Introduction To Vector Space

Internal Composition:

Let V be any set. Then the mapping $F: V \times V \to V$ is said to be internal composition and it also called vector addition.

Example: $F: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$F(a,b) = a.b$

External Composition:

Let V and F be any two non-empty set. Then the mapping $F: V \times F \to V$ is said to be external composition in V over F also called scaler multiplication.

Vector Space:

Def:-

Let $(F,+,.)$ be a field. The elements of F will be called scaler's. Let V be a non-empty set whose elements will be called vectors. Then V is a vector space over the field F, if

i) There is defined an internal composition in V called addition of vectors and denoted by '+'. Also for this composition V is an abelian group.

i-e

i) $\alpha+\beta \in V \quad \forall\ \alpha,\beta \in V$

ii) $\alpha+\beta = \beta+\alpha \quad \forall\ \alpha,\beta \in V$

iii) $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma \quad \forall\ \alpha,\beta,\gamma \in V$

iv) $\exists$ an element $0 \in V$ such that $\alpha+0=\alpha \quad \forall\ \alpha \in V$, this element $0 \in V$ will be called zero vector.

v) To every vector $\alpha \in V$, there exist a vector $-\alpha \in V$ such that $\alpha+(-\alpha)=0$

ii) There is an external composition in V over F called scaler multiplication and denoted multiplicatively i-e $a\alpha \in V\ \forall\ a \in F$ and $\forall\ \alpha \in V$.
In other words V is called closed with respect to scaler multiplication.

iii) The two compositions i-e scaler multiplication and addition of vectors satisfy the following postulates.

i) $a(\alpha+\beta) = a\alpha + a\beta \quad \forall\ a \in F$ and $\forall \in V$

ii) $(a+b)\alpha = a\alpha + b\alpha \quad \forall\ a,b \in F$ and $\alpha \in V$

iii) $(ab)\alpha = a(b\alpha) \quad \forall\ a,b \in F$ and $\alpha \in V$

iv) $1.\alpha = \alpha \quad \forall \alpha \in V$ and I is unit element of the Field. F.

Note: Some important vector space.

i) $V = \mathbb{R}^n = \{a_1, a_2, \ldots, a_n\} : a \in \mathbb{R}\}$

$F = \mathbb{R}$ i-e $\mathbb{R}^n(\mathbb{R})$

ii) $V = \{[a_{ij}]_{m \times n} : a_{ij} \in \mathbb{R}\}$

$F = (R, +, .)$ i-e $V(F)$ is a vector space.

iii) $V = P(x) = \{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \mid a_i \in \mathbb{R}$

$F = \mathbb{R}$ i-e $V(\mathbb{R})$ is vector space.

iv) $V = \{f \mid f : R \rightarrow R\}$ f is real valued Function.

$F = R$ i-e $V(R)$ is V.S

v) $V = \{\langle a_n \rangle\ a_n \in R\}$ a_n be a sequence.

$F = (R, +, .)$ Then $V(R)$ is vector space.

vi) Every Field is a vector space over its subfield.

Example:

$\mathbb{R}(Q), Q[\sqrt{2}](Q), C(\mathbb{R})$

$Z \subset Q \subset R \subset C$

$$\left.\begin{array}{l} C(R) \\ C(Q) \\ R(Q) \end{array}\right. \quad V.S$$

Problem-I

Check whether the vector
$V = \{ x, y, z \in R^3 : 2x + 3y^3 - 4z^2 = 0 \}$
is vector space or not.

Solution:

Given $V = \{ x, y, z \in R^3 : 2x + 3y^3 - 4z^2 = 0 \}$

So, take $(2, 0, 1), (8, 0, 2) \in V$

VS1:- $(2, 0, 1) + (8, 0, 2) = (10, 0, 3)$

$= -36 \neq 0$

i-e additive vector $(10, 0, 3)$ does not belongs to V.

Therefore V set is not a vector space.

Problem-2

Check wheter the set $V = \{(a, b, c) \in R^3 : (a-b)c = 0\}$ is V.S or not.

Sol:

Given that $V = \{(a, b, c) \in R^3 : (a-b)c = 0\}$

So take, $(1, 1, 2), (2, 5, 0) \in V$

VS1: $(1, 1, 2) + (2, 5, 0) = (3, 6, 2)$

So, $(3-6)2 \neq 0$

$\therefore$ the closure property is not satisfied.

