System of Linear Equation's
(Linear System)

Linear Equation:

Def:

"An equation in which variable's highest power is one is called Linear Equation."

Examples:

$$x + 2 = 3$$

$$x + y = 5$$

In general linear equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Linear System:-

Def:

"A system (group) of two or more linear equations involving same variable's is called System of linear equations or Linear system."

Examples:

(i) $\begin{cases} 2x_1 + x_2 + 3x_3 = 7 \\ 4x_1 \quad\quad + x_3 = 6 \\ 7x_1 - x_2 + 4x_3 = 2 \end{cases}$ (ii) $\begin{cases} x + 2y = 7 \\ x - 3y + z = 8 \end{cases}$

General Linear System:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

So, this is general linear system of 'm' equations in 'n' variables.

Solution of Linear System:

Basic:

$$\begin{cases} 2x_1 - x_2 + 1.5x_3 = 8 \\ x_1 - 4x_3 = -7 \end{cases}$$

$\Rightarrow (x_1, x_2, x_3) = (5, 6.5, 3)$

$\Rightarrow x_1 = 5,\ x_2 = 6.5,\ x_3 = 3$

Solution of a linear eq

$x + 2 = 3 \rightarrow (1)$

$\Rightarrow x = 1$

Putting $x = 1$ in eq (1)

$1 + 2 = 3$

$2x_1 - x_2 + 1.5x_3 = 8$, $x_1 - 4x_3 = -7$

$2(5) - 6.5 + 1.5(3) = 8$, $5 - 4(3) = -7$

$8 = 8$, $-7 = -7$

A system of linear equations has either:

i) No solution

ii) Exactly one solution } → Consistant

iii) Infinitely many solutions → Inconsistant

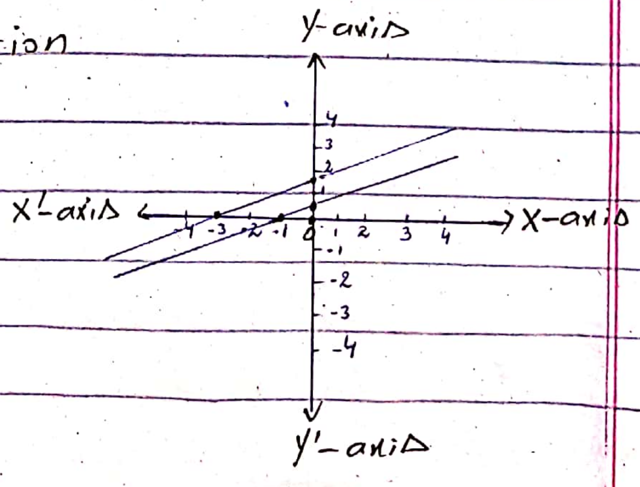
Examples:

A system of linear equations has

i) no solution

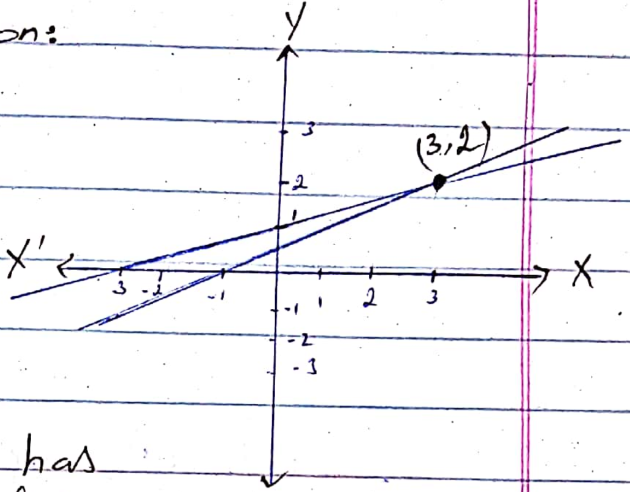
$x_1 - 2x_1 = -1$

$-x_1 + 2x_2 = 3$



ii) A system of linear equations has exactly one solution:

