Chapter 3

Plasma fluid theory

3.1 Introduction

So far, we only considered single particles moving in electromagnetic fields without modifying them. Here we start to deal with the change to the fields due to the motion of the particles, which gives rise to the great variety of plasma phenomena described in the next sections and modules of this course. As you saw in Chapter 1, real plasmas contain a huge number of particles within the Debye sphere, and it is impossible to describe analytically each particle as it moves through the plasma. We would have to:

- determine the position and velocity of N particles
- calculate the charge and current distribution
- calculate **E** and **B** from this distribution
- solve the equation of motion for $N = 10^{10} - 10^{20} cm^{-3}$ particles
- repeat the first step

This task is possible though, but can only be done numerically with massive parallel so called particle-in-cell (PIC) codes. Here we will derive a different approach by treating the plasma as a fluid. In plasma fluid theory, a plasma is characterized by a few local parameters–such as the particle density, the kinetic temperature, and the flow velocity–the time evolution of which are determined by means of fluid equations. These equations are analogous to, but generally more complicated than, the equations of hydrodynamics because here the fluid is charged.

Before we start, let's recapitulate the *convective* or *total* derivative (see Eq. (2.43)). When a function varies in space and time, the total time derivative is calculated as:

$$\frac{d\mathbf{F}(x,t)}{dt} = \frac{\partial \mathbf{F}(x,t)}{\partial t} + \frac{\partial \mathbf{F}(x,t)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial \mathbf{F}(x,t)}{\partial t} + u_x \frac{\partial \mathbf{F}(x,t)}{\partial x} \tag{3.1}$$

In three dimensions this is generalized to

$$\frac{d\mathbf{F}}{dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{F} \tag{3.2}$$

Note that the first term on the RHS represents the change of $\mathbf{F}$ on a fixed point in space. The second term represents the fact that a moving observer can see a time variation of $\mathbf{F}$ even when $\mathbf{F}$ doesn't change in time, as long as the observer moves with $\mathbf{u}$ through regions with different values of $\mathbf{F}$

In the following sections we will give an introduction to how to describe a plasma as a fluid. To do this we need to know how the characteristic quantities of the plasma are evolving in space and time. This can be done via conservation equations:

- for the particle density via **equation of continuity**
- for the mean velocity via **momentum conservation**
- for the Energy via the **equation of state**

We will focus here on a more or less direct or 'physical' description of these equations. The fluid equations can be derived mathematically by taking moments of the Boltzmann Equation. One gets the particle (0th moment), momentum (1st moment) and energy (2nd moment) conservation by

$$\int f d^3\mathbf{v} \rightarrow 0^{th}\text{moment} \tag{3.3}$$

$$\int f d^3\mathbf{v}\mathbf{v} \rightarrow 1^{st}\text{moment} \tag{3.4}$$

$$\int f d^3\mathbf{v}\mathbf{v}\mathbf{v} \rightarrow 2^{nd}\text{moment} \tag{3.5}$$

$$\tag{3.6}$$

This will be discussed in more detail in Module 913

3.2 Equation of continuity

The equation of continuity reflects the conservation of matter. Assume we have an arbitrary volume V with some particles N in it. It is clear that the number of particles in V can only change if particles are generated or annihilated, or if there is a net flux of particles though the surface S bounding that volume. For the time being we assume that inside the volume there is no generation (e.g. ionization) or annihilation (e.g. recombination) of particles. Let n be the particle density and $n\mathbf{u}$ the particle flux density through the surface S. So we get

$$\frac{\partial N}{\partial t} = \int_V \frac{\partial n}{\partial t} dV = -\int_{\partial V} n\mathbf{u} d\mathbf{S} \tag{3.7}$$

Using Gauss's theorem

$$\int_V \nabla \cdot \mathbf{A} dV = \int_{\partial V} \mathbf{A} d\mathbf{S} \tag{3.8}$$

we can rewrite the above equations as

$$\int_V \frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) dV = 0 \tag{3.9}$$

This equation must be fulfilled for any volume, so

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{3.10}$$

This is the equation of continuity for one particle species. It is our first equation to describe a plasma as a fluid. For each species in the plasma there is such an equation. Note that any sources or sinks of particles are to be added on the RHS.

3.3 Momentum conservation

To derive the equation for the mean velocity we start once again with the Lorentz force (1.4) for a single particle:

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{3.11}$$

Note that here we have to solve this equation at positions where the actual particle is. The important transition from the single particle description to the fluid description lies in the fact that for a fluid we want an equation for *fixed positions in space*. Further we deal with a number of particles. So we multiply the above equation by the density n and use the total derivative to get

$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) \tag{3.12}$$

As always in the equation of motion (which is, as you can see nothing else that the momentum conservation equation) on the RHS we have the sum of all forces. Here we have to include some very important effects reflecting collective forces on the mean velocity. The most important ones are forces due to collisions and to pressure gradients.

