General Properties of Vector Space

Let $V(F)$ be a Vector Space over the field $(F,+,\cdot)$ and 0 be the additive identity of F, then

i) $a0=0$ ii) $0\alpha=0$ iii) $a(-\alpha)=(-a\alpha)$

iv) $(-a)\alpha=-(a\alpha)$ v) $a(\alpha-\beta)=a\alpha-a\beta$

vi) $a\alpha=0 \Rightarrow a=0$ or $\alpha=0$; $\forall\ a\in F, \alpha,\beta\in V$

Proof: (i) $a0=0$

We have $a0=a(0+0)=a0+a0$ $(\because 0+0=0)$ by distributive prop.

Also $a0=a0+0$ $\because$ 0 being additive identity in V.)

$\therefore$ $a0+0=a0+a0$

$(\because V$ is abelian group therefore using cancellation law)

$\Rightarrow$ $a0=0$ Proved

ii) $0\alpha=0$

We have $0+0=0,\ \forall\ 0\in F$

Let $\alpha\in V$, then $(0+0)\alpha=0\cdot\alpha$

or $0\alpha+0\alpha=0\alpha+0$ $\because(\alpha\in V$ & $0+0\alpha=0\alpha)$

Since V is an abelian group for addition, therefore by using cancellation law we get

$0\alpha=0$

Proved

iii) $a(-\alpha)=(-a\alpha)$

Let $a\in F, \alpha\in V$

$\alpha\in V \Rightarrow -\alpha\in V$, then we have

$a[\alpha+(-\alpha)]=a\alpha+a(-\alpha)$ (by distributive law)

$a0=a\alpha+a(-\alpha)$ $(\because \alpha+(-\alpha)=0\in V)$

$0=a\alpha+a(-\alpha)$ $(\because a0=0)$

i.e $a\alpha+(-\alpha)=0$

$\Rightarrow$ $-(a\alpha)=a(-\alpha)$

which shows that $a.\alpha$ and $a(-\alpha)$ are inverse of each other w.r.t addition in V. Proved

iv) $(-a)\alpha = -(a\alpha)$

We know that, if $a \in F \Rightarrow -a \in F$

and also $a + (-a) = 0$

Now $[a + (-a)]\alpha = 0\alpha$

$a\alpha + (-a)\alpha = 0\alpha$ (by distributive law)

$a\alpha + (-a)\alpha = 0$ $\because (0\alpha = 0 \in V)$

$\therefore$ $a\alpha$ and $(-a)\alpha$ are inverse of each other w.r.t addition in V. Proved.

v) $a(\alpha - \beta) = a\alpha - a\beta$

Let $a \in F$, $\alpha, \beta \in V$, then we have

$a(\alpha - \beta) = a[\alpha + (-\beta)] = a\alpha + a(-\beta)$ (by distributive law)

$= a\alpha - a\beta$ Proved $\because a(-\beta) = -a\beta$

vi) $a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$; $\forall a \in F, \alpha, \beta \in V$

Let $a\alpha = 0$, $a \neq 0$, then $\exists\ a^{-1} \in F$ to each non-zero $a \in F$

$\therefore a\alpha = 0$

$\Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$ by associative law and $a^{-1}0 = 0$

$\Rightarrow 1.\alpha = 0$

$\Rightarrow \alpha = 0$ $(\because a^{-1}a = 1 \in F)$

Again $a\alpha = 0$ and $\alpha \neq 0$, then we are required to prove that $a = 0$

Let $a \neq 0 \in F \Rightarrow a^{-1} \in F$

Consider $a.\alpha = 0$

$\Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$ by associative law and $a^{-1}0 = 0$

$\Rightarrow 1.\alpha = 0$ $(\because a^{-1}a = 1)$

$\Rightarrow \alpha = 0$

Which is contradiction because $\alpha \neq 0$

$\therefore$ a must be equal to zero.

Hence $\alpha \neq 0$ and $a\alpha = 0 \Rightarrow a = 0$ Proved

Vector Subspace:

Let V be a vector space over the field F and let $W \subseteq V$. then W is called a subspace of V if W itself is a vector space over F w.r.t the operation's of vector addition and scaler mult-iplication in V.

Theorem-1

A subset W of a vector space $V(F)$ is a subspace of V, iff $\forall \alpha, \beta \in W$ and $a, b \in F \Rightarrow a\alpha + b\beta \in W$

Proof: The Condition is Necessary-

Let $V(F)$ be a vector space over F and W be a subspace of V, hence it must be closed for vector addition and scaler multiplication i-e

$a \in F, \alpha \in W \Rightarrow a\alpha \in W$ (closed for scaler multiplication)

$b \in F, \beta \in W \Rightarrow b\beta \in W$ (closed for scaler multiplication)

Now, $a\alpha \in W, b\beta \in W \Rightarrow (a\alpha + b\beta) \in W$

Hence the condition is necessary.