So, set V is not a vector space.

Problem-3

Show that the set of all ordered n-tuples of the element's of any Field F, is a vector space over the Field F.

Sol:

Let F be a Field, then an ordered set of all n-tuples elements of F, i-e $\alpha = (a_1, a_2, \_\_\_, a_n)$ is called an n-tuple over F, and the set of all these n-tuples over a Field F will be denoted by $V_n(F)$ or V_n.

Let us define the vector addition, Scaler multiplication, and equality of n-tuples as follows.

Addition Composition:

Let α, β be any two arbitrary element's of $V_n(F)$;

i-e

$\alpha = (a_1, a_2, \_\_\_, a_n), \beta = (b_1, b_2, \_\_\_ b_n)$

$\therefore$

$\alpha + \beta = (a_1 + b_1, a_2 + b_2, \_\_\_, a_n + b_n)$

obviously, $\alpha + \beta = V_n(F)$

Hence, the set $V_n(F)$ is closed w.r.t addition of n-tuples.

Scaler multiplication Composition:

Let,

$\alpha = (a_1, a_2, \dots, a_n) \in V_n$ and $a \in F$,
then we define scaler multiplication in $V^n(F)$ as.

$a\alpha = a a_1, a a_2, \dots, a a_n, \forall a \in F$

as $aa_1, aa_2, \dots, a a_n$ all are elements of F.

Hence, the set $V_n(F)$ is closed w.r.t scaler multiplication.

Equality of two n-tuples – two n-tuples

$\alpha = (a_1, a_2, \dots, a_n)$; $\beta = (b_1, b_2, \dots, b_n)$
are said to be equal iff
$a_i = b_i$, For each $i = 1, 2, \dots, n$

Now let us verify that the set $V_n(F)$ is a V.S For the above define composition.

V_1: $V_n(F)$ is an abelian group for addition.

i) associative addition:

Let $\alpha, \beta, \gamma \in V_n(F)$, where
$\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$
and $\gamma = (c_1, c_2, \dots, c_n)$

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Now,

$\alpha+(\beta+\gamma) = (a_1, a_2, ---, a_n) + \{(b_1, b_2, ---, b_n) + (c_1, c_2, ---, c_n)\}$

$= (a_1, a_2, ---, a_n) + (b_1+c_1, b_2+c_2, ---, b_n+c_n)$

$= \{a_1+(b_1+c_1), a_2+(b_2+c_2), ---, a_n+(b_n+c_n)\}$

$= \{(a_1+b_1)+c_1, (a_2+b_2)+c_2, ---, (a_n+b_n)+c_n\}$

$= (a_1+b_1, a_2+b_2, ---, a_n+b_n) + (c_1, c_2, ---, c_n)$

$= \{(a_1, a_2, ---, a_n)+(b_1, b_2, ---, b_n)\} + (c_1, c_2, ---, c_n)$

$= (\alpha+\beta)+\gamma$

$\therefore$ $V_n(F)$ is associative for addition.

Now, let us verify that the set $V_n(F)$ is a vector space for the above defind composition.

ii) Commutivity for addition:

Let $\alpha, \beta \in V_n(F)$, where

$\alpha = (a_1, a_2, ---, a_n)$

$\beta = (b_1, b_2, --- b_n)$

Now, $\alpha+\beta = (a_1, ---, a_n)+(b_1, b_2, ---, b_n)$

$\alpha+\beta = (a_1+b_1, a_2+b_2, ---, a_n+b_n)$

$= (b_1+a_1, b_2+a_2, ---, b_n+a_n)$

($\because$ Scalar addition is commutative

$= (b_1, b_2, ---, b_n)+(a_1, a_2, ---, a_n)$

$= \beta+\alpha$

$\therefore$ $V_n(F)$ is commutative for addition

iii) Existence of additive identity

We have

$0 = (0, 0, ---, 0) \in V_n(F)$

and also $a = (a_1, a_2, ---, a_n) \in V_n(F)$

Now

$0 + a = (0, 0, ---, 0) + (a_1, a_2, ---, a_n)$

$= 0 + a_1, 0 + a_2, ---, 0 + a_n = a$

i-e $0 = (0, 0, -, 0)$ is the additive identity in $V_n(F)$

iV) Existence of additive inverse :

