Chapter 2

Motion of Charged Particles in Fields

Plasmas are complicated because motions of electrons and ions are determined by the electric and magnetic fields but *also change* the fields by the currents they carry.

For now we shall ignore the second part of the problem and assume that *Fields are Prescribed.*

Even so, calculating the motion of a charged particle can be quite hard.

Equation of motion:

$$\underbrace{m\frac{d\mathbf{v}}{dt}}_{\text{Rate of change of momentum}} = \underbrace{\underset{\text{charge}}{q}\,(\underset{\text{E-field}}{\mathbf{E}} + \underset{\text{velocity}}{\mathbf{v}} \wedge \underset{\text{B-field}}{\mathbf{B}})}_{\text{Lorentz Force}} \tag{2.1}$$

Have to solve this differential equation, to get position $\mathbf{r}$ and velocity ($\mathbf{v}= \dot{\mathbf{r}}$) given $\mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)$.

Approach: Start simple, gradually generalize.

2.1 Uniform B field, $\mathbf{E} = 0$.

$$m\dot{\mathbf{v}} = q\mathbf{v} \wedge \mathbf{B} \tag{2.2}$$

2.1.1 Qualitatively

in the plane perpendicular to B: Accel. is perp to $\mathbf{v}$ so particle moves in a circle whose radius r_L is such as to satisfy

$$m r_L \Omega^2 = m\frac{v_\perp^2}{r_L} = |q| v_\perp B \tag{2.3}$$

Ω is the angular (velocity) frequency

1st equality shows $\Omega^2 = v_\perp^2 / r_L^2 \quad (r_L = v_\perp/\Omega)$

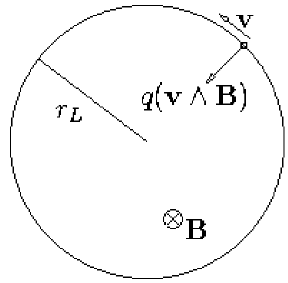

Figure 2.1: Circular orbit in uniform magnetic field.

Hence second gives $m\frac{v_\perp}{\Omega}\Omega^2 = |q|v_\perp B$

$$\text{i.e.} \quad \Omega = \frac{|\text{q}|\text{B}}{\text{m}} \quad . \tag{2.4}$$

Particle moves in a circular orbit with

$$\text{angular velocity} \quad \Omega = \frac{|q|B}{m} \quad \text{the “Cyclotron Frequency”} \tag{2.5}$$

$$\text{and radius} \quad r_l = \frac{v_\perp}{\Omega} \quad \text{the “Larmor Radius.} \tag{2.6}$$

2.1.2 By Vector Algebra

- Particle Energy is constant. *proof*: take $\mathbf{v}$. Eq. of motion then

$$m\mathbf{v}.\dot{\mathbf{v}} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q\mathbf{v}.(\mathbf{v}\wedge\mathbf{B}) = 0. \tag{2.7}$$

- Parallel and Perpendicular motions separate. $v_\parallel$ = constant because accel ($\propto \mathbf{v}\wedge\mathbf{B}$) is perpendicular to $\mathbf{B}$.

Perpendicular Dynamics:

Take B in $\hat{z}$ direction and write components

$$m\dot{v}_x = qv_yB \quad , \quad m\dot{v}_y = -qv_xB \tag{2.8}$$

Hence

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x = -\Omega^2 v_x \tag{2.9}$$

Solution: $v_x = v_\perp \cos\Omega t$ (choose zero of time)

Substitute back:

$$v_y = \frac{m}{qB}\dot{v}_x = -\frac{|q|}{q}v_\perp \sin\Omega t \tag{2.10}$$

Integrate:

$$x = x_0 + \frac{v_\perp}{\Omega} \sin \Omega t \quad , \quad y \ = \ y_0 + \frac{q}{|q|} \frac{v_\perp}{\Omega} \ \cos \Omega t \tag{2.11}$$

