7+9+11 &= 36. \end{aligned}$$

From these values, we conjecture that $\sum_{j=1}^{n}(2j-1) = 1+3+5+7+\cdots+2n-1 = n^2$ for every positive integer n.

How can we prove that this formula holds for all positive integers n?

The *principle of mathematical induction* is a valuable tool for proving results about the integers—such as the formula just conjectured for the sum of the first n odd positive integers. First, we will state this principle, and then we will show how it is used. Subsequently, we will use the well-ordering principle to show that mathematical induction is a valid proof technique. We will use the principle of mathematical induction, and the well-ordering property, many times in our study of number theory.

We must accomplish two things to prove by mathematical induction that a particular statement holds for every positive integer. Letting S be the set of positive integers for which we claim the statement to be true, we must show that 1 belongs to S; that is, that the statement is true for the integer 1. This is called the *basis step*.

Second, we must show, for each positive integer n, that $n+1$ belongs to S if n does; that is, that the statement is true for $n+1$ if it is true for n. This is called the *inductive step*. Once these two steps are completed, we can conclude by the principle of mathematical induction that the statement is true for all positive integers.

Theorem 1.5. ***The Principle of Mathematical Induction.*** A set of positive integers that contains the integer 1, and that has the property that, if it contains the integer k, then it also contains $k+1$, must be the set of all positive integers.

We illustrate the use of mathematical induction by several examples; first, we prove the conjecture made at the start of this section.

Example 1.22. We will use mathematical induction to show that

$$\sum_{j=1}^{n}(2j-1) = 1 + 3 + \cdots + (2n-1) = n^2$$

for every positive integer n. (By the way, if our conjecture for the value of this sum was incorrect, mathematical induction would fail to produce a proof!)

We begin with the basis step, which follows because

$$\sum_{j=1}^{1}(2j-1) = 2 \cdot 1 - 1 = 1 = 1^2.$$

For the inductive step, we assume the inductive hypothesis that the formula holds for n; that is, we assume that $\sum_{j=1}^{n}(2j-1) = n^2$. Using the inductive hypothesis, we have

$$\begin{aligned}
\sum_{j=1}^{n+1}(2j-1) &= \sum_{j=1}^{n}(2j-1) + (2(n+1)-1) && \textit{(splitting off the term with } j = n+1\textit{)} \\
&= n^2 + 2(n+1) - 1 && \textit{(using the inductive hypothesis)} \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}$$

Because both the basis and the inductive steps have been completed, we know that the result holds. ◀

Next, we prove an inequality via mathematical induction.

Example 1.23. We can show by mathematical induction that $n! \leq n^n$ for every positive integer n. The basis step, namely, the case where $n = 1$, holds because $1! = 1 \leq 1^1 = 1$. Now, assume that $n! \leq n^n$; this is the inductive hypothesis. To complete the proof, we must show, under the assumption that the inductive hypothesis is true, that $(n+1)! \leq (n+1)^{n+1}$. Using the inductive hypothesis, we have

The Origin of Mathematical Induction
The first known use of mathematical induction appears in the work of the sixteenth-century mathematician Francesco Maurolico (1494–1575). In his book *Arithmeticorum Libri Duo,* Maurolico presented various properties of the integers, together with proofs. He devised the method of mathematical induction so that he could complete some of the proofs. The first use of mathematical induction in his book was in the proof that the sum of the first n odd positive integers equals n^2.

$$
\begin{aligned}
(n+1)! &= (n+1)\cdot n! \\
&\le (n+1)n^n \\
&< (n+1)(n+1)^n \\
&\le (n+1)^{n+1}.
\end{aligned}
$$

This completes both the inductive step and the proof. ◀

We now show that the principle of mathematical induction follows from the well-ordering principle.

Proof. Let S be a set of positive integers containing the integer 1, and the integer $n + 1$ whenever it contains n. Assume (for the sake of contradiction) that S is not the set of all positive integers. Therefore, there are some positive integers not contained in S. By the well-ordering property, because the set of positive integers not contained in S is nonempty, there is a least positive integer n that is not in S. Note that $n \neq 1$, because 1 is in S.

Now, because $n > 1$ (as there is no positive integer n with $n < 1$), the integer $n - 1$ is a positive integer smaller than n, and hence must be in S. But because S contains $n - 1$, it must also contain $(n - 1) + 1 = n$, which is a contradiction, as n is supposedly the smallest positive integer not in S. This shows that S must be the set of all positive integers. ■

A slight variant of the principle of mathematical induction is also sometimes useful in proofs.

