Basis and Dimension's

Basis:

Def:- Any subset S of a vector space $V(F)$ is called basis of $V(F)$, if

i) S is linearly independent

ii) S generates V i-e $V = \langle S \rangle$ (S is span of V).

i-e each vector in V is uniquely expressed as a linear combination of the basis vector.

standard basis of $V_2(F) = \{(1,0),(0,1)\}$

and $V_n(F) = \{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,\ldots,1)\}$

Example: Show that the vectors $(1,0,0),(1,1,0),(1,1,1)$ form a basis for R^3.

Sol:- Let $S = \{(1,0,0),(1,1,0),(1,1,1)\}$

Again let $a_1, a_2, a_3 \in R$ be such that

$$a_1(1,0,0) + a_2(1,1,0) + a_3(1,1,1) = 0$$

$$\Rightarrow (a_1+a_2+a_3,\ a_2+a_3,\ a_3) = (0,0,0)$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1 \neq 0$$

Hence vectors are L.I and only solution of these equation is

$\Rightarrow a_1 = 0,\ a_2 = 0,\ a_3 = 0,$ clearly $L(S) = V_3(R)$

Hence

$S = \{(1,0,0),(1,1,0),(1,1,1)\}$ is a basis of R^3.

Dimension:

Def: "The number of element in the finite basis of a vector space $V(F)$ is called the dimension of the vector space. It is denoted by $\dim V$ and if $\dim V = n$, then V is called n-dimensional vector space".

Finite Dimensional Vector Space:

Def:- "A vector space $V(F)$ is said to be finite dimensinal vector space if it has a finite basis".

Infinite Dimensional Vector Space:

Def: "A vector space $V(F)$ is said to be infinite dimensional vector space if it has infinite basis".

Examples:

1. $C(R)$ is a vector space of dimension of two as its basis is $\{1, i\}$ as $a+bi = a.1+b.i$
2. Dimension of $F(F)$ is always one as basis $\{1\}$ i-e $Q(Q), R(R), C(C)$ is one-dimensional vector space.
3. Dimension of $R^2(R)$ is two as basis $B = \{(1,0),(0,1)\}$, $(a,b) = a(1,0) + b(0,1)$

4. $F^n(F)$ is of n-dimensional as its standard basis will be
$e_1 = (1,0,0,\dots,0), e_2 = (0,1,0,\dots,0), \dots, e_n = (0,0,0\dots,1)$

5. $C(Q)$ and $R(Q)$ is of infinite dimensional vector space.

6. Dimension of $A_{m\times n}(R) = mn$
i-e $\dim(M_{3\times 2}) = 6$

Example

Show that the vectors $(2,1,4),(1,-1,2),(3,1,-2)$ form a basis for $R^3(R)$.

Sol: Let $S = \{(2,1,4),(1,-1,2),(3,1,-2)\}$
again let, $a_1, a_2, a_3 \in R$ be such that

$$a_1(2,1,4) + a_2(1,-1,2) + a_3(3,1,-2) = 0$$

$$(2a_1 + a_2 + 3a_3,\ a_1 - a_2 + a_3,\ 4a_1 + 2a_2 - 2a_3) = (0,0,0)$$

$$\therefore\quad 2a_1 + a_2 + 3a_3 = 0 \longrightarrow (1)$$
$$a_1 - a_2 + a_3 = 0 \longrightarrow (2)$$
$$4a_1 + 2a_2 - 2a_3 = 0 \longrightarrow (3)$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$$

$$|A| = 24 \neq 0$$

$\therefore$ Rank$(A) = 3$ i-e the number of unknowns a_1, a_2, a_3.

Hence, the only solution of these eq is

$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0$

therefor the set is L.I

Also, the dimension of vector space R^3 is 3. Hence by theorem.

"if $V(F)$ is finitely generated VS of dim.n then any set of n L.I vectors in V. Form a basis of V."

any set of L.I vectors is a basis R^3.

$\therefore$ the set $\{(2,1,4),(1,-1,2),(3,1,-2)\}$ is a basis of R^3.

Example: Let $V_3(R)$ be a finitely-dimensional V.S. Find the co-ordinate vector of $\alpha = (3,1,-4)$ relative to the basis $B = \{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1,1,1)$, $\alpha_2 = (0,1,1)$, $\alpha_3 = (0,0,1)$

Sol: Let $a_1, a_2, a_3 \in R$ such that

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3$$

$$\therefore (3,1,-4) = a_1(1,1,1) + a_2(0,1,1) + a_3(0,0,1)$$

$$(3,1,-4) = (a_1, a_1 + a_2, a_1 + a_2 + a_3)$$

$\therefore$
$$a_1 = 3$$
$$a_1 + a_2 = 1$$
$$a_1 + a_2 + a_3 = -4$$

$\therefore a_1 = 3, \quad a_2 = -2, \quad a_3 = -5$

thus the coordinate vector of $\alpha \in V_3(R)$ relative to the given basis is $[\alpha]B = (3,-2,-5)$

Question

Show that $\{(1,2,1),(2,1,0),(1,-1,2)\}$ of R^3 form a basis of R^3. Also, express the standard basis as L.C of these vectors.

Sol:

$e_1 = a_1(1,2,1) + a_2(2,1,0) + a_3(1,-1,2) \longrightarrow (A)$

$(1,0,0) = (a_1+2a_2+a_3, 2a_1+a_2-a_3, a_1-2a_3)$

$$\Rightarrow \quad a_1+2a_2+a_3 = 1$$
$$2a_1+a_2-a_3 = 0$$
$$a_1 - 2a_3 = 0$$

$e_1 = (1,0,0)$
$e_2 = (0,1,0)$
$e_3 = (0,0,1)$

$a_1 = -2/9, \quad a_2 = 5/9, \quad a_3 = 1/6$

$(A) \Rightarrow$

$(1,0,0) = -2/9(1,2,1) + \frac{5}{9}(2,1,0) + \frac{1}{6}(1,-1,2)$

solve for e_2 and e_3 by your self.

Question

Show that the set $S = \{a+ib, c+id\}$ is a basis if $C(R)$ iff $ad-bc \neq 0$

Sol:- $a_1(a+ib) + a_2(c+id) = (0+i0)$

$(a_1 a + a_2 c) + i(a_1 b + a_2 d) = 0 + i0$

$$\Rightarrow \quad a_1 a + a_2 c = 0$$
$$a_1 b + a_2 d = 0$$

Cooficient matrix

$$|A| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc \neq 0$$

Question:

Let W be the subspace of $V_4(R)$ generated by the vectors $(1,-2,5,-3), (2,3,1,-4), (3,8,-3,5)$ then Find a basis and dim of W.

Sol

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & 5 \end{bmatrix}$$

$$\sim A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(A) = 2$

$S = \{ (1,-2,5,-3), (0,7,-9,2) \}$

$\Rightarrow \quad W = L(S)$

$\Rightarrow \quad \dim W = 2$