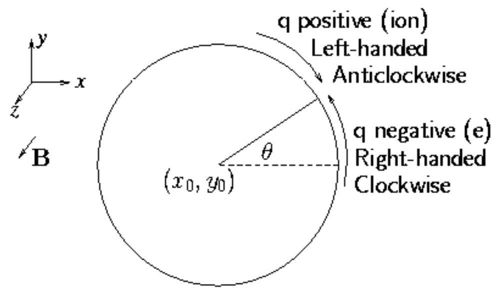

Figure 2.2: Gyro center (x_0, y_0) and orbit

This is the equation of a circle with center $\mathbf{r}_0 = (x_0, y_0)$ and radius $r_L = v_\perp/\Omega$: Gyro Radius. [Angle is $\theta = \Omega t$]

Direction of rotation is as indicated opposite for opposite sign of charge:

Ions rotate anticlockwise. Electrons clockwise about the magnetic field.

The current carried by the plasma always is in such a direction as to *reduce* the magnetic field.

This is the property of a magnetic material which is *"Diagmagnetic"*.

When $v_\parallel$ is non-zero the total motion is along a helix.

2.2 Uniform B and non-zero E

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}) \tag{2.12}$$

Parallel motion: Before, when $\mathbf{E} = 0$ this was $v_\parallel$ = const. Now it is clearly

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.13}$$

Constant acceleration along the field.

Perpendicular Motion

Qualitatively:

Speed of positive particle is greater at top than bottom so radius of curvature is greater. Result is that guiding center moves perpendicular to both $\mathbf{E}$ and $\mathbf{B}$. It 'drifts' across the field.

Algebraically: It is clear that if we can find a constant velocity $\mathbf{v}_d$ that satisfies

$$\mathbf{E} + \mathbf{v}_d \wedge \mathbf{B} = 0 \tag{2.14}$$

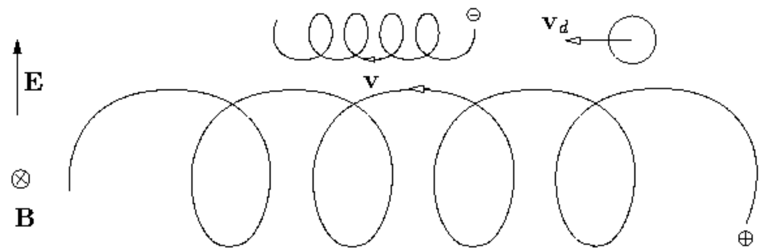

Figure 2.3: $\mathbf{E} \wedge \mathbf{B}$ drift orbit

then the sum of this drift velocity plus the velocity

$$\mathbf{v}_L = \frac{d}{dt}[\mathbf{r}_L e^{i\Omega(t-t_0)}] \tag{2.15}$$

which we calculated for the $\mathbf{E} = 0$ gyration will satisfy the equation of motion.
Take $\wedge\mathbf{B}$ the above equation:

$$0 = \mathbf{E} \wedge \mathbf{B} + (\mathbf{v}_d \wedge \mathbf{B}) \wedge \mathbf{B} = \mathbf{E} \wedge \mathbf{B} + (v_d.\mathbf{B})\mathbf{B} - B^2\mathbf{v}_d \tag{2.16}$$

so that

$$\mathbf{v}_d = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{2.17}$$

does satisfy it.
Hence the full solution is

$$\mathbf{v} = \underset{\text{parallel}}{\mathbf{v}_\parallel} + \underset{\text{cross-field drift}}{\mathbf{v}_d} + \underset{\text{Gyration}}{\mathbf{v}_L} \tag{2.18}$$

where

$$\dot{v}_\parallel = \frac{qE_\parallel}{m} \tag{2.19}$$

and
$\mathbf{v}_d$ (eq 2.17) is the "E × B drift" of the gyrocenter.

Comments on E × B drift:

1. It is *independent* of the properties of the drifting particle (q, m, v, whatever).

2. Hence it is in the *same* direction for electrons and ions.

3. Underlying physics for this is that in the frame moving at the E × B drift E = 0. We have 'transformed away' the electric field.

4. Formula given above is exact except for the fact that relativistic effects have been ignored. They would be important if $v_d \sim c$.