Theorem 1.6. ***The Second Principle of Mathematical Induction.*** A set of positive integers that contains the integer 1, and that has the property that, for every positive integer n, if it contains all the positive integers $1, 2, \ldots, n$, then it also contains the integer $n + 1$, must be the set of all positive integers.

The second principle of mathematical induction is sometimes called *strong induction* to distinguish it from the principle of mathematical induction, which is also called *weak induction.*

Before proving that the second principle of mathematical induction is valid, we will give an example to illustrate its use.

Example 1.24. We will show that any amount of postage more than one cent can be formed using just two-cent and three-cent stamps. For the basis step, note that postage of two cents can be formed using one two-cent stamp and postage of three cents can be formed using one three-cent stamp.

For the inductive step, assume that every amount of postage not exceeding n cents, $n \geq 3$, can be formed using two-cent and three-cent stamps. Then a postage amount of $n + 1$ cents can be formed by taking stamps of $n - 1$ cents together with a two-cent stamp. This completes the proof. ◀

We will now show that the second principle of mathematical induction is a valid technique.

Proof. Let T be a set of integers containing 1 and such that for every positive integer n, if it contains $1, 2, \ldots, n$, it also contains $n + 1$. Let S be the set of all positive integers n such that all the positive integers less than or equal to n are in T. Then 1 is in S, and by the hypotheses, we see that if n is in S, then $n + 1$ is in S. Hence, by the principle of mathematical induction, S must be the set of all positive integers, so clearly T is also the set of all positive integers, because S is a subset of T. ■

Recursive Definitions

The principle of mathematical induction provides a method for defining the values of functions at positive integers. Instead of explicitly specifying the value of the function at n, we give the value of the function at 1 and give a rule for finding, for each positive integer n, the value of the function at $n + 1$ from the value of the function at n.

Definition. We say that the function f is *defined recursively* if the value of f at 1 is specified and if for each positive integer n a rule is provided for determining $f(n + 1)$ from $f(n)$.

The principle of mathematical induction can be used to show that a function that is defined recursively is defined uniquely at each positive integer (see Exercise 25 at the end of this section). We illustrate how to define a function recursively with the following definition.

Example 1.25. We will recursively define the *factorial function* $f(n) = n!$. First, we specify that

$$f(1) = 1.$$

Then we give a rule for finding $f(n + 1)$ from $f(n)$ for each positive integer, namely,

$$f(n + 1) = (n + 1) \cdot f(n).$$

These two statements uniquely define $n!$ for the set of positive integers.

To find the value of $f(6) = 6!$ from the recursive definition, use the second property successively, as follows:

$$f(6) = 6 \cdot f(5) = 6 \cdot 5 \cdot f(4) = 6 \cdot 5 \cdot 4 \cdot f(3) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot f(2) = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot f(1).$$

Then use the first statement of the definition to replace $f(1)$ by its stated value 1, to conclude that

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720.$$ ◀

The second principle of mathematical induction also serves as a basis for recursive definitions. We can define a function whose domain is the set of positive integers by specifying its value at 1 and giving a rule, for each positive integer n, for finding $f(n)$

from the values $f(j)$ for each integer j with $1 \leq j \leq n-1$. This will be the basis for the definition of the sequence of Fibonacci numbers discussed in Section 1.4.

