Chapter 2

Charged particle motion

Plasmas are complicated because motions of electrons and ions are determined by the electric and magnetic fields but also change the fields by the currents they carry. As we already mentioned (see Equ. (1.4) the fundamental equation of motion of an individual particle takes the form

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.1}$$

In this section we shall ignore the back-reaction of the particles and assume that fields are prescribed, e.g. we forget for a moment that the particles are itself parts of the plasma and hence responsible for the generation and modification of the fields. Even so, calculating the motion of a charged particle can be quite hard. We will first of all consider the motion of charged particles in spatially and temporally uniform electromagnetic fields, followed by spatially varying field. At the end of this chapter we will study briefly time varying fields.

2.1 Motion in uniform fields

2.1.1 E=const, B=0

In this easiest case the Lorentz force is reduced to:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{E} \tag{2.2}$$

We will set the the x-coordinate in the direction of the electric field. This simple case has some traps tough: if we would simply assume:

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = m_e \frac{\mathrm{d}v_x}{\mathrm{d}t} = qE_x \tag{2.3}$$

we would get after the integration:

$$v_x = \frac{e}{m_e} E_x t \tag{2.4}$$

with would lead to $v_x \to \infty$ for $t \to \infty$, which is of course forbidden by the special theory of relativity. To solve this problem correctly we have to include the change of the mass according to $m_e = m_0\gamma$ with $\gamma = 1/\sqrt{1-(v/c_0)^2}$. So we have to solve this equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{v_x}{\sqrt{1-(v_x/c_0)^2}} = \frac{q}{m_0} E_x \tag{2.5}$$

which is still straight forward to integrate. After some rearrangements we get the correct velocity as:

$$v_x = \frac{e}{m_0} E_x t \frac{1}{\sqrt{1+\left(\frac{eE_x t}{m_0 c_0}\right)^2}} \tag{2.6}$$

Figure (2.1) shows the importance of the correct mass description of an electron in a field of 100kV/m. Of course for $t \ll m_0 c_0/eE_x$ the velocity can be approximated by Equ. (2.4).

Discussion task 5: According to special relativity the kinetic energy of a particle is $E_{kin} = m_0(\gamma - 1)c^2$, but almost always the kinetic energy is calculated as: $E_{kin} = mv^2/2$. How is this contradiction solved?
Assignment task 6: Prove that Eq. (2.6) can be approximated by Eq. (2.4) for $t \ll m_0 c_0/eE_x$

2.1.2 E=0, B=const

The next case is a constant **B**-field which we define as a field in z-direction. The equation of motion is here reduced to

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} \tag{2.7}$$

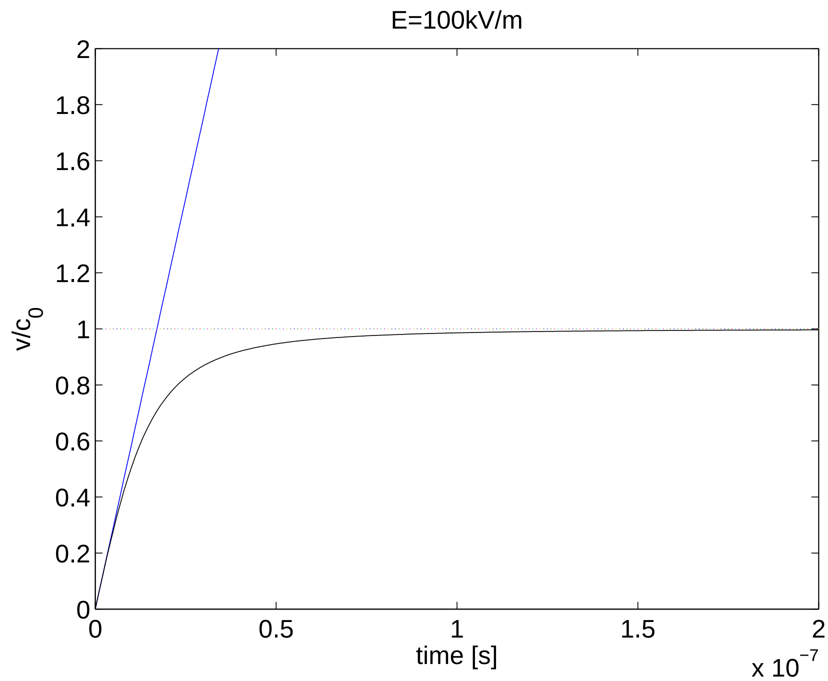

Figure 2.1: Velocity for a electron in a constant E-field of 100kV/m. Blue without relativistic mass correction. Black: correct description

or for the the components of the momentum:

$$\begin{aligned} \dot{p}_x &= q v_y B_z \\ \dot{p}_y &= -q v_x B_z \\ \dot{p}_x &= 0 \end{aligned} \tag{2.8}$$