2.2.1 Drift due to Gravity or other Forces

Suppose particle is subject to some other force, such as gravity. Write it **F** so that

$$m\dot{\mathbf{v}} = \mathbf{F} + q\ \mathbf{v} \wedge \mathbf{B} = q(\frac{1}{q}\mathbf{F} + \mathbf{v} \wedge \mathbf{B}) \tag{2.20}$$

This is just like the Electric field case except with $\mathbf{F}/q$ replacing $\mathbf{E}$.

The drift is therefore

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F} \wedge \mathbf{B}}{B^2} \tag{2.21}$$

In this case, if force on electrons and ions is same, they drift in *opposite* directions.

This general formula can be used to get the drift velocity in some other cases of interest (see later).

2.3 Non-Uniform B Field

If B-lines are straight but the magnitude of B varies in space we get orbits that look qualitatively similar to the E ⊥ B case:

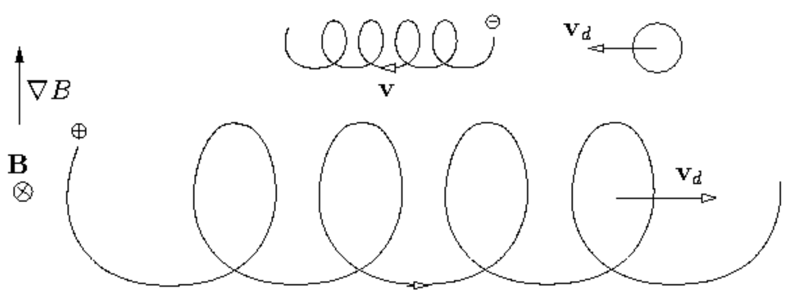

Figure 2.4: ∇B drift orbit

Curvature of orbit is greater where B is greater causing loop to be small on that side. Result is a drift perpendicular to both **B** and ∇B. Notice, though, that electrons and ions go in *opposite* directions (unlike $\mathbf{E} \wedge \mathbf{B}$).

Algebra

We try to find a decomposition of the velocity as before into $\mathbf{v} = \mathbf{v}_d + \mathbf{v}_L$ where $\mathbf{v}_d$ is constant.

We shall find that this can be done only approximately. Also we must have a simple expression for B. This we get by assuming that the Larmor radius is much smaller than the scale length of B variation i.e.,

$$r_L << B/|\nabla B| \tag{2.22}$$

in which case we can express the field approximately as the first two terms in a Taylor expression:

$$\mathbf{B} \simeq \mathbf{B}_0 + (\mathbf{r}.\nabla)\mathbf{B} \tag{2.23}$$

Then substituting the decomposed velocity we get:

$$m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}}_L = q(\mathbf{v} \wedge \mathbf{B}) = q[\mathbf{v}_L \wedge \mathbf{B}_0 + \mathbf{v}_d \wedge \mathbf{B}_0 + (\mathbf{v}_L + \mathbf{v}_d) \wedge (\mathbf{r}.\nabla)\mathbf{B}] \tag{2.24}$$

$$\text{or} \quad 0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} + \mathbf{v}_d \wedge (\mathbf{r}.\nabla)\mathbf{B} \tag{2.25}$$

Now we shall find that v_d/v_L is also small, like $r|\nabla B|/B$. Therefore the last term here is second order but the first two are first order. So we drop the last term.

Now the awkward part is that $\mathbf{v}_L$ and $\mathbf{r}_L$ are periodic. Substitute for $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$ so

$$0 = \mathbf{v}_d \wedge \mathbf{B}_0 + \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} + \mathbf{v}_L \wedge (\mathbf{r}_0.\nabla)\mathbf{B} \tag{2.26}$$

We now average over a cyclotron period. The last term is $\propto e^{-i\Omega t}$ so it averages to zero:

$$0 = \mathbf{v}_d \wedge \mathbf{B} + \langle \mathbf{v}_L \wedge (\mathbf{r}_L.\nabla)\mathbf{B} \rangle \ . \tag{2.27}$$

To perform the average use

$$\mathbf{r}_L = (x_L, y_L) = \frac{v_\perp}{\Omega}\left(\sin \Omega t, \frac{q}{|q|}\cos \Omega t\right) \tag{2.28}$$

$$\mathbf{v}_L = (\dot{x}_L, \dot{y}_L) = v_\perp\left(\cos \Omega t, \frac{-q}{|q|}\sin \Omega t\right) \tag{2.29}$$

$$\text{So} \quad [v_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_x = v_y y \frac{d}{dy} B \tag{2.30}$$

$$[\mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B}]_y = -v_x y \frac{d}{dy} B \tag{2.31}$$

(Taking ∇B to be in the y-direction).

