Chapter 2

Charged particle motion

Plasmas are complicated because motions of electrons and ions are determined by the electric and magnetic fields but also change the fields by the currents they carry. As we already mentioned (see Equ. (1.4) the fundamental equation of motion of an individual particle takes the form

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.1}$$

In this section we shall ignore the back-reaction of the particles and assume that fields are prescribed, e.g. we forget for a moment that the particles are itself parts of the plasma and hence responsible for the generation and modification of the fields. Even so, calculating the motion of a charged particle can be quite hard. We will first of all consider the motion of charged particles in spatially and temporally uniform electromagnetic fields, followed by spatially varying field. At the end of this chapter we will study briefly time varying fields.

2.1 Motion in uniform fields

2.1.1 E=const, B=0

In this easiest case the Lorentz force is reduced to:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} \tag{2.2}$$

We will set the the x-coordinate in the direction of the electric field. This simple case has some traps tough: if we would simply assume:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = m_e \frac{\mathrm{d}v_x}{\mathrm{d}t} = qE_x \tag{2.3}$$

we would get after the integration:

$$v_x = \frac{e}{m_e} E_x t \tag{2.4}$$

with would lead to $v_x \to \infty$ for $t \to \infty$, which is of course forbidden by the special theory of relativity. To solve this problem correctly we have to include the change of the mass according to $m_e = m_0\gamma$ with $\gamma = 1/\sqrt{1-(v/c_0)^2}$. So we have to solve this equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{v_x}{\sqrt{1-(v_x/c_0)^2}} = \frac{q}{m_0} E_x \tag{2.5}$$

which is still straight forward to integrate. After some rearrangements we get the correct velocity as:

$$v_x = \frac{e}{m_0} E_x t \frac{1}{\sqrt{1+\left(\frac{eE_x t}{m_0 c_0}\right)^2}} \tag{2.6}$$

Figure (2.1) shows the importance of the correct mass description of an electron in a field of 100kV/m. Of course for $t \ll m_0 c_0 / eE_x$ the velocity can be approximated by Equ. (2.4).

Discussion task 5: According to special relativity the kinetic energy of a particle is $E_{kin} = m_0(\gamma - 1)c^2$, but almost always the kinetic energy is calculated as: $E_{kin} = mv^2/2$. How is this contradiction solved?
Assignment task 6: Prove that Eq. (2.6) can be approximated by Eq. (2.4) for $t \ll m_0 c_0 / eE_x$

2.1.2 E=0, B=const

The next case is a constant **B**-field which we define as a field in z-direction. The equation of motion is here reduced to

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} \tag{2.7}$$

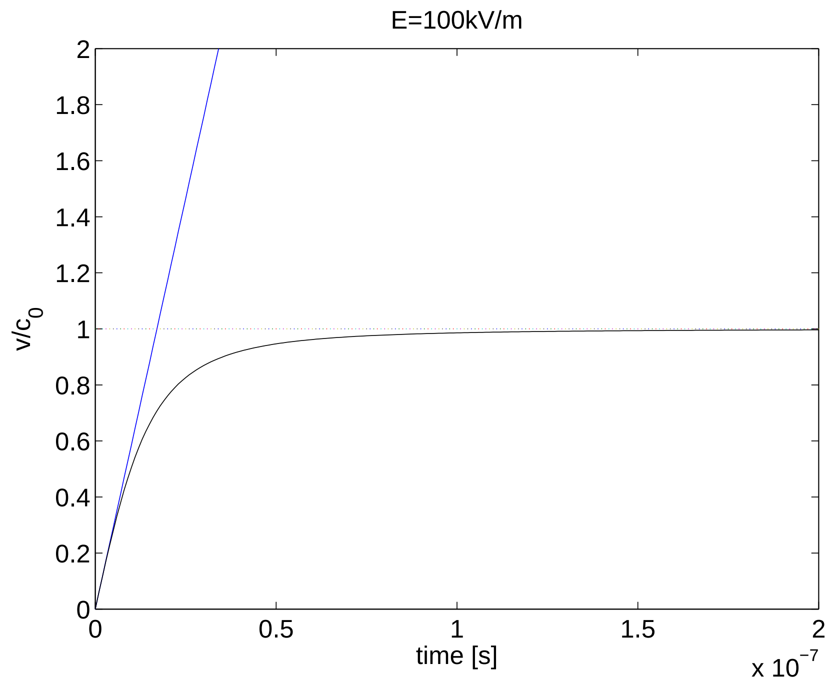

Figure 2.1: Velocity for a electron in a constant E-field of 100kV/m. Blue without relativistic mass correction. Black: correct description

or for the the components of the momentum:

$$\begin{aligned} \dot{p}_x &= q v_y B_z \\ \dot{p}_y &= -q v_x B_z \\ \dot{p}_x &= 0 \end{aligned} \tag{2.8}$$

where the dot represents the time derivative. First of all we see that there is no acceleration in the direction of the **B**-field. For the further analysis of this problem we assume $v \ll c_0$. After performing a second time derivative for e.g. the x components we can substitute the y components and we get for v_x:

$$\ddot{v}_x = \frac{q}{m_e}\dot{v}_y B_z = -\left(\frac{qB_z}{m_e}\right)^2 v_x \tag{2.9}$$

which is the well know equation for an harmonic oscillator with the characteristic frequency $\Omega = |q|B_z/m_e$ and the general solution:

$$v_x = v_0 \cos(\Omega t) + v_1 \sin(\Omega t) \tag{2.10}$$

with the two constants v_0 and v_1 terminated by our choice of the initial velocity for $t = 0$ as $v(t = 0) = v_0\mathbf{e_x} + 0\mathbf{e_y} + 0\mathbf{e_z}$, leading to

$$v_x = v_0 \cos(\Omega t) \tag{2.11}$$

for the x component of the velocity. Inserting this in Equ. (2.8) leads to

$$\dot{v}_y = -\frac{q}{|q|}\Omega v_0 \cos(\Omega t) \tag{2.12}$$

which gives

$$v_y = -\frac{q}{|q|} v_0 \sin(\Omega t) \tag{2.13}$$

where we used again our choice of the initial velocity. Equ. (2.11) and (2.13) describe the velocity of a charged particle in a constant magnetic field. An initially present velocity v_z is not modified by a magnetic field parallel to this velocity component. So in general we can write:

$$\mathbf{v} = v_{\perp 0} \cos(\Omega t)\mathbf{e_x} - \frac{q}{|q|} v_{\perp 0} \sin(\Omega t)\mathbf{e_y} + v_z\mathbf{e_z} \tag{2.14}$$

To get the trajectory of the particle we integrate Equ. (2.14) resulting in

$$\mathbf{r} = \mathbf{r_0} + \frac{v_{\perp 0}}{\Omega} \sin(\Omega t)\mathbf{e_x} + \frac{q}{|q|}\frac{v_{\perp 0}}{\Omega} \cos(\Omega t)\mathbf{e_y} + v_z t\mathbf{e_z} \tag{2.15}$$

This is a circular trajectory with radius $\rho = v_{\perp 0}/\Omega.$, which is referred to as *Gyroradius*, and Ω is known as the *Gyrofrequency*. Equ. (2.15) shows that the sign of the charge defines the direction of the rotation. Ions rotate anticlockwise and electrons clockwise about the magnetic field (see figure 2.2). Note that a particle gyrating as described here produces a magnetic field counteracting the external field resulting in a reduction of the total field. This is the property of a magnetic material which is *Diagmagnetic*.