1.3 Exercises

1. Use mathematical induction to prove that $n < 2^n$ whenever n is a positive integer.
2. Conjecture a formula for the sum of the first n even positive integers. Prove your result using mathematical induction.
3. Use mathematical induction to prove that $\sum_{k=1}^{n} \frac{1}{k^2} = \frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ whenever n is a positive integer.
4. Conjecture a formula for $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$ from the value of this sum for small integers n. Prove that your conjecture is correct using mathematical induction. (Compare this to Exercise 17 in Section 1.2.)
5. Conjecture a formula for $\mathbf{A}^n$ where $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove your conjecture using mathematical induction.
6. Use mathematical induction to prove that $\sum_{j=1}^{n} j = 1 + 2 + 3 + \cdots + n = n(n+1)/2$ for every positive integer n. (Compare this to Example 1.19 in Section 1.2.)
7. Use mathematical induction to prove that $\sum_{j=1}^{n} j^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for every positive integer n.
8. Use mathematical induction to prove that $\sum_{j=1}^{n} j^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = [n(n+1)/2]^2$ for every positive integer n.
9. Use mathematical induction to prove that $\sum_{j=1}^{n} j(j+1) = 1\cdot 2 + 2\cdot 3 + \cdots + n\cdot (n+1) = n(n+1)(n+2)/3$ for every positive integer n.
10. Use mathematical induction to prove that $\sum_{j=1}^{n}(-1)^{j-1}j^2 = 1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1}n(n+1)/2$ for every positive integer n.
11. Find a formula for $\prod_{j=1}^{n} 2^j$.
12. Show that $\sum_{j=1}^{n} j \cdot j! = 1\cdot 1! + 2\cdot 2! + \cdots + n\cdot n! = (n+1)! - 1$ for every positive integer n.
13. Show that any amount of postage that is an integer number of cents greater than 11 cents can be formed using just 4-cent and 5-cent stamps.
14. Show that any amount of postage that is an integer number of cents greater than 53 cents can be formed using just 7-cent and 10-cent stamps.

Let H_n be the nth partial sum of the harmonic series, that is, $H_n = \sum_{j=1}^{n} 1/j$.

* 15. Use mathematical induction to show that $H_{2^n} \geq 1 + n/2$.

* 16. Use mathematical induction to show that $H_{2^n} \leq 1 + n$.

17. Show by mathematical induction that if n is a positive integer, then $(2n)! < 2^{2n}(n!)^2$.
18. Use mathematical induction to prove that $x - y$ is a factor of $x^n - y^n$, where x and y are variables.

➢ **19.** Use the principle of mathematical induction to show that a set of integers that contains the integer k, such that this set contains $n+1$ whenever it contains n, contains the set of integers that are greater than or equal to k.

20. Use mathematical induction to prove that $2^n < n!$ for $n \geq 4$.

21. Use mathematical induction to prove that $n^2 < n!$ for $n \geq 4$.

22. Show by mathematical induction that if $h \geq -1$, then $1 + nh \leq (1+h)^n$ for all nonnegative integers n.

23. A jigsaw puzzle is solved by putting its pieces together in the correct way. Show that exactly $n-1$ moves are required to solve a jigsaw puzzle with n pieces, where a move consists of putting together two blocks of pieces, with a block consisting of one or more assembled pieces. (*Hint:* Use the second principle of mathematical induction.)

24. Explain what is wrong with the following proof by mathematical induction that all horses are the same color: Clearly all horses in any set of 1 horse are all the same color. This completes the basis step. Now assume that all horses in any set of n horses are the same color. Consider a set of $n+1$ horses, labeled with the integers $1, 2, \ldots, n+1$. By the induction hypothesis, horses $1, 2, \ldots, n$ are all the same color, as are horses $2, 3, \ldots, n, n+1$. Because these two sets of horses have common members, namely, horses $2, 3, 4, \ldots, n$, all $n+1$ horses must be the same color. This completes the induction argument.

25. Use the principle of mathematical induction to show that the value at each positive integer of a function defined recursively is uniquely determined.

26. What function $f(n)$ is defined recursively by $f(1) = 2$ and $f(n+1) = 2f(n)$ for $n \geq 1$? Prove your answer using mathematical induction.

27. If g is defined recursively by $g(1) = 2$ and $g(n) = 2^{g(n-1)}$ for $n \geq 2$, what is $g(4)$?

28. Use the second principle of mathematical induction to show that if $f(1)$ is specified and a rule for finding $f(n+1)$ from the values of f at the first n positive integers is given, then $f(n)$ is uniquely determined for every positive integer n.

29. We define a function recursively for all positive integers n by $f(1) = 1$, $f(2) = 5$, and for $n \geq 2$, $f(n+1) = f(n) + 2f(n-1)$. Show that $f(n) = 2^n + (-1)^n$, using the second principle of mathematical induction.

30. Show that $2^n > n^2$ whenever n is an integer greater than 4.

31. Suppose that $a_0 = 1, a_1 = 3, a_2 = 9$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Show that $a_n \leq 3^n$ for every nonnegative integer n.