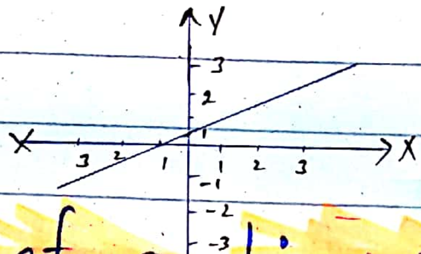
$$\begin{cases} x_1 - 2x_2 = -1 \\ -x_1 + 3x_2 = 3 \end{cases}$$



iii) A linear system has infinitely many solutions:

$$x_1 - 2x_2 = -1$$

$$-x_1 + 2x_2 = 1$$



Matrix Notation of a Linear system

$$2x_1 - x_2 + x_3 = 0$$

$$0x_1 + 2x_2 - 8x_3 = 8$$

$$-4x_1 + 5x_2 + 9x_3 = -9$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

$$\Rightarrow \quad AX = b$$

$$A_b = [A \mid b]$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & -2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

Linear Combination of Vector's

Let

$$X_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$$

Is $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination

of $X_1, X_2, X_3 = ?$

$\Rightarrow \quad \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = b$

$$\left[\begin{array}{ccc|c} 0 & 3 & 3 & 1 \\ 2 & 5 & 7 & 2 \\ 4 & 1 & 5 & 3 \end{array}\right]$$

R.R $\Rightarrow$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ No solution

b is not a linear combination of x_1, x_2, x_3.

Rank of a matrix by reducing to Echelon Form:

Note: only row operations

Reduce to upper triangular matrix

No. of non-zero rows = Rank of matrix

Example:

Find rank of matrix by reducing to echelon form.

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 4 & 3 & 5 \\ -1 & -2 & 6 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ -1 & -2 & 6 & -7 \end{bmatrix} \sim R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 5 & -3 \end{bmatrix} \sim R_3 + R_1$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim R_3 - R_2$$

$$\rho(A) = 2$$

Permutation Matrix

Def:- "Permutation matrix is a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere".

Example:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$PA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$AP = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a & b & c \\ 2 & d & e & f \\ 3 & g & h & i \end{array}$$

This can be written in $3!$ ways.

i-e

$(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)$

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{bmatrix}$$

System of Homogeneous Linear Equations

A system of 'm' homogenous linear equations in 'n' unknown $(x_1, x_2, x_3, \ldots, x_n)$ is represented by

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0$$

Above eq's in matrix form.

$$AX = 0$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix}$ is called the co-efficient matrix of order $m \times n$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a vector of $n \times 1$

and 0 is a null vector of order $m \times 1$

The no. of linearly independent solution's of eq's $AX=0$ is $n-r$

where $n \to$ no of unknown

and $r \to$ rank of coefficient matrix A.

Case-I

If $n=r$, then the eq's $AX=0$ will have non-linearly independent solution's

$\Rightarrow X=0$

$\Rightarrow x_1 = x_2 = x_3 = \ldots = x_n = 0$

(This is the only solution)

↳ It is also called Trivial Solution.

Case-II

If $n>r$, then the eq's $AX=0$ will have $n-r$ linearly independent solutions.

→ It is also called Non-trivial Solution

→ the eq's $AX=0$ will have infinite sol.

Note:

The system of eq's have a non-trivial solution if $\det(A)=0$

Example:

Solve the following system of eq's

$2x_1 + x_2 + 3x_3 = 0$; $x_1 + 2x_2 = 0$; $x_2 + x_3 = 0$

Sol:

The above eq's in matrix form is

$$AX = 0$$

where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Reducing matrix 'A' to echelon form

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 3 \\ 0 & 1 & 1 \end{bmatrix} \sim R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 3 \end{bmatrix} \sim R_2 \sim R_3$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \sim R_3 + 3R_2$$

$\Rightarrow \rho(A) = 3 \Rightarrow \underset{\text{no. of unknown}}{n} = r \rightarrow$ rank of coefficient matrix A

Hence, the system of eq's has a trivial solution

$$\Rightarrow \left.\begin{matrix} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{matrix}\right\} \text{Ans}//$$

Scanned with CamScanner

Example:

Solve the following system of eq's

$3x - y - z = 0$; $x + y + 2z = 0$; $5x + y + 3z = 0$

Sol:

The above eq's in matrix form

$$AX = 0$$

where,

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & 2 \\ 5 & 1 & 3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Reducing matrix A to echelon form

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & -1 \\ 5 & 1 & 3 \end{bmatrix} \sim R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -7 \\ 0 & -4 & -7 \end{bmatrix} \begin{matrix} \sim R_2 - 3R_1 \\ \sim R_3 - 5R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow \rho(A) = 2$ i.e $n > r$ ($n = 3$, $r = 2$)

$\Rightarrow$ system of eq's has a non-trivial solution.