Discussion task 6: Calculate the energy gain of a charged particle in a constant magnetic field

2.1.3 E=const, B=const

When both electric and magnetic fields are present the motion of a charged particle it the superposition of the acceleration in the **E** direction and a

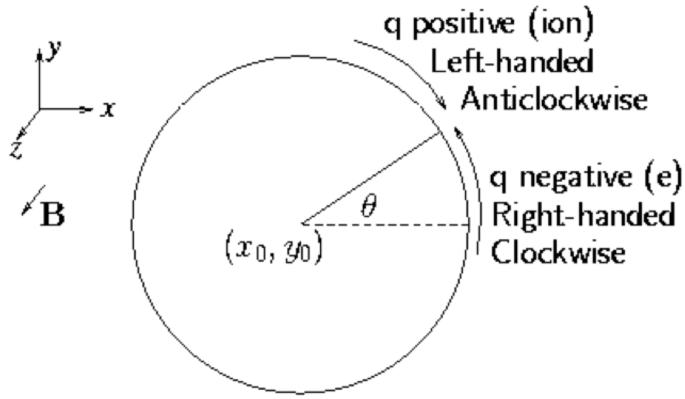

Figure 2.2: Gyro centre (x_0, y_0 and orbit)

circular motion perpendicular to the **B** direction. It is important for the further analysis to split the the electric field into a component parallel to the magnetic field $E_\parallel$ and a component perpendicular to the magnetic field $\mathbf{E}_\perp$. As we saw in the last section the velocity component of a particle in the direction of the magnetic field is not affected by it. So there is just $E_\parallel$ to change the velocity, as we described in the first case. For the remaining perpendicular field we will solve this problem with a common trick, by finding a coordinate system in which $\mathbf{E}_\perp = 0$. We restrict ourselves again to the non-relativistic case where the **E** field is transformed to a moving coordinate system as $\mathbf{E}' = \mathbf{E} + \mathbf{v_d} \times \mathbf{B}$. The goal is now to find this velocity $\mathbf{v_d}$. We multiply this equation with $\times\mathbf{B}$ to get

$$0 = \mathbf{E}_\perp \times \mathbf{B} + (\mathbf{v_d} \times \mathbf{B}) \times \mathbf{B} = \mathbf{E}_\perp \times \mathbf{B} + (\mathbf{v_d} \cdot \mathbf{B})\mathbf{B} - B^2 \mathbf{v}_d \tag{2.16}$$

We can solve this equation only if we set $\mathbf{v}_d$ perpendicular to **B**. With that assumption we get:

$$\mathbf{v}_d = \frac{\mathbf{E}_\perp \times \mathbf{B}}{B^2}. \tag{2.17}$$

This drift, which is termed the E-cross-B drift in plasma physics, is identical for all plasma species. Inside this frame $\mathbf{E}_\perp = \mathbf{0}$, so this frame can properly be regarded as the rest frame of the plasma. This also so called *guiding centre* is an important concept in analyzing complicated particle motions. Here the advantage of this description is obvious: since the electric field is zero the particle gyrates around the magnetic field at frequency Ω exactly in the same way as described above for the E=0; B=const case.

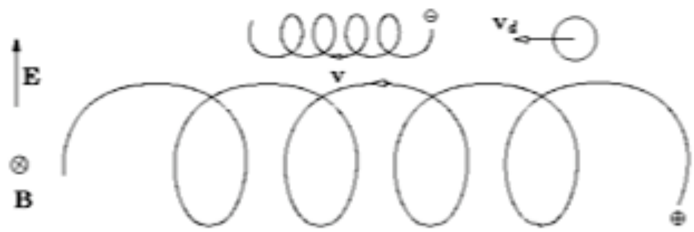

Figure 2.3: $\mathbf{E} \times \mathbf{B}$ drift orbit

Hence the full solution for the particle trajectory is:

$$\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_d + \mathbf{v}_{Gyration} \tag{2.18}$$

This separation gives us a clue to simplify the description for some cases. Sometimes when analyzing charged particle motion in non-uniform electromagnetic fields, we can somehow neglect the rapid, and relatively uninteresting, gyromotion, and focus, instead, on the far slower motion of the guiding centre. Clearly, what we need to do in order to achieve this goal is to somehow average the equation of motion over gyrophase, so as to obtain a reduced equation of motion for the guiding centre. This method was introduced by Hans Alfén and in known as *guiding centre approximation*

2.1.4 Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity. Write it $\mathbf{F}$ so that

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} + q\mathbf{v} \times \mathbf{B} = q\left(\frac{1}{q}\mathbf{F} + \mathbf{v} \times \mathbf{B}\right) \tag{2.19}$$

This is just like the previous case except with $\mathbf{F}$/q replacing $\mathbf{E}$. The drift is therefore

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}. \tag{2.20}$$

In this case, if force on electrons and ions is same, they drift in opposite directions. This general formula can be used to get the drift velocity in some other cases of interest.

2.2 Motion in nonuniform fields

In the case of nonuniform, inhomogeneous and/or time dependent electromagnetic field the equation of motion becomes nonlinear and can be solved in general only by numeric integration. However in some cases we can use the guiding centre approximation to find reasonable solutions. As mentioned before we can use this approximation if the spacial inhomogeneity is so small or the time dependence of the fields is so slow, that during one gyro period the fields can be approximately treated as constant. This is in most laboratory plasmas possible, but only seldom in interstellar plasmas.

2.2.1 E=0, B=Non-Uniform

Lets assume that the magnetic field varies only along one spacial coordinate. Then we get orbits that look qualitatively similar to the $\mathbf{E} \perp \mathbf{B}$

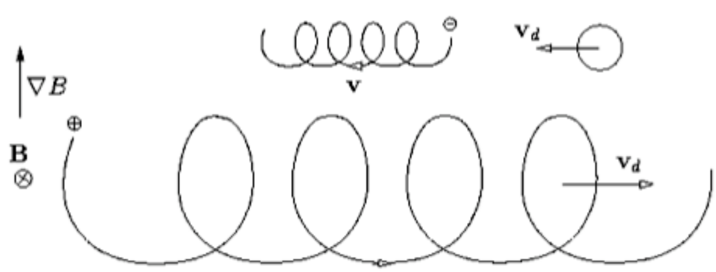

Figure 2.4: $\nabla\mathbf{B}$-Drift

Curvature of orbit is greater where $\mathbf{B}$ is greater causing loop to be small on that side. Result is a drift perpendicular to both $\mathbf{B}$ and $\nabla\mathbf{B}$ Notice, though, that electrons and ions go in opposite directions (unlike the $\mathbf{E}\times\mathbf{B}$ case). We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v_d} + \mathbf{v_L}$ where $\mathbf{v_d}$ is constant. We shall find that this can be done only approximately, by assuming that the the velocity is small compared to c_0 and the field gradient is small compared to the gyroradius ρ. i.e.,

$$\rho \ll B/|\nabla B| \tag{2.21}$$

in which case we can express the field approximately as the first two terms in a Taylor expression:

$$\mathbf{B} \approx \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B} \tag{2.22}$$