Let $\alpha = (a_1, a_2, ---, a_n) \in V_n(F)$

then $-\alpha = (-a_1, -a_2, ---, -a_n) \in V_n(F)$

Also

$\alpha + (-\alpha) = (a_1, a_2, --, a_n) + (-a_1, -a_2, ---, -a_n)$

$= (a_1 - a_1, a_2 - a_2, ---, a_n - a_n)$

$= (0, 0, --, 0)$

i-e $\alpha + (-\alpha) = 0$

i-e $-\alpha = (-a_1, -a_2, ---, -a_n)$ is the additive inverse of $\alpha = (a_1, a_2, ---, a_n)$

Thus, $V_n(F)$ is an abelian group of addition

V2 : Scalar multiplication is distributive over vector addition.

Let

$\alpha = (a_1, a_2, ---, a_n)$ and $\beta = (b_1, b_2, ---, b_n) \in V_n$ and $a, b \in F$

then

$a.(\alpha+\beta) = a.\{(a_1,a_2,\dots,a_n)+(b_1,b_2,\dots,b_n)\}$

$= a.(a_1+b_1, a_2+b_2, \dots, a_n+b_n)$

$= aa_1+ab_1, aa_2+ab_2, \dots, aa_n+ab_n$

$= a(a_1,a_2,\dots,a_n) + a(b_1,b_2,\dots,b_n)$

$= a\alpha + a\beta$

i-e

Scaler multiplication is distributive over vector addition in $V_n(F)$

V3: Multiplication by vector is distributive over Scaler addition

Let

$\alpha = (a_1,a_2,\dots,a_n) \in V_n$ and $a,b \in F$.

Consider

$(a+b).\alpha = (a+b).(a_1,a_2,\dots,a_n)$

$= \{(a+b).a_1, (a+b).a_2, (a+b).a_3, \dots (a+b).a_n)\}$

$= (aa_1+ba_1, aa_2+ba_2, \dots, aa_n+ba_n)$

$= (aa_1, aa_2, \dots, aa_n) + (ba_1, ba_2, \dots, ba_n)$

$= a.(a_1,a_2,\dots,a_n) + b(a_1,a_2,\dots,a_n)$

$= a.\alpha + b.\alpha$

i-e multiplication by vector is distributive over scaler addition in $V_n(F)$.

V4: Scalar multiplication is associative

Let

$\alpha = (a_1, a_2, \ldots, a_n) \in V_n$ and $a, b \in F$

Consider

$$(ab).\alpha = (ab).(a_1, a_2, \ldots, a_n)$$
$$= (aba_1, aba_2, \ldots, aba_n)$$
$$= a.(ba_1, ba_2, \ldots, ba_n)$$
$$= a.\{(b(a_1, a_2, \ldots, a_n)\}$$
$$= a.(b\alpha)$$

i-e scalar multiplication is associative in $V_n(F)$

V5:

Let 1 be the unit element of F and $\alpha = (a_1, a_2, \ldots, a_n) \in V_n$

then

$$1.\alpha = 1.(a_1, a_2, \ldots, a_n) = (1.a_1, 1.a_2, \ldots, 1.a_n)$$
$$= \alpha$$

Since all the Postulates of vector space are satisfied and so that the set $V_n(F)$ of all ordered n-tuples is a V.S over F.

proved

Problem-4

If $V = \{(a,b);\ a,b \in R\}$ and R is a Field, show that V is not a V.S over R under the addition and scalar multiplication defined by,

$(a,b) + (c,d) = (0, b+d)$

$\alpha(a,b) = \alpha a, \alpha b$

Sol: V is not a V.S under the composition defined by

$(a,b) + (c,d) = (0, b+d)$

$\alpha(a,b) = (\alpha a, \alpha b)$

because,

$(\alpha_1 + \alpha_2)(a,b) = [(\alpha_1 + \alpha_2)a, (\alpha_1 + \alpha_2)b]$

$[\because \alpha(a,b) = (\alpha a, \alpha b)]$

$$\left\{\begin{array}{l} \alpha_1(a,b) + \alpha_2(a,b) \\ (\alpha_1 a, \alpha_1 b) + (\alpha_2 a, \alpha_2 b) \\ 0, \alpha_1 b + \alpha_2 b \end{array}\right.$$