32. The tower of Hanoi was a popular puzzle of the late nineteenth century. The puzzle includes three pegs and eight rings of different sizes placed in order of size, with the largest on the bottom, on one of the pegs. The goal of the puzzle is to move all of the rings, one at a time, without ever placing a larger ring on top of a smaller ring, from the first peg to the second, using the third as an auxiliary peg.

a) Use mathematical induction to show that the minimum number of moves to transfer n rings from one peg to another, with the rules we have described, is $2^n - 1$.

b) An ancient legend tells of the monks in a tower with 64 gold rings and 3 diamond pegs. They started moving the rings, one move per second, when the world was created. When they finish transferring the rings to the second peg, the world will end. How long will the world last?

* **33.** The *arithmetic mean* and the *geometric mean* of the positive real numbers $a_1, a_2, \ldots, a_n$ are $A = (a_1 + a_2 + \cdots + a_n)/n$ and $G = (a_1 a_2 \cdots a_n)^{1/n}$, respectively. Use mathematical induction to prove that $A \geq G$ for every finite sequence of positive real numbers. When does equality hold?

34. Use mathematical induction to show that a $2^n \times 2^n$ chessboard with one square missing can be covered with L-shaped pieces, where each L-shaped piece covers three squares.

* **35.** A *unit fraction* is a fraction of the form $1/n$, where n is a positive integer. Because the ancient Egyptians represented fractions as sums of distinct unit fractions, such sums are called *Egyptian fractions*. Show that every rational number p/q, where p and q are integers with $0 < p < q$, can be written as a sum of distinct unit fractions, that is, as an Egyptian fraction. (*Hint:* Use strong induction on the numerator p to show that the greedy algorithm that adds the largest possible unit fraction at each stage always terminates. For example, running this algorithm shows that $5/7 = 1/2 + 1/5 + 1/70$.)

36. Using the algorithm in Exercise 35, write each of these numbers as Egyptian fractions.

a) $2/3$ b) $5/8$ c) $11/17$ d) $44/101$

Computations and Explorations

1. Complete the basis and inductive steps, using both numerical and symbolic computation, to prove that $\sum_{j=1}^{n} j = n(n+1)/2$ for all positive integers n.

2. Complete the basis and inductive steps, using both numerical and symbolic computation, to prove that $\sum_{j=1}^{n} j^2 = n(n+1)(2n+1)/6$ for all positive integers n.

3. Complete the basis and inductive steps, using both numerical and symbolic computation, to prove that $\sum_{j=1}^{n} j^3 = (n(n+1)/2)^2$ for all positive integers n.

4. Use the values $\sum_{j=1}^{n} j^4$ for $n = 1, 2, 3, 4, 5, 6$ to conjecture a formula for this sum that is a polynomial of degree 5 in n. Attempt to prove your conjecture via mathematical induction using numerical and symbolic computation.

5. Paul Erdős and E. Strauss have conjectured that the fraction $4/n$ can be written as the sum of three unit fractions, that is, $4/n = 1/x + 1/y + 1/z$, where x, y, and z are distinct positive integers for all integers n with $n > 1$. Find such representation for as many positive integers n as you can.

6. It is conjectured that the rational number p/q, where p and q are integers with $0 < p < q$ and q is odd, can be expressed as an Egyptian fraction that is the sum of unit fractions with odd denominators. Explore this conjecture using the greedy algorithm that successively adds the unit fraction with the least positive odd denominator q at each stage. (For example, $2/7 = 1/5 + 1/13 + 1/115 + 1/10{,}465$.)

Programming Projects

* **1.** List the moves in the tower of Hanoi puzzle (see Exercise 32). If you can, animate these moves.

** **2.** Cover a $2^n \times 2^n$ chessboard that is missing one square using L-shaped pieces (see Exercise 34).

3. Given a rational number p/q, express p/q as an Egyptian fraction using the algorithm described in Exercise 35.

1.4 The Fibonacci Numbers

In his book *Liber Abaci*, written in 1202, the mathematician *Fibonacci* posed a problem concerning the growth of the number of rabbits in a certain area. This problem can be phrased as follows: A young pair of rabbits, one of each sex, is placed on an island. Assuming that rabbits do not breed until they are two months old and after they are two months old, each pair of rabbits produces another pair each month, how many pairs are there after n months?

Let f_n be the number of pairs of rabbits after n months. We have $f_1 = 1$ because only the original pair is on the island after one month. As this pair does not breed during the second month, $f_2 = 1$. To find the number of pairs after n months, add the number on the island the previous month, f_{n-1}, to the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old. This leads to the following definition.