Then substituting the decomposed velocity we get:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} = q\left(\mathbf{v}_L \times \mathbf{B}_0 + \mathbf{v}_d \times \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \times (\mathbf{r} \cdot \nabla)\mathbf{B}\right) \tag{2.23}$$

or

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{v}_d \times (\mathbf{r} \cdot \nabla)\mathbf{B} \tag{2.24}$$

Keep in mind that $\mathbf{v}_d/\mathbf{v}_L \ll 1$, like $r|\nabla B|/B \ll 1$ Therefore the last term here is much smaller than the first two and can be dropped (e.g. the last term is of second order, whereas the first two are of first order). The problem here is that $\mathbf{v}_L$ and $\mathbf{r}_L$ are periodic. Similar to the velocity, we substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so we get

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} + \mathbf{v}_{\mathbf{L}d} \times (\mathbf{r_0} \cdot \nabla)\mathbf{B} \tag{2.25}$$

We now average over a cyclotron period Ω. The last term is $\propto \exp(i\Omega t)$ so it averages to zero. So this it the remaining equation we have to solve:

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \langle \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} \rangle \tag{2.26}$$

To perform the time average denote here with the brackets $\langle ... \rangle$ we use

$$\begin{aligned}
\mathbf{r}_L = \binom{x_L}{y_L} &= \frac{v_\perp}{\Omega}\binom{\sin(\Omega t)}{\frac{q}{|q|}\cos(\Omega t)} \\
\mathbf{v}_L = \binom{v_{xL}}{v_{yL}} &= v_\perp\binom{\cos(\Omega t)}{-\frac{q}{|q|}\sin(\Omega t)} \\
\text{So } [\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]_x &= v_y y\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}y} \\
[\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]_y &= -v_x y\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}y}
\end{aligned} \tag{2.27}$$

(Taking $\nabla\mathbf{B}$ to be in the y-direction). Then

$$\begin{aligned} \langle v_y y\rangle &= -\langle\cos\Omega t\sin\Omega t\rangle\frac{v_\perp^2}{\Omega} = 0 \\ \langle v_x y\rangle &= \langle\cos^2\Omega t\rangle\frac{v_\perp^2 q}{\Omega|q|} = \frac{v_\perp^2 q}{2\Omega|q|} \end{aligned} \tag{2.28}$$

So

$$\langle\mathbf{v}_L\times(\mathbf{r}_L\cdot\nabla)\mathbf{B}\rangle = -\frac{v_\perp^2 q}{2\Omega|q|}\nabla\mathbf{B} \tag{2.29}$$

Substitute in the remaining equation we had to solve:

$$0 = \mathbf{v}_d\times\mathbf{B}_0 - \frac{v_\perp^2 q}{2\Omega|q|}\nabla\mathbf{B} \tag{2.30}$$

and solve as before to get

$$\mathbf{v}_d = \frac{\left(-\frac{v_\perp^2}{2\Omega|q|}\nabla\mathbf{B}\right)\times\mathbf{B}}{B^2} = \frac{v_\perp^2 q}{2\Omega|q|}\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2} \tag{2.31}$$

or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2} \tag{2.32}$$

This is called the "Grad B drift".

2.2.2 E=0, B $\|$ ∇B; The Mirror Effect of Parallel Field Gradients

In the situation outlined in Figure 2.5 we have a magnetic field which increases in the direction of the field lines. Again we are only interested in the average movements of the particles and not on the detailed gyration. There is a net force on average along **B** which is.

$$\begin{aligned} \langle F_\|\rangle &= -|q\mathbf{v}\times\mathbf{B}|\sin\alpha = -|q|v_\perp B\sin\alpha \\ \text{with}\sin\alpha &= -B_r/B \end{aligned} \tag{2.33}$$

To calculate B_r as function of B_z we use Maxwell's Equation $\nabla\cdot\mathbf{B} = 0$. We assume here rotation symmetry along the z-axis as well as that the field

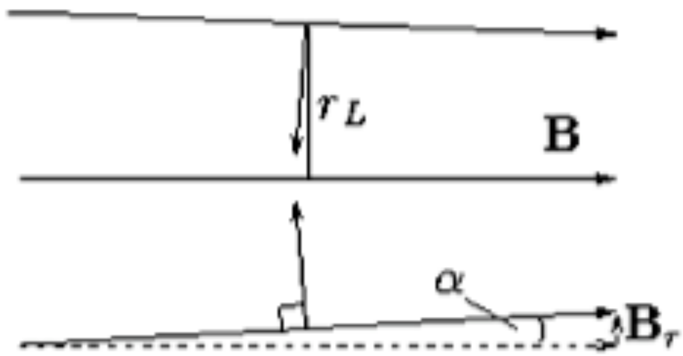

Figure 2.5: Basis of parallel mirror force

gradient points mainly in z direction. With this assumption we can write the Nabla operator in cylindric coordinates as

$$\nabla \cdot \mathbf{B} = \frac{1}{r}\partial_r(rB_r) + \partial_z B_z = 0 \tag{2.34}$$

Hence

$$rB_r = -\int r\partial_z B_z \mathrm{d}r \tag{2.35}$$

Supposing the gyroradius ρ is small enough that $\partial_z B_z \approx$ const. we can solve the integral as

$$rB_r = -\int_0^\rho r\partial_z B_z \mathrm{d}r \approx -\partial_z B_z \int_0^\rho r\mathrm{d}r = -\partial_z B_z \frac{1}{2}\rho^2 \tag{2.36}$$

So the radial component of the magnetic field is then

$$B_r = -\partial_z B_z \frac{1}{2}\rho \tag{2.37}$$

which gives for net force:

$$\langle F_\parallel \rangle = -\frac{1}{2}|q|v_\perp \rho \partial_z B_z = -\frac{mv_\perp^2}{2B}\partial_z B_z \tag{2.38}$$

where we used the definition of the gyroradius as $\rho = v_\perp m/|q|B$. As a charged particle enters a region with increasing field it experiences despite the sign of the charge a net parallel retarding force. It's worth to mention an other aspect here. Since any gyrating charge represents a loop current which generates a magnetic field, we can define a magnetic moment μ associated with this current as

$$\mu = AI = \pi\rho^2 \frac{|q|v_\perp}{2\pi\rho} = \frac{|q|v_\perp\rho}{2} = \frac{mv_\perp^2}{2B} = \frac{W_\perp}{B} \tag{2.39}$$

Where $W_\perp$ represents the kinetic energy of the parallel motion. In conclusion the force of a field gradient on a charged particle gyrating in this field can be seen as the force on a magnet dipole of moment μ

$$F_\parallel = -\mu\nabla_\parallel \mathbf{B} \tag{2.40}$$

The force always points along $\mathbf{B}$ but towards lower fields (against the gradient of $\mathbf{B}$).

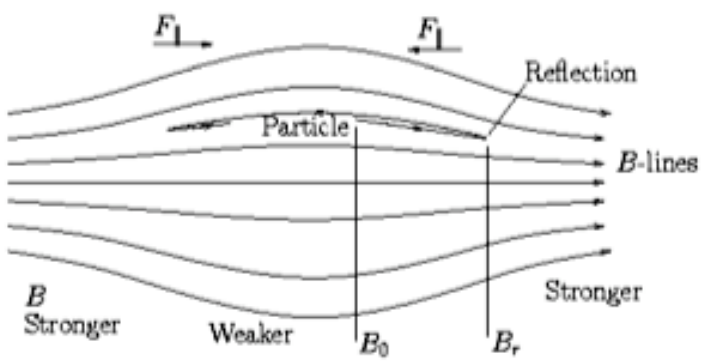

Figure 2.6: Magnetic Trap

With the use of two gradients as depicted in fig. 2.6 one can build a magnetic trap for charged particles. This is of particular interest for storing antimatter since in this magnet trap the particles are not in contact with the walls of the trap which would lead to annihilation of the antimatter.