where the dot represents the time derivative. First of all we see that there is no acceleration in the direction of the **B**-field. For the further analysis of this problem we assume $v \ll c_0$. After performing a second time derivative for e.g. the x components we can substitute the y components and we get for v_x:

$$\ddot{v}_x = \frac{q}{m_e}\dot{v}_y B_z = -\left(\frac{qB_z}{m_e}\right)^2 v_x \tag{2.9}$$

which is the well know equation for an harmonic oscillator with the characteristic frequency $\Omega = |q|B_z/m_e$ and the general solution:

$$v_x = v_0 \cos(\Omega t) + v_1 \sin(\Omega t) \tag{2.10}$$

with the two constants v_0 and v_1 terminated by our choice of the initial velocity for $t = 0$ as $v(t = 0) = v_0\mathbf{e_x} + 0\mathbf{e_y} + 0\mathbf{e_z}$, leading to

$$v_x = v_0 \cos(\Omega t) \tag{2.11}$$

for the x component of the velocity. Inserting this in Equ. (2.8) leads to

$$\dot{v}_y = -\frac{q}{|q|}\Omega v_0 \cos(\Omega t) \tag{2.12}$$

which gives

$$v_y = -\frac{q}{|q|}v_0 \sin(\Omega t) \tag{2.13}$$

where we used again our choice of the initial velocity. Equ. (2.11) and (2.13) describe the velocity of a charged particle in a constant magnetic field. An initially present velocity v_z is not modified by a magnetic field parallel to this velocity component. So in general we can write:

$$\mathbf{v} = v_{\perp 0} \cos(\Omega t)\mathbf{e_x} - \frac{q}{|q|}v_{\perp 0} \sin(\Omega t)\mathbf{e_y} + v_z\mathbf{e_z} \tag{2.14}$$

To get the trajectory of the particle we integrate Equ. (2.14) resulting in

$$\mathbf{r} = \mathbf{r_0} + \frac{v_{\perp 0}}{\Omega} \sin(\Omega t)\mathbf{e_x} + \frac{q}{|q|}\frac{v_{\perp 0}}{\Omega} \cos(\Omega t)\mathbf{e_y} + v_z t\mathbf{e_z} \tag{2.15}$$

This is a circular trajectory with radius $\rho = v_{\perp 0}/\Omega$., which is referred to as *Gyroradius*, and Ω is known as the *Gyrofrequency*. Equ. (2.15) shows that the sign of the charge defines the direction of the rotation. Ions rotate anticlockwise and electrons clockwise about the magnetic field (see figure 2.2). Note that a particle gyrating as described here produces a magnetic field counteracting the external field resulting in a reduction of the total field. This is the property of a magnetic material which is *Diagmagnetic*.

Discussion task 6: Calculate the energy gain of a charged particle in a constant magnetic field

2.1.3 E=const, B=const

When both electric and magnetic fields are present the motion of a charged particle it the superposition of the acceleration in the **E** direction and a

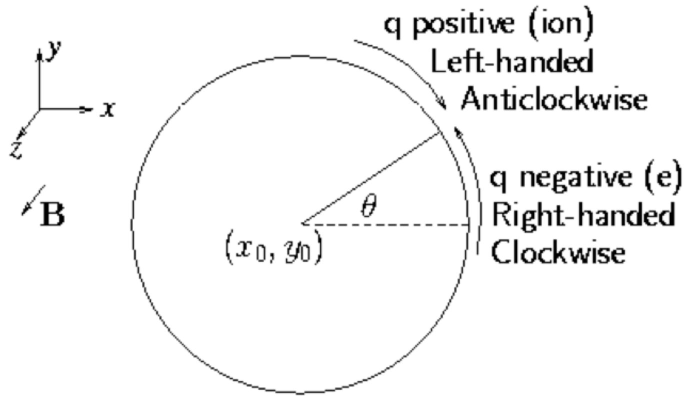

Figure 2.2: Gyro centre (x_0, y_0 and orbit)

circular motion perpendicular to the **B** direction. It is important for the further analysis to split the the electric field into a component parallel to the magnetic field $E_{\|}$ and a component perpendicular to the magnetic field $\mathbf{E}_{\perp}$. As we saw in the last section the velocity component of a particle in the direction of the magnetic field is not affected by it. So there is just $E_{\|}$ to change the velocity, as we described in the first case. For the remaining perpendicular field we will solve this problem with a common trick, by finding a coordinate system in which $\mathbf{E}_{\perp} = 0$. We restrict ourselves again to the non-relativistic case where the **E** field is transformed to a moving coordinate system as $\mathbf{E}' = \mathbf{E} + \mathbf{v_d} \times \mathbf{B}$. The goal is now to find this velocity $\mathbf{v_d}$. We multiply this equation with $\times\mathbf{B}$ to get

$$0 = \mathbf{E}_{\perp} \times \mathbf{B} + (\mathbf{v_d} \times \mathbf{B}) \times \mathbf{B} = \mathbf{E}_{\perp} \times \mathbf{B} + (\mathbf{v_d} \cdot \mathbf{B})\mathbf{B} - B^2 \mathbf{v}_d \qquad (2.16)$$