Then

$$\langle v_y y \rangle = -\langle \cos \Omega t \sin \Omega t \rangle \frac{v_\perp^2}{\Omega} = 0 \tag{2.32}$$

$$\langle v_x y \rangle = \frac{q}{|q|}\langle \cos \Omega t \cos \Omega t \rangle \frac{v_\perp^2}{\Omega} = \frac{1}{2}\frac{v_\perp^2}{\Omega}\frac{q}{|q|} \tag{2.33}$$

So

$$\langle \mathbf{v}_L \wedge (\mathbf{r}.\nabla)\mathbf{B} \rangle = -\frac{q}{|q|}\,\frac{1}{2}\frac{v_\perp^2}{\Omega}\nabla B \tag{2.34}$$

Substitute in:

$$0 = \mathbf{v}_d \wedge \mathbf{B} - \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\nabla B \tag{2.35}$$

and solve as before to get

$$\mathbf{v}_d = \frac{\left(\frac{-1}{|q|}\frac{v_\perp^2}{2\Omega}\ \nabla B\right)\wedge \mathbf{B}}{B^2} = \frac{q}{|q|}\frac{v_\perp^2}{2\Omega}\frac{\mathbf{B}\wedge\nabla B}{B^2} \tag{2.36}$$

or equivalently

$$\mathbf{v}_d = \frac{1}{q}\frac{mv_\perp^2}{2B}\frac{\mathbf{B}\wedge\nabla B}{B^2} \tag{2.37}$$

This is called the 'Grad B drift'.

2.4 Curvature Drift

When the B-field lines are curved and the particle has a velocity $v_\parallel$ along the field, another drift occurs.

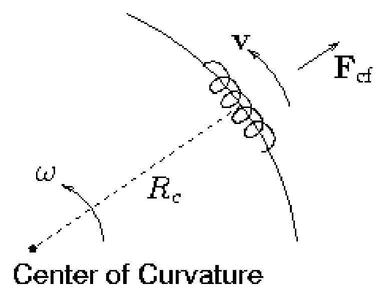

Figure 2.5: Curvature and Centrifugal Force

Take $|B|$ constant; radius of curvature R_e.

To 1st order the particle just spirals along the field.

In the frame of the guiding center a force appears because the plasma is rotating about the center of curvature.

This centrifugal force is F_{cf}

$$F_{cf} = m\frac{v_\parallel^2}{R_c} \quad \text{pointing outward} \tag{2.38}$$

as a vector

$$\mathbf{F}_{cf} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \tag{2.39}$$

[There is also a coriolis force $2m(\omega\wedge\mathbf{v})$ but this averages to zero over a gyroperiod.]

Use the previous formula for a force

$$\mathbf{v}_d = \frac{1}{q}\frac{\mathbf{F}_{cf}\wedge\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2} \tag{2.40}$$

This is the "Curvature Drift".

It is often convenient to have this expressed in terms of the field gradients. So we relate $\mathbf{R}_c$ to ∇B etc. as follows:

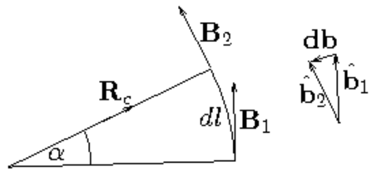

Figure 2.6: Differential expression of curvature

(Carets denote unit vectors)

From the diagram

$$\mathbf{db} = \hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_1 = -\hat{\mathbf{R}}_c \alpha \tag{2.41}$$

and

$$d\ell = \propto R_c \tag{2.42}$$

So

$$\frac{d\mathbf{b}}{dl} = -\frac{\hat{\mathbf{R}}_c}{R_c} = -\frac{\mathbf{R}_c}{R_c^2} \tag{2.43}$$

But (by definition)

$$\frac{d\mathbf{b}}{dl} = (\hat{\mathbf{B}}.\nabla)\hat{\mathbf{b}} \tag{2.44}$$