Assignment task 7:

- Verify that a charge rotating with a radius ρ generate the current I described in Equ. (2.39)

- Calculate how the magnetic flux Φ through the gyro orbit is related to the magnet moment μ

2.2.3 Invariance of the magnetic moment

At this point we will introduce an important fact which will help us to analyze the particle motion in more complicated cases. There are three invariance of motion for a particle in electro-magnetic fields, which we will introduce in the course of the next sections. Let us here demonstrate that the magnetic moment $\mu = m\,v_\perp^2/2\,B$ is one of the invariant, e.g. it is a constant of the motion, at least to lowest order. We start with the force on the rotating particle in an inhomogeneous magnetic field (equ. (2.40))

$$F_\parallel = m\frac{dv_\parallel}{dt} = -\mu\nabla_\parallel\mathbf{B} \tag{2.41}$$

We assume that $\mathbf{B}$ point in z-direction. After multiplying this equation with $v_\parallel = dz/dt$ we get

$$\frac{m}{2}\frac{dv_\parallel^2}{dt} = -\mu\partial_z B\frac{dz}{dt} \tag{2.42}$$

The left side of this equation is just the kinetic energy of the parallel motion $W_\parallel$. Note that in general the total differential of B is

$$\frac{dB}{dt} = \partial_t B + v_z\partial_z B. \tag{2.43}$$

Here in our case B is constant in time, so

$$\partial_z B\frac{dz}{dt} = \frac{dB}{dt} \tag{2.44}$$

Here you can see that in the case of a time independent but spatial changing field, a *moving* observer (here the particle) registers a *time-dependent* field. We can now rewrite Equ. (2.42) as

$$\frac{dW_\parallel}{dt} = -\frac{W_\perp}{B}\frac{dB}{dt} \tag{2.45}$$

As we already saw in previous sections that a charged particle cannot gain energy from a magnetic field. So

$$W_{\|} + W_{\perp} = \text{const} \tag{2.46}$$

$$\frac{dW_{\|}}{dt} + \frac{dW_{\perp}}{dt} = 0 \tag{2.47}$$

$$\frac{dW_{\perp}}{dt} = \frac{W_{\perp}}{B}\frac{dB}{dt} \tag{2.48}$$

On the other hand when we just look at the time derivative of $W_{\perp}$ we have

$$\frac{dW_{\perp}}{dt} = \frac{d}{dt}\left(\frac{W_{\perp}}{B}B\right) = \frac{W_{\perp}}{B}\frac{dB}{dt} + B\frac{d}{dt}\left(\frac{W_{\perp}}{B}\right) \tag{2.49}$$

Comparing this with Equ. (2.48) we find that

$$B\frac{d}{dt}\left(\frac{W_{\perp}}{B}\right) = B\frac{d}{dt}\mu = 0 \tag{2.50}$$

thus

$$\mu = \text{const} \tag{2.51}$$

The magnetic moment μ is a constant as long as the guiding centre approximation is valid. Or in other words μ is a constant of the motion to the lowest order. This invariance follows directly for the energy conservation law. More general, it can be shown that $mv_{\perp}^2/2B$ is the lowest order approximation to a quantity which is a constant of the motion to all orders in the perturbation expansion. Such a quantity is called an adiabatic invariant.

Assignment task 8: Research magnetic mirror and traps and explain their principle in terms of energy conservation. Further explain mirror trapping and the loss cone.

2.2.4 Magnetized plasmas

In the last sections we derived the typical behavior of a *single* charged particle gyrating in a magnetic field. To conclude this section we give here some estimates how particles inside a *plasma* gyrate when a magnetic field is present. When the plasma is in equilibrium we can use the mean velocity to describe the gyration according to the last section as

$$\rho \equiv \frac{v_t}{\Omega}, \tag{2.52}$$

where $\Omega = eB/m$. As usual, there is a distinct gyroradius for each species. When species temperatures are comparable, the electron gyroradius is distinctly smaller than the ion gyroradius:

$$\rho_e \sim \left(\frac{m_e}{m_i}\right)^{1/2} \rho_i. \tag{2.53}$$

A plasma system, or process, is said to be magnetized if its characteristic length-scale L is large compared to the gyroradius. In the opposite limit, $\rho \gg L$, charged particles have essentially straight-line trajectories. Thus, the ability of the magnetic field to significantly affect particle trajectories is measured by the magnetization parameter

$$\delta \equiv \frac{\rho}{L}. \tag{2.54}$$

There are some cases of interest in which the electrons are magnetized, but the ions are not. However, a "magnetized" plasma conventionally refers to one in which both species are magnetized. This state is generally achieved when

$$\delta_i \equiv \frac{\rho_i}{L} \ll 1. \tag{2.55}$$

In conclusion, all descriptions of plasma behaviour are based, ultimately, on the motions of the constituent particles. For the case of an unmagnetized plasma, the motions are fairly trivial, since the constituent particles move essentially in straight lines between collisions. The motions are also trivial in a magnetized plasma where the collision frequency ν greatly exceeds the gyrofrequency Ω: in this case, the particles are scattered after executing only a small fraction of a gyro-orbit, and, therefore, still move essentially in straight lines between collisions. The situation of primary interest in this section is that of a collisionless (i.e., $\nu \ll \Omega$), magnetized plasma, where the gyroradius ρ is much smaller than the typical variation length-scale L of the $\mathbf{E}$ and $\mathbf{B}$ fields, and the gyroperiod Ω^{-1} is much less than the typical time-scale τ on which these fields change. In such a plasma, we expect the motion of the constituent particles to consist of a rapid gyration perpendicular to magnetic field-lines, combined with free-streaming parallel to the field-lines. We are particularly interested in calculating how this motion is affected by the spatial and temporal gradients in the $\mathbf{E}$ and $\mathbf{B}$ fields. In general, the motion of charged particles in spatially and temporally non-uniform electromagnetic fields is extremely complicated: however, we hope to considerably simplify this motion by exploiting the assumed smallness of the parameters ρ/L and $(\Omega\,\tau)$

2.2.5 Example of a magnetic mirror: The Van Allen radiation belts

Plasma confinement via magnetic mirroring occurs in nature as well as in unsuccessful fusion devices. For instance, the Van Allen radiation belts, which surround the Earth, consist of energetic particles trapped in the Earth's dipole-like magnetic field. These belts were discovered by James A. Van Allen and co-workers using data taken from Geiger counters which flew on the early U.S. satellites, Explorer 1 (which was, in fact, the first U.S. satellite), Explorer 4, and Pioneer 3. Van Allen was actually trying to measure the flux of cosmic rays (high energy particles whose origin is outside the Solar System) in outer space, to see if it was similar to that measured on Earth. However, the flux of energetic particles detected by his instruments so greatly exceeded the expected value that it prompted one of his co-workers to exclaim, "My God, space is radioactive!" It was quickly realized that this flux was due to energetic particles trapped in the Earth's magnetic field, rather than to cosmic rays.