Definition. The *Fibonacci sequence* is defined recursively by $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. The terms of this sequenceare called the *Fibonacci numbers*.

The mathematician Edouard Lucas named this sequence after Fibonacci in the nineteenth century when he established many of its properties. The answer to Fibonacci's question is that there are f_n rabbits on the island after n months.

Examining the initial terms of the Fibonacci sequence will be useful as we study their properties.

Example 1.26. We compute the first ten Fibonacci numbers as follows:

FIBONACCI (c. 1180–1228) (short for *filus Bonacci*, son of Bonacci), also known as Leonardo of Pisa, was born in the Italian commercial center of Pisa. Fibonacci was a merchant who traveled extensively throughout the Mideast, where he came into contact with mathematical works from the Arabic world. In bis *Liber Abaci* Fibonacci introduced Arabic notation for numerals and their algorithms for arithmetic into the European world. It was in this book that his famous rabbit problem appeared. Fibonacci also wrote *Practica geometriae,* a treatise on geometry and trigonometry, and *Liber quadratorum,* a book on diophantine equations.

$$\begin{aligned}
f_3 &= f_2 + f_1 = 1 + 1 = 2, \\
f_4 &= f_3 + f_2 = 2 + 1 = 3, \\
f_5 &= f_4 + f_3 = 3 + 2 = 5, \\
f_6 &= f_5 + f_4 = 5 + 3 = 8, \\
f_7 &= f_6 + f_5 = 8 + 5 = 13, \\
f_8 &= f_7 + f_6 = 13 + 8 = 21, \\
f_9 &= f_8 + f_7 = 21 + 13 = 34, \\
f_{10} &= f_9 + f_8 = 34 + 21 = 55.
\end{aligned}$$ ◀

We can define the value of $f_0 = 0$, so that $f_2 = f_1 + f_0$. We can also define f_n where n is a negative number so that the equality in the recursive definition is satisfied (see Exercise 37).

The Fibonacci numbers occur in an amazing variety of applications. For example, in botany the number of spirals in plants with a pattern known as phyllotaxis is always a Fibonacci number. They occur in the solution of a tremendous variety of counting problems, such as counting the number of bit strings with no two consecutive 1s (see [Ro07]).

The Fibonacci numbers also satisfy an extremely large number of identities. For example, we can easily find an identity for the sum of the first n consecutive Fibonacci numbers.

Example 1.27. The sum of the first n Fibonacci numbers for $3 \leq n \leq 8$ equals 1, 2, 4, 7, 12, 20, 33, and 54. Looking at these numbers, we see that they are all just 1 less than the Fibonacci number f_{n+2}. This leads us to the conjecture that

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1.$$

Can we prove this identity for all positive integers n?

We will show, in two different ways, that this identity does hold for all integers n. We provide two different demonstrations, to show that there is often more than one way to prove that an identity is true.

First, we use the fact that $f_n = f_{n-1} + f_{n-2}$ for $n = 2, 3, \ldots$ to see that $f_k = f_{k+2} - f_{k+1}$ for $k = 1, 2, 3, \ldots$. This means that

$$\sum_{k=1}^{n} f_k = \sum_{k=1}^{n} (f_{k+2} - f_{k+1}).$$

We can easily evaluate this sum because it is telescoping. Using the formula for a telescoping sum found in Section 1.2, we have

$$\sum_{k=1}^{n} f_k = f_{n+2} - f_2 = f_{n+2} - 1.$$

This proves the result.

We can also prove this identity using mathematical induction. The basis step holds because $\sum_{k=1}^{1} f_k = 1$ and this equals $f_{1+2} - 1 = f_3 - 1 = 2 - 1 = 1$. The inductive hypothesis is

$$\sum_{k=1}^{n} f_k = f_{n+2} - 1.$$

We must show that, under this assumption,

$$\sum_{k=1}^{n+1} f_k = f_{n+3} - 1.$$

To prove this, note that by the inductive hypothesis we have

$$\begin{aligned}\sum_{k=1}^{n+1} f_k &= \left(\sum_{k=1}^{n} f_k\right) + f_{n+1} \\ &= (f_{n+2} - 1) + f_{n+1} \\ &= (f_{n+1} + f_{n+2}) - 1 \\ &= f_{n+3} - 1.\end{aligned}$$ ◀

The exercise set at the end of this section asks you to prove many other identities of the Fibonacci numbers.

How Fast Do the Fibonacci Numbers Grow?