So the curvature drift can be written

$$\mathbf{v}_d = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c}{R_c^2} \wedge \frac{\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{q}\frac{\mathbf{B} \wedge (\hat{\mathbf{b}}.\nabla)\hat{\mathbf{b}}}{B^2} \tag{2.45}$$

2.4.1 Vacuum Fields

Relation between ∇B & $\mathbf{R}_c$ drifts

The curvature and ∇B are related because of Maxwell's equations, their relation depends on the current density $\mathbf{j}$. A particular case of interest is $\mathbf{j} = 0$: vacuum fields.

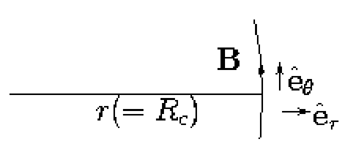

Figure 2.7: Local polar coordinates in a vacuum field

$$\nabla \wedge \mathbf{B} = 0 \quad \text{(static case)} \tag{2.46}$$

Consider the z-component

$$0 = (\nabla \wedge \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) \quad (B_r = 0 \text{ by choice}). \tag{2.47}$$

$$= \frac{\partial B_\theta}{\partial r} + \frac{B_\theta}{r} \tag{2.48}$$

or, in other words,

$$(\nabla B)_r = -\frac{B}{R_c} \tag{2.49}$$

[Note also $0 = (\nabla \wedge \mathbf{B})_\theta = \partial B_\theta/\partial z \ : (\nabla B)_z = 0$]
and hence $(\nabla B)_{\text{perp}} = -B\ \mathbf{R}_c/R_c^2$.
Thus the grad B drift can be written:

$$\mathbf{v}_{\nabla B} = \frac{mv_\perp^2}{2q}\frac{\mathbf{B}\wedge\nabla B}{B^3} = \frac{mV_\perp^2}{2q}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2 B^2} \tag{2.50}$$

and the total drift across a vacuum field becomes

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2 B^2}\ . \tag{2.51}$$

Notice the following:

1. R_c & ∇B drifts are in the *same* direction.
2. They are in *opposite* directions for opposite charges.
3. They are proportional to particle *energies*
4. Curvature $\leftrightarrow$ Parallel Energy ($\times$ 2)
   $\nabla B \leftrightarrow$ Perpendicular Energy
5. As a result one can very quickly calculate the average drift over a thermal distribution of particles because

$$\langle\frac{1}{2}mv_\parallel^2\rangle = \frac{T}{2} \tag{2.52}$$

$$\langle\frac{1}{2}mv_\perp^2\rangle = T \qquad \text{2 degrees of freedom} \tag{2.53}$$

Therefore

$$\langle\mathbf{v}_R + \mathbf{v}_{\nabla B}\rangle = \frac{2T}{q}\frac{\mathbf{R}_c\wedge\mathbf{B}}{R_c^2 B^2}\left(= \frac{2T}{q}\frac{\mathbf{B}\wedge\left(\hat{\mathbf{b}}.\nabla\right)\hat{\mathbf{b}}}{B^2}\right) \tag{2.54}$$

2.5 Interlude: Toroidal Confinement of Single Particles

Since particles can move freely along a magnetic field even if not across it, we cannot obviously confine the particles in a straight magnetic field. Obvious idea: bend the field lines into circles so that they have no ends.

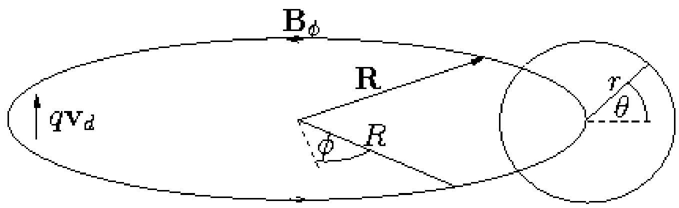

Figure 2.8: Toroidal field geometry

Problem

Curvature & ∇B drifts

$$\mathbf{v}_d = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{\mathbf{R}\wedge\mathbf{B}}{R^2B^2} \tag{2.55}$$

$$|\mathbf{v}_d| = \frac{1}{q}\left(mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right)\frac{1}{BR} \tag{2.56}$$

Ions drift *up*. Electrons down. There is no confinement. When there is finite density things

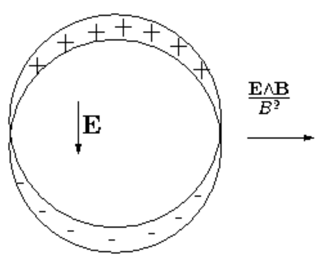

Figure 2.9: Charge separation due to vertical drift

are even worse because charge separation occurs → E → $E \wedge B$ → Outward Motion.