There are, in fact, two radiation belts surrounding the Earth. The inner belt, which extends from about 1-3 Earth radii in the equatorial plane is mostly populated by protons with energies exceeding 10 MeV. The origin of these protons is thought to be the decay of neutrons which are emitted from the Earth's atmosphere as it is bombarded by cosmic rays. The inner belt is fairly quiescent. Particles eventually escape due to collisions with neutral atoms in the upper atmosphere above the Earth's poles. However, such collisions are sufficiently uncommon that the lifetime of particles in the belt range from a few hours to 10 years. Clearly, with such long trapping times only a small input rate of energetic particles is required to produce a region of intense radiation.

The outer belt, which extends from about 3-9 Earth radii in the equatorial plane, consists mostly of electrons with energies below 10 MeV. The origin of these electrons is via injection from the outer magnetosphere. Unlike the inner belt, the outer belt is very dynamic, changing on time-scales of a few hours in response to perturbations emanating from the outer magnetosphere.

In regions not too far distant (i.e., less than 10 Earth radii) from the

Earth, the geomagnetic field can be approximated as a dipole field,

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{M_E}{r^3}(-2\cos\theta, -\sin\theta,\, 0), \tag{2.56}$$

where we have adopted conventional spherical polar coordinates (r, θ, φ) aligned with the Earth's dipole moment, whose magnitude is $M_E = 8.05 \times 10^{22}\ \mathrm{A\,m^2}$. It is usually convenient to work in terms of the latitude, $\vartheta = \pi/2 - \theta$, rather than the polar angle, θ. An individual magnetic field-line satisfies the equation

$$r = r_{\rm eq}\,\cos^2\vartheta, \tag{2.57}$$

where $r_{\rm eq}$ is the radial distance to the field-line in the equatorial plane ($\vartheta = 0^\circ$). It is conventional to label field-lines using the L-shell parameter, $L = r_{\rm eq}/R_E$. Here, $R_E = 6.37 \times 10^6\,\mathrm{m}$ is the Earth's radius. Thus, the variation of the magnetic field-strength along a field-line characterized by a given L-value is

$$B = \frac{B_E}{L^3}\frac{(1 + 3\,\sin^2\vartheta)^{1/2}}{\cos^6\vartheta}, \tag{2.58}$$

where $B_E = \mu_0 M_E/(4\pi\,R_E^3) = 3.11 \times 10^{-5}\,\mathrm{T}$ is the equatorial magnetic field-strength on the Earth's surface.

Consider, for the sake of simplicity, charged particles located on the equatorial plane ($\vartheta = 0^\circ$) whose velocities are predominately directed perpendicular to the magnetic field. The proton and electron gyrofrequencies are written

$$\Omega_p = \frac{e\,B}{m_p} = 2.98\,L^{-3}\ \mathrm{kHz}, \tag{2.59}$$

and

$$|\Omega_e| = \frac{e\,B}{m_e} = 5.46\,L^{-3}\ \mathrm{MHz}, \tag{2.60}$$

respectively. The proton and electron gyroradii, expressed as fractions of the Earth's radius, take the form

$$\frac{\rho_p}{R_E} = \frac{\sqrt{2\,\mathcal{E}\,m_p}}{e\,B\,R_E} = \sqrt{\mathcal{E}(\mathrm{MeV})}\left(\frac{L}{11.1}\right)^3, \tag{2.61}$$

and

$$\frac{\rho_e}{R_E} = \frac{\sqrt{2\,\mathcal{E}\,m_e}}{e\,B\,R_E} = \sqrt{\mathcal{E}(\mathrm{MeV})}\left(\frac{L}{38.9}\right)^3, \tag{2.62}$$

respectively. It is clear that MeV energy charged particles in the inner magnetosphere (i.e, $L \ll 10$) gyrate at frequencies which are much greater than the typical rate of change of the magnetic field (which changes on time-scales which are, at most, a few minutes). Likewise, the gyroradii of such particles are much smaller than the typical variation length-scale of the magnetospheric magnetic field. Under these circumstances, we expect the magnetic moment to be a conserved quantity: i.e., we expect the magnetic moment to be a good adiabatic invariant. It immediately follows that any MeV energy protons and electrons in the inner magnetosphere which have a sufficiently large magnetic moment are trapped on the dipolar field-lines of the Earth's magnetic field, bouncing back and forth between mirror points located just above the Earth's poles–see Fig. 2.7.

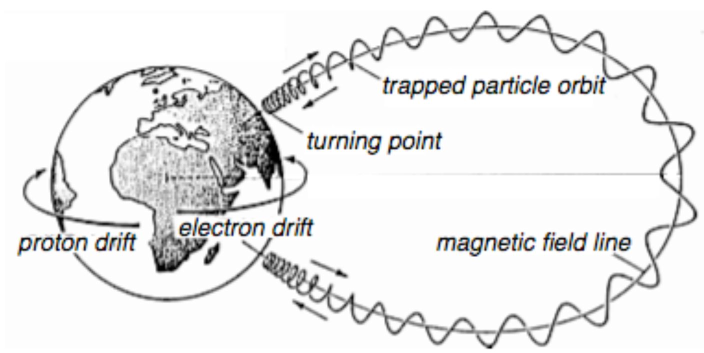

Figure 2.7: A typical trajectory of a charged particle trapped in the Earth's magnetic field

It is helpful to define the pitch-angle field

$$\alpha = \tan^{-1}(v_\perp/v_\parallel), \tag{2.63}$$

of a charged particle in the magnetosphere. If the magnetic moment is a conserved quantity then a particle of fixed energy drifting along a field-line satisfies

$$\frac{\sin^2\alpha}{\sin^2\alpha_{\rm eq}} = \frac{B}{B_{\rm eq}}, \tag{2.64}$$

where $\alpha_{\rm eq}$ is the equatorial pitch-angle (i.e., the pitch-angle on the equatorial plane) and $B_{\rm eq} = B_E/L^3$ is the magnetic field-strength on the equatorial plane. It is clear from Equ. (2.58) that the pitch-angle increases (i.e., the parallel component of the particle velocity decreases) as the particle drifts off the equatorial plane towards the Earth's poles.

The mirror points correspond to $\alpha = 90^\circ$ (i.e., $v_\parallel = 0$). It follows from Equs. (2.58) and (2.64) that

$$\sin^2\alpha_{\rm eq} = \frac{B_{\rm eq}}{B_m} = \frac{\cos^6\vartheta_m}{(1+3\,\sin^2\vartheta_m)^{1/2}}, \tag{2.65}$$

where B_m is the magnetic field-strength at the mirror points, and ϑ_m is the latitude of the mirror points. Clearly, the latitude of a particle's mirror point depends only on its equatorial pitch-angle, and is independent of the L-value of the field-line on which it is trapped.