The condition is Sufficient-

Let us assume that W is a non-empty subset of V, subjected to the condition.

$a\alpha + b\beta \in W.\ \forall\ \alpha, \beta \in W$ and $a, b \in F$.

i-e W is closed under vector addition and scaler multiplication. Then we are required to prove that W is a vector subspace of $V(F)$; W satisfies all the postulates of vector space.

Since, $\forall a, b \in F$. So let us take $a=1, b=1$, then from the given condition, we have,

$1.\alpha + 1.\beta = \alpha + \beta \in W, \forall \alpha, \beta \in W$ $(\because \alpha \in W \Rightarrow \alpha \in V,\ 1.\alpha = \alpha \in V)$

Hence, W is closed under vector addition.

Now, taking $a=-1, b=0$, we see if $\alpha \in W$ then $(-1)\alpha + 0\alpha \in W$

[In place of β we have taken α].

$\Rightarrow -1(\alpha) + 0 \in W \Rightarrow -\alpha \in W$

Hence the additive inverse of each element of W is also in W.

Now, taking $a=0, b=0$, then from the given condition, we have, $0.\alpha + 0.\beta \in W \Rightarrow 0+0 \in W \Rightarrow 0 \in W$

Thus the zero vector of V belongs to W. It will be the zero vector of W.

As $W \subseteq V$, therefore, vector addition will be, associative as well as commutative in W.

Again, taking $b=0$, then from the given condition, We, have,

$a.\alpha + b.0 \in W \Rightarrow a\alpha + 0 \in W \Rightarrow a\alpha \in W, \forall a \in F, \alpha \in W$

and hence W is closed under scaler multiplication. So, the remaining postulates of a vector space hold in W as $W \subseteq V$.

Hence, W(F) is a vector subspace of V(F).

Proved.

Theorem-II

The necessary and sufficient conditions for a non-empty subset W of a vector space V(F) to be a subspace of V are

i) $\alpha \in W, \beta \in W \Rightarrow \alpha - \beta \in W$

and ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$

Proof: The condition is necessary-

Suppose that W is a subspace of V(F), then we are to prove that condition (i) and (ii).

Since W is a subspace, consequently W is an abelian group w.r.t addition. Thus,

$\beta \in W \Rightarrow -\beta \in W$

$\alpha \in W, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$

$\Rightarrow \alpha + (-\beta) \in W$ (By vector addition in W)

$\Rightarrow \alpha - \beta \in W$

Again $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.
[$\because$ W is closed under scaler mult.)

Hence condition (i) and (ii) are necessary.

<u>The Conditions are Sufficient</u>:

Now suppose that W is a non-empty subset of V such that the condition's (i) and (ii) hold.
We are to prove that W is a subspace of V(F).

From condition (i), we get

$\alpha \in W, \alpha \in W \Rightarrow \alpha - \alpha \in W \Rightarrow 0 \in W$

Thus the zero vector (additive identity) of V is also the zero vector of W. Again from condition (i), we get,

$0 \in W, \beta \in W \Rightarrow 0 - \beta \in W \Rightarrow -\beta \in W$.

$\Rightarrow$ The additive inverse of each element of W also belongs to W.

Again $\alpha \in W, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$ by [1]

$\Rightarrow \alpha - (-\beta) \in W$ by [1]

$\Rightarrow \alpha + \beta \in W$

Hence W is closed under vector addition.
Since, $W \subset V$, Consequently, the vector addition is commutative as well as associative in W. Therefore, (W, +) is an abelian group.
From Condition (ii), we have $a \in F, \alpha \in W \Rightarrow a\alpha \in W$
i.e W is closed w.r.t scaler multiplication.
The remaining postulates of a vector will hold in W as they hold in $V \supseteq W$.

Hence, W is itself a vector space.
Thus W is a subspace of V(F).
This proves the sufficient condition.

Problem-1

If V is a set of all $(n \times n)$ matrices over any field F, then show that a set W of all $(n \times n)$ symmetric matrix from a vector subspace of V(F).

Solution: In order to show that the set W of all $(n \times n)$ symmetric matrices form a vector subspace of V(F), It is enough to show that W is closed for vector addition and scaler multiplication.

Any matrix $A = [A_{ij}]_{n \times n}$ will be called symmetric matrix, if $A = A^t$.

