Linear Combination

Def:- Let $V(F)$ be vector space and
$S = \{v_1, v_2, v_3, ---, v_n\}$ be a subset of V,
then the vector,

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + ---- + \alpha_n v_n$$

where, $\alpha_i \in F$, $\{i = 1, 2, 3, ---, n\}$
$F \rightarrow$ scaler

is called linear combination of the vector S.

Example: In the vector space $\mathbb{R}^3$ express the vector $(1, -2, 5)$ as linear combination of the vectors. $(1,1,1), (1,2,3), (2,-1,1)$

Sol:- Let $V = (1, -2, 5)$

and $v_1 = (1, 1, 1)$, $v_2 = (1, 2, 3)$, $v_3 = (2, -1, 1)$

Let $V = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

$$(1, -2, 5) = \alpha_1(1, 1, 1) + \alpha_2(1, 2, 3) + \alpha_3(2, -1, 1) \longrightarrow (A)$$

$$(1, -2, 5) = (\alpha_1, \alpha_1, \alpha_1) + (\alpha_2, 2\alpha_2, 3\alpha_2) + (2\alpha_3, -\alpha_3, \alpha_3)$$

$$(1, -2, 5) = (\alpha_1 + \alpha_2 + 2\alpha_3, \alpha_1 + 2\alpha_2 - \alpha_3, \alpha_1 + 3\alpha_2 + \alpha_3)$$

By comparing, we get

$$\alpha_1 + \alpha_2 + 2\alpha_3 = 1 \longrightarrow (1)$$

$$\alpha_1 + 2\alpha_2 - \alpha_3 = -2 \longrightarrow (2)$$

$$\alpha_1 + 3\alpha_2 + \alpha_3 = 5 \longrightarrow (3)$$

Solve Eq. (1), (2), (3), We get

$\alpha_1 = -6, \alpha_2 = 3, \alpha_3 = 2$

Put the above values in Eq. (A)

$(1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$

Linear Span

Def:- Let $V(F)$ be a vector space and $S = \{v_1, v_2, v_3, ---, v_n\} \subseteq V$ then linear span of S is the set of all linear combination of finitly many elements of S. It is denoted by $L(S)$.
or set of vector generated by S. It is also called linear span or spanning set or generating set.

i-e
$$L(S) = \{S\}$$
$$= \{\alpha_1 v_1 + \alpha_2 v_2 + ---- + \alpha_n v_n\}$$
where $\alpha_i \in F$

is any arbitrary finite subset of S. and $\{\alpha_1, \alpha_2, --, \alpha_n\}$ is any finite subset of F.

Example:- Determine whether the given under is in the span(S). $(2,-1,1)$
$$S = \{(1,0,2), (-1,1,1)\}$$

Sol: $Span(S) = \{av_1 + bv_2 \mid a, b \in \mathbb{R}\}$

$$(2,-1,1) = \{a(1,0,2) + b(-1,1,1)\}$$
$$= \{(a,0,2a) + (-b,b,b)\}$$

$$(2,-1,1) = (a-b, b, 2a+b)$$

By comparing, we get,

$a-b=2$, $b=-1$, $2a+b=1$

$a-(-1)=2$, , $2(1)-1=1$

$a=2-1$, , $2-1=1$

$a=1$, $b=-1$, $1=1$

Example: $V=\mathbb{R}^3$, vectors form a spanning set of $\mathbb{R}^3$.

$V_1=(1,0,0)$, $V_2=(0,1,0)$, $V_3=(0,0,1)$

Sol: if $V=(a,b,c)\subseteq\mathbb{R}^3$

then

$a(1,0,0)+b(0,1,0)+c(0,0,1)$.