Charged particles with large equatorial pitch-angles have small parallel velocities, and mirror points located at relatively low latitudes. Conversely, charged particles with small equatorial pitch-angles have large parallel velocities, and mirror points located at high latitudes. Of course, if the pitch-angle becomes too small then the mirror points enter the Earth's atmosphere, and the particles are lost via collisions with neutral particles. Neglecting the thickness of the atmosphere with respect to the radius of the Earth, we can say that all particles whose mirror points lie inside the Earth are lost via collisions. It follows from Equ. (2.65) that the equatorial loss cone is of approximate width

$$\sin^2\alpha_l = \frac{\cos^6\vartheta_E}{(1+3\sin^2\vartheta_E)^{1/2}}, \tag{2.66}$$

where ϑ_E is the latitude of the point where the magnetic field-line under investigation intersects the Earth. Note that all particles with $|\alpha_{\rm eq}| < \alpha_l$ and $|\pi - \alpha_{\rm eq}| < \alpha_l$ lie in the loss cone. It is easily demonstrated from Eq. (2.57) that

$$\cos^2\vartheta_E = L^{-1}. \tag{2.67}$$

It follows that

$$\sin^2\alpha_l = (4\,L^6 - 3\,L^5)^{-1/2}. \tag{2.68}$$

Thus, the width of the loss cone is independent of the charge, the mass, or the energy of the particles drifting along a given field-line, and is a function only of the field-line radius on the equatorial plane. The loss cone is surprisingly small. For instance, at the radius of a geostationary orbit ($6.6\,R_E$), the loss cone is less than 3° degrees wide. The smallness of the loss cone is a consequence of the very strong variation of the magnetic field-strength along field-lines in a dipole field–see Equ. (2.58).

A dipole field is clearly a far more effective configuration for confining a collisionless plasma via magnetic mirroring than the more traditional linear configuration shown in Fig. 2.6. In fact, M.I.T has recently constructed a dipole mirror machine. The dipole field is generated by a superconducting current loop levitating in a vacuum chamber.

The bounce period, τ_b, is the time it takes a particle to move from the equatorial plane to one mirror point, then to the other, and then return to the equatorial plane. It follows that

$$\tau_b = 4 \int_0^{\vartheta_m} \frac{d\vartheta}{v_\parallel} \frac{ds}{d\vartheta}, \tag{2.69}$$

where ds is an element of arc length along the field-line under investigation, and $v_\parallel = v\,[1 - B/B_m]^{1/2}$. The above integral cannot be performed analytically. However, it can be solved numerically, and is conveniently approximated as

$$\tau_b \simeq \frac{L\,R_E}{(\mathcal{E}/m)^{1/2}}\,(3.7 - 1.6 \sin\alpha_{\rm eq}). \tag{2.70}$$

Thus, for protons

$$(\tau_b)_p \simeq 2.41\,\frac{L}{\sqrt{\mathcal{E}(\mathrm{MeV})}}\,(1 - 0.43 \sin\alpha_{\rm eq})\ \ \text{secs}, \tag{2.71}$$

whilst for electrons

$$(\tau_b)_e \simeq 5.62\times 10^{-2}\,\frac{L}{\sqrt{\mathcal{E}(\mathrm{MeV})}}\,(1 - 0.43 \sin\alpha_{\rm eq})\ \ \text{secs}. \tag{2.72}$$

It follows that MeV electrons typically have bounce periods which are less than a second, whereas the bounce periods for MeV protons usually lie in the range 1 to 10 seconds. The bounce period only depends weakly on equatorial pitch-angle, since particles with small pitch angles have relatively large parallel velocities but a comparatively long way to travel to their mirror points, and vice versa. Naturally, the bounce period is longer for longer field-lines (i.e., for larger L).

2.2.6 The second adiabatic invariant

We have seen that there is an adiabatic invariant associated with the periodic gyration of a charged particle around magnetic field-lines. Thus, it is reasonable to suppose that there is a second adiabatic invariant associated with the periodic bouncing motion of a particle trapped between two mirror points on a magnetic field-line. This is indeed the case.

Recall that an adiabatic invariant is the lowest order approximation to a Poincaré invariant:

$$\mathcal{J} = \oint_C \mathbf{p} \cdot d\mathbf{q}. \tag{2.73}$$

In this case, let the curve C correspond to the trajectory of a guiding centre as a charged particle trapped in the Earth's magnetic field executes a bounce orbit. Of course, this trajectory does not quite close, because of the slow azimuthal drift of particles around the Earth. However, it is easily demonstrated that the azimuthal displacement of the end point of the trajectory, with respect to the beginning point, is of order the gyroradius. Thus, in the limit in which the ratio of the gyroradius, ρ, to the variation length-scale of the magnetic field, L, tends to zero, the trajectory of the guiding centre can be regarded as being approximately closed, and the actual particle trajectory conforms very closely to that of the guiding centre. Thus, the adiabatic invariant associated with the bounce motion can be written

$$\mathcal{J} \simeq J = \oint p_\parallel \, ds, \tag{2.74}$$

where the path of integration is along a field-line: from the equator to the upper mirror point, back along the field-line to the lower mirror point, and then back to the equator. Furthermore, ds is an element of arc-length along the field-line, and $p_\parallel \equiv \mathbf{p} \cdot \mathbf{b}$. Using $\mathbf{p} = m\,\mathbf{v} + e\,\mathbf{A}$, the above expression yields

$$J = m \oint v_\parallel \, ds + e \oint \mathbf{A}_\parallel \, ds = m \oint v_\parallel \, ds + e\,\Phi. \tag{2.75}$$

Here, Φ is the total magnetic flux enclosed by the curve–which, in this case, is obviously zero. Thus, the so-called second adiabatic invariant or longitudinal adiabatic invariant takes the form

$$J = m \oint v_\parallel \, ds. \tag{2.76}$$

In other words, the second invariant is proportional to the loop integral of the parallel (to the magnetic field) velocity taken over a bounce orbit. Actually, the above "proof" is not particularly rigorous: the rigorous proof that J is an adiabatic invariant was first given by Northrop and Teller. It should be noted, of course, that J is only a constant of the motion for particles trapped in the inner magnetosphere provided that the magnetospheric magnetic field varies on time-scales much longer than the bounce time, τ_b. Since the bounce time for MeV energy protons and electrons is, at most, a few seconds, this is not a particularly onerous constraint.