We can solve this equation only if we set $\mathbf{v}_d$ perpendicular to **B**. With that assumption we get:

$$\mathbf{v}_d = \frac{\mathbf{E}_{\perp} \times \mathbf{B}}{B^2}. \qquad (2.17)$$

This drift, which is termed the E-cross-B drift in plasma physics, is identical for all plasma species. Inside this frame $\mathbf{E}_{\perp} = \mathbf{0}$, so this frame can properly be regarded as the rest frame of the plasma. This also so called *guiding centre* is an important concept in analyzing complicated particle motions. Here the advantage of this description is obvious: since the electric field is zero the particle gyrates around the magnetic field at frequency Ω exactly in the same way as described above for the E=0; B=const case.

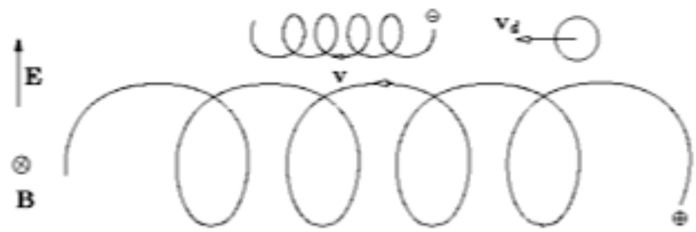

Figure 2.3: $\mathbf{E} \times \mathbf{B}$ drift orbit

Hence the full solution for the particle trajectory is:

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_d + \mathbf{v}_{Gyration} \tag{2.18}$$

This separation gives us a clue to simplify the description for some cases. Sometimes when analyzing charged particle motion in non-uniform electromagnetic fields, we can somehow neglect the rapid, and relatively uninteresting, gyromotion, and focus, instead, on the far slower motion of the guiding centre. Clearly, what we need to do in order to achieve this goal is to somehow average the equation of motion over gyrophase, so as to obtain a reduced equation of motion for the guiding centre. This method was introduced by Hans Alfén and in known as *guiding centre approximation*

2.1.4 Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity. Write it $\mathbf{F}$ so that

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} + q\mathbf{v} \times \mathbf{B} = q\left(\frac{1}{q}\mathbf{F} + \mathbf{v} \times \mathbf{B}\right) \tag{2.19}$$

This is just like the previous case except with $\mathbf{F}$/q replacing $\mathbf{E}$. The drift is therefore

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}. \tag{2.20}$$

In this case, if force on electrons and ions is same, they drift in opposite directions. This general formula can be used to get the drift velocity in some other cases of interest.

2.2 Motion in nonuniform fields

In the case of nonuniform, inhomogeneous and/or time dependent electromagnetic field the equation of motion becomes nonlinear and can be solved in general only by numeric integration. However in some cases we can use the guiding centre approximation to find reasonable solutions. As mentioned before we can use this approximation if the spacial inhomogeneity is so small or the time dependence of the fields is so slow, that during one gyro period the fields can be approximately treated as constant. This is in most laboratory plasmas possible, but only seldom in interstellar plasmas.

2.2.1 E=0, B=Non-Uniform

Lets assume that the magnetic field varies only along one spacial coordinate. Then we get orbits that look qualitatively similar to the $\mathbf{E} \perp \mathbf{B}$

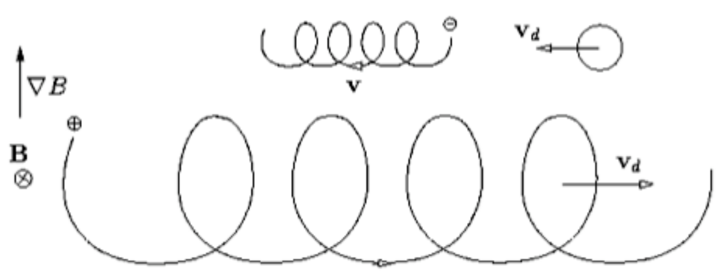

Figure 2.4: $\nabla\mathbf{B}$-Drift

Curvature of orbit is greater where $\mathbf{B}$ is greater causing loop to be small on that side. Result is a drift perpendicular to both $\mathbf{B}$ and $\nabla\mathbf{B}$ Notice, though, that electrons and ions go in opposite directions (unlike the $\mathbf{E} \times \mathbf{B}$ case). We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v_d} + \mathbf{v_L}$ where $\mathbf{v_d}$ is constant. We shall find that this can be done only approximately, by assuming that the the velocity is small compared to c_0 and the field gradient is small compared to the gyroradius ρ. i.e.,

$$\rho \ll B/|\nabla B| \tag{2.21}$$

in which case we can express the field approximately as the first two terms in a Taylor expression:

$$\mathbf{B} \approx \mathbf{B}_0 + (\mathbf{r} \cdot \nabla)\mathbf{B} \tag{2.22}$$