$\therefore$ The corresponding system of eq's is

$$x + y + 2z = 0$$

$$-4y - 7z = 0$$

Let $z = t$

Scanned with CamScanner

$\Rightarrow 4y = -7z = -7t$

$y = -7t/4$

Also,

$x = -y - 2z$

$= \frac{7t}{4} - 2t$

$= \frac{7t - 8t}{4}$

$x = -t/4$

Hence,

$$\left.\begin{aligned} x &= -t/4 \\ y &= -7t/4 \\ z &= t \end{aligned}\right\}$$ non-trivial solution of given eq's

where 't' is a parameter.

System of Non-Homogeneous Linear Equations

A system of 'm' non-homogeneous linear eq's in 'n' unknown $(x_1, x_2, x_3, \dots, x_n)$ is represented by

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

The above equation in matrix form

$AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called the coefficient matrix of order $m \times n$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any vector of order $n \times 1$

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ is any vector of order $m \times 1$

Then, the aumented matrix of given system of linear eq's can be written as.

$$[A : B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

Condition For Consistency:

The system of eq's

$$AX = B$$

is consistent if and only if the coefficient matrix A and the augmented matrix $[A : B]$ are of the same rank i.e $\rho(A) = \rho(A : B)$

Case-I

If $\rho(A) = \rho(A:B) = n$ (no of unknowns)
then system of linear eq's has

a unique solution.

Case II

if $\rho(A) = \rho(A:B) < n$

then system of linear equations has many solutions

Now,

if $\rho(A) \neq \rho(A:B)$

$\Rightarrow$ Eq's are inconsistent and the system of linear eq's has no sol.

Example: Show that the following eq's are inconsistent.

$x + y + z = -3 \; ; \; 3x + y - 2z = -2 \; ; \; 2x + 4y + 7z = 7$

Sol: The above given eq's can be written in matrix form as

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix}$$

Let $[A:B]$ be the augmented matrix, then

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & 2 & -5 & \vdots & 7 \\ 0 & 2 & 5 & \vdots & 13 \end{bmatrix} \quad \begin{matrix} \sim R_2 - 3R_1 \\ \sim R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \vdots & -3 \\ 0 & -2 & -5 & \vdots & 7 \\ 0 & 0 & 0 & \vdots & 20 \end{bmatrix} \quad \sim R_3 + R_2$$

Hence

$\rho(A) = 3$ and $\rho(A:B) = 2$

$\because \rho(A) \neq \rho(A:B)$

Hence, the given eq's are inconsistent and has no-solution.

Example: Examine the consistency of the system and if found consistent, then solve the eq's:

$x+y+2z=9$; $2x+4y-3z=1$; $3x+6y-5z=0$

Sol:

The above eq's in matrix form

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 9 \\ 1 \\ 0 \end{bmatrix}$$

Let $[A:B]$ be the augmented matrix

$$[A:B] = \begin{bmatrix} 1 & 1 & 2 & \vdots & 9 \\ 2 & 4 & -3 & \vdots & 1 \\ 3 & 6 & -5 & \vdots & 0 \end{bmatrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & -27 \end{array}\right] \quad \begin{array}{l} \sim R_2 - 2R_1 \\ \sim R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 3 & -11 & -27 \end{array}\right] \quad \sim R_2/2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -7/2 & -17/2 \\ 0 & 0 & -1/2 & -3/2 \end{array}\right] \quad \sim R_3 - 3R_2$$

Here

$\rho(A) = \rho(A:B) = 3 =$ no of unknowns

Hence, the given system of eq's are inconsistent and has unique solution.

The corresponding system of eq's is

$$x + y + 2z = 9$$
$$y - 7/2 z = -17/2$$
$$\frac{-1}{2} z = -3/2$$

$$\Rightarrow z = 3$$

then, $y = -\frac{17}{2} - \frac{7}{2}(3) \Rightarrow y = 2$

and, $x = 9 - y - 2z$
$= 9 - 2 - (2 \times 3)$
$x = 1$

$\Rightarrow \left.\begin{array}{l} x = 1 \\ y = 2 \\ z = 3 \end{array}\right\}$ solution of given system of l. Eq's.

Scanned with CamScanner