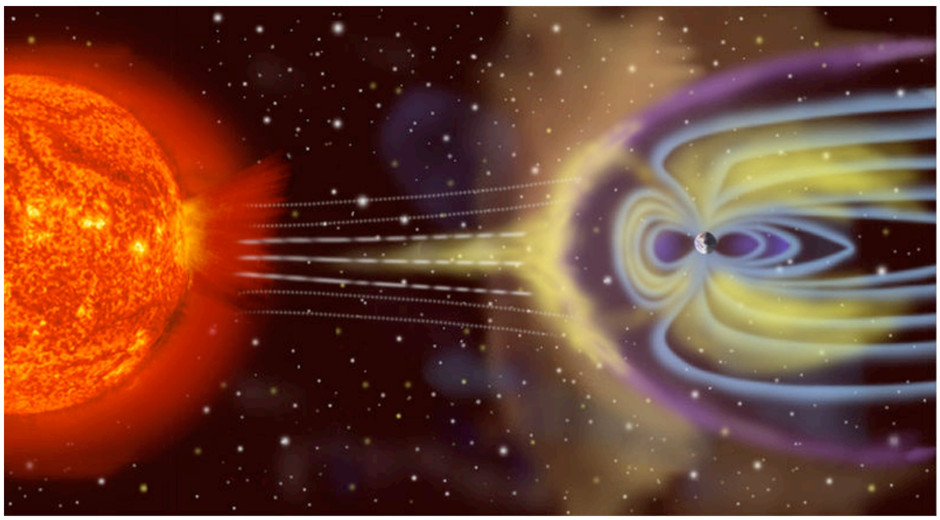

Figure 2.8: The distortion of the Earth's magnetic field by the solar wind

The invariance of J is of great importance for charged particle dynamics in the Earth's inner magnetosphere. It turns out that the Earth's magnetic field is distorted from pure axisymmetry by the action of the solar wind, as illustrated in Fig. 2.8. Because of this asymmetry, there is no particular reason to believe that a particle will return to its earlier trajectory as it makes a full rotation around the Earth. In other words, the particle may well end up on a different field-line when it returns to the same azimuthal angle. However, at a given azimuthal angle, each field-line has a different length between mirror points, and a different variation of the field-strength B between the mirror points, for a particle with given energy $\mathcal{E}$ and magnetic moment μ. Thus, each field-line represents a different value of J for that particle. So, if J is conserved, as well as $\mathcal{E}$ and μ, then the particle must return to the same field-line after precessing around the Earth. In other words, the conservation of J prevents charged particles from spiraling radially in or out of the Van Allen belts as they rotate around the Earth. This helps to explain the persistence of these belts.

2.2.7 The third adiabatic invariant

It is clear, by now, that there is an adiabatic invariant associated with every periodic motion of a charged particle in an electromagnetic field. Now, we have just demonstrated that, as a consequence of J-conservation, the drift orbit of a charged particle precessing around the Earth is approximately closed, despite the fact that the Earth's magnetic field is non-axisymmetric. Thus, there must be a third adiabatic invariant associated with the precession of particles around the Earth. Just as we can define a guiding centre associated with a particle's gyromotion around field-lines, we can also define a bounce centre associated with a particle's bouncing motion between mirror points. The bounce centre lies on the equatorial plane, and orbits the Earth once every drift period, τ_d. We can write the third adiabatic invariant as

$$K \simeq \oint p_\phi \, ds, \tag{2.77}$$

where the path of integration is the trajectory of the bounce centre around the Earth. Note that the drift trajectory effectively collapses onto the trajectory of the bounce centre in the limit in which $\rho/L \to 0$–all of the particle's gyromotion and bounce motion averages to zero. Now $p_\phi = m\, v_\phi + e\, A_\phi$ is dominated by its second term, since the drift velocity v_ϕ is very small. Thus,

$$K \simeq e \oint A_\phi \, ds = e\, \Phi, \tag{2.78}$$

where Φ is the total magnetic flux enclosed by the drift trajectory (i.e., the flux enclosed by the orbit of the bounce centre around the Earth). The above "proof" is, again, not particularly rigorous–the invariance of Φ is demonstrated rigorously by Northrup. Note, of course, that Φ is only a constant of the motion for particles trapped in the inner magnetosphere provided that the magnetospheric magnetic field varies on time-scales much longer than the drift period, τ_d. Since the drift period for MeV energy protons and electrons is of order an hour, this is only likely to be the case when the magnetosphere is relatively quiescent (i.e., when there are no geomagnetic storms in progress).

The invariance of Φ has interesting consequences for charged particle dynamics in the Earth's inner magnetosphere. Suppose, for instance, that the strength of the solar wind were to increase slowly (i.e., on time-scales significantly longer than the drift period), thereby, compressing the Earth's magnetic field. The invariance of Φ would cause the charged particles which constitute the Van Allen belts to move radially inwards, towards the Earth, in order to conserve the magnetic flux enclosed by their drift orbits. Likewise, a slow decrease in the strength of the solar wind would cause an outward radial motion of the Van Allen belts.

2.3 Motion in time dependent fields

In this section we will briefly discuss time dependent fields. You saw already in the previous sections, cases with non constant fields have to be solved in general via numeric integration of the equation of motion. It is not surprising that the same goes for time depending fields, especially when we consider arbitrary functions in time. Nevertheless in most of the practical situations the electron-magnetic fields are periodic and can be described via the complex notation $E = E_0 \exp(-i\omega t)$. This will be important in the following sections, especially when we will describe waves in plasmas. We start in this section with E(t) followed by B(t). After that we will give an overview over particles in periodic electro-magnetic fields.

Exercise: Refresh your knowledge about the complex notation of periodic fields

2.3.1 Motion in time varying E-field

Recall the E-cross-B drift

$$\mathbf{v}_d = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.79}$$

when $\mathbf{E}$ varies so does $\mathbf{v_d}$. Thus the guiding centre experiences an acceleration according to

$$\dot{\mathbf{v}}_d = \frac{d}{dt}\frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.80}$$

This acceleration leads to the force which is felt in the frame of the guiding centre as

$$\mathbf{F}_a = -m\frac{d}{dt}\frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2.81}$$

We saw in one of the previous sections how an additional force (similar to the gravitation) produces an additional drift:

$$
\begin{aligned}
\mathbf{v}_a &= \frac{1}{q}\frac{\mathbf{F}_a \times \mathbf{B}}{B^2} = -\frac{m}{qB^2}\frac{d}{dt}\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right)\times\mathbf{B} \\
&= -\frac{m}{qB^2}\frac{d}{dt}\left(\frac{(\mathbf{E}\cdot\mathbf{B})\mathbf{B} - B^2\mathbf{E}}{B^2}\right) \\
&= \frac{m}{qB^2}\dot{\mathbf{E}}_\perp = \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp
\end{aligned}
\tag{2.82}
$$

This it called the *polarization drift*. The total drift in this case is

$$
\mathbf{v}_d = \frac{\mathbf{E}\times\mathbf{B}}{B^2} + \frac{1}{\Omega B}\dot{\mathbf{E}}_\perp \tag{2.83}
$$

Consider a periodic **E**-field with frequency ω, then $\dot{\mathbf{E}}_\perp \propto \omega\mathbf{E}_\perp$. Again our calculation here is only valid for $\omega \ll \Omega$

2.3.2 Motion in time varying B-field

In this section we consider a time varying B-field which leads to the inductive generation of an E-field via

$$
\nabla \times \mathbf{E} = -\partial_t \mathbf{B} \tag{2.84}
$$

$$
\text{or} \oint \mathbf{E}\cdot d\mathbf{l} = -\int_S \dot{\mathbf{B}}\cdot d\mathbf{l} = -\dot{\Phi} \tag{2.85}
$$

Hence the work done on a particle during one revolution is

$$
\delta w = -\oint |q|\mathbf{E}\cdot d\mathbf{l} = |q|\int_S \dot{\mathbf{B}}\cdot d\mathbf{l} = -|q|\dot{\Phi} = |q|\dot{B}\pi\rho^2 = \frac{2\pi\dot{B}}{\Omega}\mu \tag{2.86}
$$

Hence

$$
\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{\Omega}{2\pi}\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\dot{B} \tag{2.87}
$$

but on the other hand the change of the kinetic energy in terms of the magnetic moment and the magnetic field is

$$
\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}(\mu B) = \mu\dot{B} + B\dot{\mu} \tag{2.88}
$$