Let, $A = [a_{ij}]_{n \times n}$ and $B = [b_{ij}]_{n \times n}$ be two symmetric matrix, then we have,

$$A + B = [a_{ij}]_{n \times n} + [b_{ij}]_{n \times n}$$
$$= [a_{ij} + b_{ij}]_{n \times n} \quad \text{[by addition of matrix]}$$
$$= [a_{ji} + b_{ji}]_{n \times n} \quad (\because A \text{ and } B \text{ are symmetric})$$
$$= (A + B)^t$$

i-e the sum of any two symmetric matrices is again symmetric matrix. Hence, W is closed under matrix. i-e,

For all $A, B \in W$, We have $A + B \in W$,

Also, if $a \in F$ and $A = [a_{ij}]_{n \times n} \in W$

Now, $a \cdot A = a \cdot [a_{ij}]_{n \times n} = [a \cdot a_{ij}]_{n \times n} \quad \therefore (A = A^t)$

$$= [a \cdot a_{ji}]_{n \times n} = (a \cdot A)^t.$$

i-e

$a \cdot A$ is a symmetric matrix. Hence W is closed under scaler multiplication.

Hence, W(F) is a vector subspace of V(F).

Proved.

Problem.2

Show that the set $W = \{(a,b,0) : a,b \in F\}$ is a subspace of $V_3(F)$.

Solution:

Let $\alpha, \beta \in W$, $c \in F$

then $\alpha = (a_1, b_1, 0)$

and $\beta = (a_2, b_2, 0)$

For some $a_1, b_1 \in F$

$$\alpha - \beta = (a_1 - a_2, b_1 - b_2, 0-0) \in W$$

and $c\alpha = (ca_1, cb_1, 0) \in W$

Hence, W is a subspace of $V_3(F)$. Proved

Theorem:

An arbitrary intersection of subspaces i-e the intersection of any family subspaces of a vector space is a sub-space.

Proof: Let $V(F)$ be a V.S and let $W = \bigcap_{i=I} W_i$ be the intersection of any family of subspaces of V. I being any index set.

$\because$ $0 \in W$, $\forall i \in I$ and so $0 \in \bigcap_{i \in I} W_i$

Hence, $\bigcap_{i \in I} W_i$ i-e $W \neq \phi$

Let $a, b \in F$ and $\alpha, \beta \in \bigcap_{i \in I} W_i$, then

$\alpha, \beta \in W_i$, $\forall i \in I$

$\because$ every W_i is a subspace of V.

Consequently, $a, b \in F$ and $\alpha, \beta \in W_i$

$\Rightarrow$ $a\alpha + b\beta \in W_i$, $\forall i \in I$

$\Rightarrow$ $a\alpha + b\beta \in \bigcap_{i=I} W_i$

$\therefore$ $\forall a, b \in F$ and $\alpha, \beta \in \bigcap_{i \in I} W_i$

$a\alpha + b\beta \in \bigcap_{i \in I} W_i$

Hence $\bigcap_{i \in I} W_i$ is a subspace of $V(F)$. Proved

Theorem: The union of two subspaces iff one is contain in the other.

Proof: Let $V(F)$ be a vector space and let W_1 and W_2 be two subspaces of $V(F)$.

Firstly, let us suppose $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2 = W_2$ or W_1

Therefore; $W_1 \cup W_2$ is also a subspace of $V(F)$.

Since W_1, W_2 are subspaces.

Conversely, Let $W_1 \cup W_2$ be a subspace of $V(F)$. then we are to prove $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

We shall prove it by contradiction.

Suppose that W_1 is not a subspace of W_2, and W_2 is not a subspace of W_1.

$\because$ $W_1 \nsubseteq W_2 \Rightarrow \exists\ \alpha \in W_1,\ \alpha \notin W_2 \longrightarrow (1)$

$W_2 \nsubseteq W_1 \Rightarrow \exists\ \beta \in W_2,\ \beta \notin W_1 \longrightarrow (2)$

Now from (1) and (2), we get,

$\alpha \in W_1 \Rightarrow \alpha \in W_1 \cup W_2$ and $\beta \in W_2 \Rightarrow \beta \in W_1 \cup W_2$

Again since $W_1 \cup W_2$ is a subspace and so, we have

$\alpha, \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1 \cup W_2 \Rightarrow \alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$

if $\alpha + \beta \in W_1$ then $(\alpha + \beta) - \alpha = \beta \in W_1$ [$\because$ W_1 is a subspace and $\beta \in W_1$)

But from (2), we see that $\beta \notin W_1$. which is contradiction.

Hence either W_1 is a subset of W_2 or W_2 is a subset of W_1.

Proved