First notice that *like* particle collisions *do not* change the total momentum (which is averaged over all particles of that species). Collisions between *unlike* particles *do* exchange momentum between the species. Therefore once we realize that any quasi-neutral plasma consists of at least two different species (electrons and ions) and hence two different interpenetrating fluids we may need to account for another momentum loss (gain) term. The rate of momentum density loss by species 1 colliding with species 2 is:

$$\mathbf{F}_c = -\nu_{12} n_1 m_1 (\mathbf{u}_1 - \mathbf{u}_2) \tag{3.13}$$

where ν_{12} is the collision frequency between species 1 and 2. For the force due to pressure variations we give, without going into details at this point, the result as

$$\mathbf{F_p} = -\nabla\cdot\mathbf{P} \tag{3.14}$$

note that in general $\mathbf{P}$ is the stress tensor. Here we will describe the simplest case, where all off diagonal elements are 0 and we can rewrite $\mathbf{F_p}$ as

$$\mathbf{F}_p = -\nabla p \tag{3.15}$$

with p as the scalar pressure. So including the most important collective forces the equation of motion for the plasma fluid is

$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = nq(\mathbf{E} + \mathbf{u}\times\mathbf{B}) - \nu_{12} n_1 m_1 (\mathbf{u}_1 - \mathbf{u}_2) - \nabla p \tag{3.16}$$

3.4 Equation of State

We need the equation of state to calculate the pressure. Basically $p = nT$ is determined by energy balance, which will tell how T varies. We could write an energy equation in the same way as momentum. However, this would then contain a term for heat flux, which would be unknown. You see already the problem in describing the plasma as a fluid. In order to solve Eq. (3.10) we need $\mathbf{u}$. To get $\mathbf{u}$ we have to solve Eq. (3.16). But to solve this equation we need an equation for the pressure, to solve this equation we need the heat flux and so on. At one point we must stop and assume a quantity. The reason for that is that we deal here with mean quantities which are only approximations. As it was mentioned in the introduction to this chapter, the fluid equations can be derived formally by taking moments of the Boltzmann Equation. It turned out that for most problems it is sufficient to find an equation of the pressure and stop the approximations after that point. We do this by some sort of assumption about the heat flux. This will lead to an Equation of State:

$$pn^{-\gamma} = \text{const} \tag{3.17}$$

The value of γ to be taken depends on the heat flux assumption and on the isotropy (or otherwise) of the energy distribution. Examples

- Isothermal: $T = $ const.: $\gamma = 1$.
- Adiabatic/Isotropic: 3 degrees of freedom $\gamma = 5/3$.
- Adiabatic/1 degree of freedom: $\gamma = 3$.
- Adiabatic/2 degrees of freedom: $\gamma = 2$.

In general, $n(l/2)\delta T = -p(\delta V/V)$ (adiabatic l degrees)

$$\frac{l}{2}\frac{\delta T}{T} = -\frac{\delta V}{V} = \frac{\delta n}{n} \tag{3.18}$$

so

$$\frac{\delta p}{p} = \frac{\delta n}{n} + \frac{\delta T}{T} = \left(1 + \frac{2}{l}\right)\frac{\delta n}{n} \tag{3.19}$$

i.e.

$$pn^{-(1+2/l)} = \text{const} \tag{3.20}$$

In a normal gas, which 'holds together' by collisions, energy is rapidly shared between 3 space-degrees of freedom. Plasmas are often rather collisionless so compression in 1 dimension often stays confined to 1-degree of freedom. Sometimes heat transport is so rapid that the isothermal approach is valid. It depends on the exact situation; so lets leave γ undefined for now.

3.5 Summary of the Fluid Equations

In principle a plasma can have any number of species, but let's assume for simplicity it consists only of ions and electrons. So the complete set of equations is:

$$\begin{aligned} \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} & \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 & \nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \end{aligned} \tag{3.21}$$

the coupling between the fields and the plasma comes via current and charge densities $\mathbf{j}$ and ρ

$$\rho = e(-n_e + Zn_i) \tag{3.22}$$

$$\mathbf{j} = e(-n_e \mathbf{u}_e + Zn_i \mathbf{u}_i) \tag{3.23}$$

with Z as the ionisation state of the ions. n_j and $\mathbf{u}_j$ $(j = i, e)$ are calculated for each species via:

$$\begin{aligned} \frac{\partial n_j}{\partial t} + \nabla \cdot (n_j \mathbf{u_j}) &= 0 \\ m_j n_j \left(\frac{\partial \mathbf{u_j}}{\partial t} + (\mathbf{u}_j \cdot \nabla) \mathbf{u}_j \right) &= n_j q_j (\mathbf{E} + \mathbf{u}_j \times \mathbf{B}) - \nu_{jk} n_j m_j (\mathbf{u}_j - \mathbf{u}_k) - \nabla p \\ p_j n_j^{-\gamma} &= \text{const} \end{aligned} \tag{3.24}$$

Discussion task 8: How many unknown variables and how many equations are there in this set of equations for the fluid approximation of the plasma?

So far we have been using fluid equations which apply to electrons and ions separately. These are called 'Two Fluid' equations because we always have to keep track of both fluids separately. On simplification would occur when we are only interested in time scales where the electrons are moving, but the much heavier ions are a rest. That happens in the interaction of a fast oscillating field such as a laser pulse with a plasma. To describe such an interactions it is sometimes justified to follow only the evolution of the electrons. But this *must not* mixed up with the 'Single Fluid' description. Sometimes it's possible and useful to further simplify the fluid model by combining the electron and ion equations *together* to obtain equations governing the plasma viewed as a Single Fluid. This description will be given in Module 913.