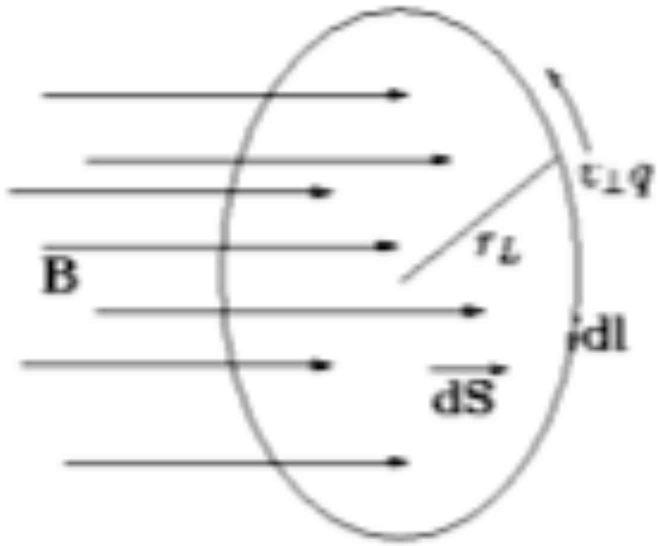

Figure 2.9: Particle orbits round **B** so as to perform a line integral of the electric field

so again also in the case of a time varying B-field that

$$\frac{d\mu}{dt} = 0 \tag{2.89}$$

Notice that since the magnetic flux $\Phi = 2\pi m\mu/q^2$, this is just another way of saying that the flux through the gyro orbit is conserved.

Discussion task 7: What happens if the frequency ω of an oscillating B-field is much larger that the Gyrofrequency Ω?

2.3.3 Motion in oscillating fields

We have seen that charged particles can be confined by a static magnetic field. A somewhat more surprising fact is that charged particles can also be confined by a rapidly oscillating, inhomogeneous electromagnetic wave-field. In order to demonstrate this, we again make use of our averaging technique. To lowest order, a particle executes simple harmonic motion in response to an oscillating wave-field. However, to higher order, any weak inhomogeneity in the field causes the restoring force at one turning point to exceed that at the other. On average, this yields a net force which acts on the centre of oscillation of the particle.

Consider a spatially inhomogeneous electromagnetic wave-field oscillating at frequency ω:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r})\,\cos\omega t. \tag{2.90}$$

The equation of motion of a charged particle placed in this field is written

$$m\,\frac{d\mathbf{v}}{dt} = e\,\left[\mathbf{E}_0(\mathbf{r})\,\cos\omega t + \mathbf{v}\times\mathbf{B}_0(\mathbf{r})\,\sin\omega t\right], \tag{2.91}$$

where

$$\mathbf{B}_0 = -\omega^{-1}\,\nabla\times\mathbf{E}_0, \tag{2.92}$$

according to Faraday's law.

In order for our averaging technique to be applicable, the electric field $\mathbf{E}_0$ experienced by the particle must remain approximately constant during an oscillation. Thus,

$$(\mathbf{v}\cdot\nabla)\,\mathbf{E} \ll \omega\,\mathbf{E}. \tag{2.93}$$

When this inequality is satisfied, Eq. (2.92) implies that the magnetic force experienced by the particle is smaller than the electric force by one order in the expansion parameter. In fact, Eq. (2.93) is equivalent to the requirement, $\Omega \ll \omega$, that the particle be unmagnetized.

We now apply the averaging technique. We make the substitution $t \to \tau$ in the oscillatory terms, and seek a change of variables,

$$\mathbf{r} = \mathbf{R} + \boldsymbol{\xi}(\mathbf{R},\mathbf{U}\,t,\tau), \tag{2.94}$$

$$\mathbf{v} = \mathbf{U} + \mathbf{u}(\mathbf{R},\mathbf{U}\,t,\tau), \tag{2.95}$$

such that $\boldsymbol{\xi}$ and $\mathbf{u}$ are periodic functions of τ with vanishing mean. Averaging $d\mathbf{r}/dt = \mathbf{v}$ again yields $d\mathbf{R}/dt = \mathbf{U}$ to all orders. To lowest order, the momentum evolution equation reduces to

$$\frac{\partial\mathbf{u}}{\partial\tau} = \frac{e}{m}\,\mathbf{E}_0(\mathbf{R})\,\cos\omega\tau. \tag{2.96}$$

The solution, taking into account the constraints $\langle\mathbf{u}\rangle = \langle\boldsymbol{\xi}\rangle = \mathbf{0}$, is

$$\mathbf{u} = \frac{e}{m\,\omega}\,\mathbf{E}_0\,\sin\omega\tau, \tag{2.97}$$

$$\boldsymbol{\xi} = -\frac{e}{m\,\omega^2}\,\mathbf{E}_0\,\cos\omega\tau. \tag{2.98}$$

Here, $\langle\cdots\rangle \equiv (2\pi)^{-2}\oint(\cdots)\,d(\omega\tau)$ represents an oscillation average.

Clearly, there is no motion of the centre of oscillation to lowest order. To first order, the oscillation average of Eq. (2.91) yields

$$\frac{d\mathbf{U}}{dt} = \frac{e}{m}\,\langle(\boldsymbol{\xi}\cdot\nabla)\mathbf{E} + \mathbf{u}\times\mathbf{B}\rangle\,, \tag{2.99}$$

which reduces to

$$\frac{d\mathbf{U}}{dt} = -\frac{e^2}{m^2\,\omega^2}\left[(\mathbf{E}_0\cdot\nabla)\mathbf{E}_0\,\langle\cos^2\omega\tau\rangle + \mathbf{E}_0\times(\nabla\times\mathbf{E}_0)\,\langle\sin^2\omega\tau\rangle\right]. \tag{2.100}$$

The oscillation averages of the trigonometric functions are both equal to 1/2. Furthermore, we have $\nabla(|\mathbf{E}_0|^2/2) \equiv (\mathbf{E}_0\cdot\nabla)\mathbf{E}_0 + \mathbf{E}_0\times(\nabla\times\mathbf{E}_0)$. Thus, the equation of motion for the centre of oscillation reduces to

$$m\,\frac{d\mathbf{U}}{dt} = -e\,\nabla\Phi_{\rm pond}, \tag{2.101}$$

where

$$\Phi_{\rm pond} = \frac{1}{4}\frac{e}{m\,\omega^2}\,|\mathbf{E}_0|^2. \tag{2.102}$$

It is clear that the oscillation centre experiences a force, called the ponderomotive force, which is proportional to the gradient in the amplitude of the wave-field. The ponderomotive force is independent of the sign of the charge, so both electrons and ions can be confined in the same potential well.

The total energy of the oscillation centre,

$$\mathcal{E}_{\rm oc} = \frac{m}{2}\,U^2 + e\,\Phi_{\rm pond}, \tag{2.103}$$

is conserved by the equation of motion (2.100). Note that the ponderomotive potential energy is equal to the average kinetic energy of the oscillatory motion:

$$e\,\Phi_{\rm pond} = \frac{m}{2}\,\langle u^2\rangle. \tag{2.104}$$

Thus, the force on the centre of oscillation originates in a transfer of energy from the oscillatory motion to the average motion.

Most of the important applications of the ponderomotive force occur in laser plasma physics. For instance, a laser beam can propagate in a plasma provided that its frequency exceeds the plasma frequency. If the beam is sufficiently intense then plasma particles are repulsed from the centre of the beam by the ponderomotive force. The resulting variation in the plasma density gives rise to a cylindrical well in the index of refraction which acts as a wave-guide for the laser beam.