The following inequality, which shows that the Fibonacci numbers grow faster than a geometric series with common ratio $\alpha = (1 + \sqrt{5})/2$, will be used in Chapter 3.

Example 1.28. We can use the second principle of mathematical induction to prove that $f_n > \alpha^{n-2}$ for $n \geq 3$ where $\alpha = (1 + \sqrt{5})/2$. The basis step consists of verifying this inequality for $n = 3$ and $n = 4$. We have $\alpha < 2 = f_3$, so the theorem is true for $n = 3$. Because $\alpha^2 = (3 + \sqrt{5})/2 < 3 = f_4$, the theorem is true for $n = 4$.

The inductive hypothesis consists of assuming that $\alpha^{k-2} < f_k$ for all integers k with $k \leq n$. Because $\alpha = (1 + \sqrt{5})/2$ is a solution of $x^2 - x - 1 = 0$, we have $\alpha^2 = \alpha + 1$. Hence,

$$\alpha^{n-1} = \alpha^2 \cdot \alpha^{n-3} = (\alpha + 1) \cdot \alpha^{n-3} = \alpha^{n-2} + \alpha^{n-3}.$$

By the inductive hypothesis, we have the inequalities

$$\alpha^{n-2} < f_n, \quad \alpha^{n-3} < f_{n-1}.$$

By adding these two inequalities, we conclude that

$$\alpha^{n-1} < f_n + f_{n-1} = f_{n+1}.$$

This finishes the proof. ◀

We conclude this section with an explicit formula for the nth Fibonacci number. We will not provide a proof in the text, but Exercises 41 and 42 at the end of this section outline how this formula can be found using linear homogeneous recurrence relations and generating functions, respectively. Furthermore, Exercise 40 asks that you prove this identity by showing that the terms satisfy the same recursive definition as the Fibonacci numbers do, and Exercise 45 asks for a proof via mathematical induction. The advantage of the first two approaches is that they can be used to find the formula, while the second two approaches cannot.

Theorem 1.7. Let n be a positive integer and let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then the nth Fibonacci number f_n is given by

$$f_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n).$$

We have presented a few important results involving the Fibonacci numbers. There is a vast literature concerning these numbers and their many applications to botany, computer science, geography, physics, and other areas (see [Va89]). There is even a scholarly journal, *The Fibonacci Quarterly,* devoted to their study.

1.4 Exercises

1. Find the following Fibonacci numbers.
   a) f_{10} c) f_{15} e) f_{20}
   b) f_{13} d) f_{18} f) f_{25}
2. Find each of the following Fibonacci numbers.
   a) f_{12} c) f_{24} e) f_{32}
   b) f_{16} d) f_{30} f) f_{36}
3. Prove that $f_{n+3} + f_n = 2f_{n+2}$ whenever n is a positive integer.
4. Prove that $f_{n+3} - f_n = 2f_{n+1}$ whenever n is a positive integer.
5. Prove that $f_{2n} = f_n^2 + 2f_{n-1}f_n$ whenever n is a positive integer. (Recall that $f_0 = 0$.)
6. Prove that $f_{n-2} + f_{n+2} = 3f_n$ whenever n is an integer with $n \geq 2$. (Recall that $f_0 = 0$.)
7. Find and prove a simple formula for the sum of the first n Fibonacci numbers with odd indices when n is a positive integer. That is, find a simple formula for $f_1 + f_3 + \cdots + f_{2n-1}$.
8. Find and prove a simple formula for the sum of the first n Fibonacci numbers with even indices when n is a positive integer. That is, find a simple formula for $f_2 + f_4 + \cdots + f_{2n}$.
9. Find and prove a simple formula for the expression $f_n - f_{n-1} + f_{n-2} - \cdots + (-1)^{n+1}f_1$ when n is a positive integer.
10. Prove that $f_{2n+1} = f_{n+1}^2 + f_n^2$ whenever n is a positive integer.
11. Prove that $f_{2n} = f_{n+1}^2 - f_{n-1}^2$ whenever n is a positive integer. (Recall that $f_0 = 0$.)
12. Prove that $f_n + f_{n-1} + f_{n-2} + 2f_{n-3} + 4f_{n-4} + 8f_{n-5} + \cdots + 2^{n-3} = 2^{n-1}$ whenever n is an integer with $n \geq 3$.
13. Prove that $\sum_{j=1}^{n} f_j^2 = f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$ for every positive integer n.