Linear Independence / Dependence of Vectors:

Def:- Let $V(F)$ be the vector space and $S=\{\alpha_1,\alpha_2,\alpha_3,\ldots\alpha_n\}$ be a finite set of vectors of $V(F)$. Then vectors $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n$ are said to be linearly Independent if for scalers $a_1,a_2,\ldots,a_n\in F$

$a_1\alpha_1+a_2\alpha_2+\ldots+a_n\alpha_n=0 \Rightarrow a_1=a_2=\ldots=a_n=0$

If at least one $a_i\neq 0$ then vectors $\alpha_1,\alpha_2,\ldots,\alpha_n$ are called linearly Dependent.

Remark.

If two vectors are L.D. then one can be represent as scaler multiple of other.

Proof: $\alpha, \beta \in V(F)$; $a, b \in F$

s.t $a\alpha + b\beta = 0$

If $a \neq 0$ $\quad a\alpha = -b\beta$

$\Rightarrow \alpha = -\frac{b}{a}\beta$

If $b \neq 0$ $\quad b\beta = -a\alpha$

$\Rightarrow \beta = -\frac{a}{b}\alpha$

Question: $\alpha_1 = (1,2,3)$, $\alpha_2 = (1,0,0)$, $\alpha_3 = (0,1,0)$, $\alpha_4 = (0,0,1)$ in $V_3(\mathbb{R})$

Sol: Let $a_1, a_2, a_3, a_4 \in F$ and

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

$$a_1(1,2,3) + a_2(1,0,0) + a_3(0,1,0) + a_4(0,0,1)$$

$$\Rightarrow (a_1 + a_2, 2a_1 + a_3, 3a_1 + a_4) = (0,0,0)$$

$$\begin{aligned} a_1 + a_2 &= 0 \\ 2a_1 + a_3 &= 0 \\ 3a_1 + a_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a_2 &= -a_1 \\ a_3 &= -2a_1 \\ a_4 &= -3a_1 \end{aligned}$$

Let $a_1 = 1$, $a_2 = -1$, $a_3 = -2$, $a_4 = -3$

$$1(1,2,3) + (-1)(1,0,0) + (-2)(0,1,0) + (-3)(0,0,1) = 0$$

$$(0,0,0) = (0,0,0)$$

Hence the given set of vectors are L.D.

Question:

$(2,3,-1), (-1,4,-2), (1,18,-4) \in V_3(\mathbb{R})$

Find L.D or L.Ind?

Sol: $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0$

$\Rightarrow \quad a_1(2,3,-1) + a_2(-1,4,-2) + a_3(1,18,-4) = 0$

$\Rightarrow \quad (2a_1 - a_2 + a_3,\ 3a_1 + 4a_2 + 18a_3,\ -a_1 - 2a_2 - 4a_3) = (0,0,0)$

$$\Rightarrow \left.\begin{aligned} 2a_1 - a_2 + a_3 &= 0 \\ 3a_1 + 4a_2 + 18a_3 &= 0 \\ -a_1 - 2a_2 - 4a_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Let $A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 4 & 18 \\ -1 & -2 & -4 \end{bmatrix}$

$$|A| = 2(-16+36) + 1(-12+18) + 1(-6+4)$$
$$= 40 + 6 - 2$$
$$|A| = 44 \neq 0$$

$\Rightarrow$ Rank of $A = 3$

The only solution of system (1) is

$$a_1 = a_2 = a_3 = 0$$

Hence Vectors are L.Ind.

Question

Show that $(1,3,2), (1,-7,-8), (2,1,-1)$ of $V_3(\mathbb{R})$ is L.D

Sol: Let $a, b, c \in F$ (scalars).

$a\alpha + b\beta + c\gamma = 0$, $\quad \alpha = (1,3,2)$

$\beta = (1,-7,-8)$

$\gamma = (2,1,-1)$

$a(1,3,2) + b(1,-7,-8) + c(2,1,-1) = (0,0,0)$

$(a, 3a, 2a) + (b, -7b, -8b) + (2c, c, -c) = (0,0,0)$

$(a + b + 2c,\ 3a - 7b + c,\ 2a - 8b - c) = (0,0,0)$

then

$$a+b+2c=0 \longrightarrow (1)$$
$$3a-7b+c=0 \longrightarrow (2)$$
$$2a-8b-c=0 \longrightarrow (3)$$