Then substituting the decomposed velocity we get:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{v}_L}{\mathrm{d}t} = q\mathbf{v} \times \mathbf{B} = q\left(\mathbf{v}_L \times \mathbf{B}_0 + \mathbf{v}_d \times \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \times (\mathbf{r} \cdot \nabla)\mathbf{B}\right) \tag{2.23}$$

or

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{v}_d \times (\mathbf{r} \cdot \nabla)\mathbf{B} \tag{2.24}$$

Keep in mind that $\mathbf{v}_d/\mathbf{v}_L \ll 1$, like $r|\nabla B|/B \ll 1$ Therefore the last term here is much smaller than the first two and can be dropped (e.g. the last term is of second order, whereas the first two are of first order). The problem here is that $\mathbf{v}_L$ and $\mathbf{r}_L$ are periodic. Similar to the velocity, we substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so we get

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} + \mathbf{v}_{\mathbf{L}d} \times (\mathbf{r_0} \cdot \nabla)\mathbf{B} \tag{2.25}$$

We now average over a cyclotron period Ω. The last term is $\propto \exp(i\Omega t)$ so it averages to zero. So this it the remaining equation we have to solve:

$$0 = \mathbf{v}_d \times \mathbf{B}_0 + \langle \mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B} \rangle \tag{2.26}$$

To perform the time average denote here with the brackets $\langle ... \rangle$ we use

$$\begin{aligned}
\mathbf{r}_L = \binom{x_L}{y_L} &= \frac{v_\perp}{\Omega}\binom{\sin(\Omega t)}{\frac{q}{|q|}\cos(\Omega t)} \\
\mathbf{v}_L = \binom{v_{xL}}{v_{yL}} &= v_\perp \binom{\cos(\Omega t)}{-\frac{q}{|q|}\sin(\Omega t)} \\
\text{So } [\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]_x &= v_y y \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}y} \\
[\mathbf{v}_L \times (\mathbf{r}_L \cdot \nabla)\mathbf{B}]_y &= -v_x y \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}y}
\end{aligned} \tag{2.27}$$

(Taking $\nabla\mathbf{B}$ to be in the y-direction). Then

$$\begin{aligned}\langle v_y y\rangle &= -\langle\cos\Omega t\sin\Omega t\rangle\frac{v_\perp^2}{\Omega} = 0 \\ \langle v_x y\rangle &= \langle\cos^2\Omega t\rangle\frac{v_\perp^2 q}{\Omega|q|} = \frac{v_\perp^2 q}{2\Omega|q|}\end{aligned} \tag{2.28}$$

So

$$\langle\mathbf{v}_L\times(\mathbf{r}_L\cdot\nabla)\mathbf{B}\rangle = -\frac{v_\perp^2 q}{2\Omega|q|}\nabla\mathbf{B} \tag{2.29}$$

Substitute in the remaining equation we had to solve:

$$0 = \mathbf{v}_d\times\mathbf{B}_0 - \frac{v_\perp^2 q}{2\Omega|q|}\nabla\mathbf{B} \tag{2.30}$$

and solve as before to get

$$\mathbf{v}_d = \frac{\left(-\frac{v_\perp^2}{2\Omega|q|}\nabla\mathbf{B}\right)\times\mathbf{B}}{B^2} = \frac{v_\perp^2 q}{2\Omega|q|}\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2} \tag{2.31}$$

or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B}\times\nabla\mathbf{B}}{B^2} \tag{2.32}$$

This is called the "Grad B drift".

2.2.2 E=0, B $\parallel$ ∇B; The Mirror Effect of Parallel Field Gradients

In the situation outlined in Figure 2.5 we have a magnetic field which increases in the direction of the field lines. Again we are only interested in the average movements of the particles and not on the detailed gyration. There is a net force on average along **B** which is.

$$\begin{aligned}\langle F_\parallel\rangle &= -|q\mathbf{v}\times\mathbf{B}|\sin\alpha = -|q|v_\perp B\sin\alpha \\ \text{with}\sin\alpha &= -B_r/B\end{aligned} \tag{2.33}$$

To calculate B_r as function of B_z we use Maxwell's Equation $\nabla\cdot\mathbf{B} = 0$. We assume here rotation symmetry along the z-axis as well as that the field