$(I)\times(3) - (II)$

$$\cancel{3a}+3b+6c=0$$
$$-\cancel{3a} \mp 7b \pm c = 0$$
$$10b+5c=0$$

$$2b+c=0 \longrightarrow (4)$$

$(I)\times 2 - (III)$

$$\cancel{2a}+2b+4c=0$$
$$-\cancel{2a} \mp 8b \mp c = 0$$
$$10b+5c=0$$

$$2b+c=0 \longrightarrow (5)$$

then as (4) and (5) are the same and give,

$$c=-2b$$
$$a+b+2(-2b)=0$$
$$a+b-4b=0$$
$$a-3b=0 \Rightarrow a=3b$$

take $b=1$, $a=3$, $c=-2$

hence set of vectors are L.D.

Question: Show that the set $\{1, x, 1+x+x^2\}$ is a L.I set of vectors in the V.S of all Polynomials over the real no. field.

Proof: Let $a,b,c \in \mathbb{R}$, such that

$$a\alpha+b\beta+c\gamma=0$$

$$a(1)+b(x)+c(1+x+x^2)=0$$

$a+bx+c+cx+cx^2=0$

$(a+c)+(bx+cx)+cx^2=0$

$(a+c)1+(b+c)x+cx^2 = 0.1+0.x+0.x^2$

$$\begin{aligned} a+c&=0 \\ b+c&=0 \\ c&=0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} a&=0 \\ b&=0 \\ c&=0 \end{aligned}$$

$\therefore$ the vectors $1, x, 1+x+x^2$ are L.I

Question: Find whether the set of vectors $V_1=(1,2,1), V_2=(3,1,5), V_3=(3,-4,7)$ is L.I or L.D?

Sol: Let a_1, a_2, a_3 be three scalers such that

$$a_1V_1+a_2V_2+a_3V_3=0$$

$a_1(1,2,1)+a_2(3,1,5)+a_3(3,-4,7)=0$

$(a_1+3a_2+3a_3,\ 2a_1+a_2-4a_3,\ a_1+5a_2+7a_3)=0$

$$\Rightarrow \quad \begin{aligned} a_1+3a_2+3a_3&=0 \\ 2a_1+a_2-4a_3&=0 \\ a_1+5a_2+7a_3&=0 \end{aligned}$$

$A=\begin{bmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{bmatrix}$ $\therefore |A|=\begin{vmatrix} 1 & 3 & 3 \\ 2 & 1 & -4 \\ 1 & 5 & 7 \end{vmatrix}$

$=1(7+20)-3(14+4)+3(10-1)=27-54+27=0$

i-e So the Rank of matrix A < no. of unknown quantities. hence the set of vectors are L.D.

Question:-

Are the vectors $(2,2,2,4),(2,-2,-4,0),(4,-2,-5,2)$, $(4,2,1,6)$ L.I.

Sol:

Let $a_1, a_2, a_3 \in F$

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$$

$$\therefore a_1(2,2,2,4) + a_2(2,-2,-4,0) + a_3(4,-2,-5,2) + a_4(4,2,1,6) = 0$$

$$(2a_1 + 2a_2 + 4a_3 + 4a_4,\ 2a_1 - 2a_2 - 2a_3 + 2a_4,\ 2a_1 - 4a_2 - 5a_3 + a_4,\ 4a_1 + 2a_3 + 6a_4) = (0,0,0,0)$$

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 2 & -2 & -2 & 2 \\ 2 & -4 & -5 & 1 \\ 4 & 0 & 2 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & -6 & -9 & -3 \\ 0 & -4 & -6 & -2 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$A = \begin{bmatrix} 2 & 2 & 4 & 4 \\ 0 & -4 & -6 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_3 \to 2R_3 - 3R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$\therefore \rho(A) = 2$

i-e so the rank of matrix $A <$ no. of unknown quantities.

Hence, the set of vectors are L.D.

Hence, given vectors arent L.I.