2.5.1 How to solve this problem?

Consider a beam of electrons $v_\parallel \neq 0$ $v_\perp = 0$. Drift is

$$v_d = \frac{mv_\parallel^2}{q}\frac{1}{B_T R} \tag{2.57}$$

What B_z is required to cancel this?

Adding B_z gives a compensating vertical velocity

$$v = v_\parallel \frac{B_z}{B_T} \qquad \text{for } \mathrm{B_z} \ << \ \mathrm{B_T} \tag{2.58}$$

We want total

$$v_z = 0 = v_\parallel \frac{B_z}{B_T} + \frac{mv_\parallel^2}{q} \frac{q}{B_T R} \tag{2.59}$$

So $B_z = -mv_\parallel / Rq$ is the right amount of field.

Note that this is such as to make

$$r_L(B_z) = \frac{|mv_\parallel|}{|qB_z|} \ = R \ \ . \tag{2.60}$$

But B_z required depends on $v_\parallel$ and q so we can't compensate for all particles simultaneously. Vertical field along cannot do it.

2.5.2 The Solution: Rotational Transform

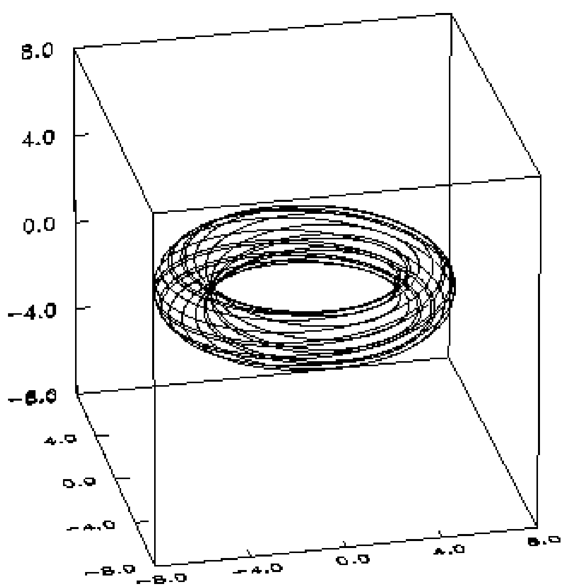

Figure 2.10: Tokamak field lines with rotational transform

Toroidal Coordinate system (r, θ, ϕ) (minor radius, poloidal angle, toroidal angle), see figure 2.8.

Suppose we have a *poloidal field* B_θ

Field Lines become helical and wind around the torus: figure 2.10.

In the poloidal cross-section the field describes a circle as it goes round in ϕ. Equation of motion of a particle *exactly* following the field is:

$$r\frac{d\theta}{dt} = \frac{B_\theta}{B_\phi}v_\phi = \frac{B_\theta}{B_\phi}\frac{B_\phi}{B}v_\parallel \ = \frac{B_\theta}{B}\ v_\parallel \tag{2.61}$$

and

$$r = \text{constant.} \tag{2.62}$$

Now add on to this motion the cross field drift in the $\hat{\mathbf{z}}$ direction.

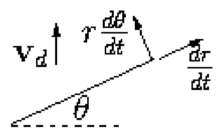

Figure 2.11: Components of velocity

$$r\frac{d\theta}{dt} \quad = \quad \frac{B_\theta}{B}v_\parallel + v_d\cos\theta \tag{2.63}$$

$$\frac{dr}{dt} \quad = \quad v_d\sin\theta \tag{2.64}$$

Take ratio, to eliminate time:

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u_d\sin\theta}{\frac{B_\theta}{B}v_\parallel + v_d\cos\theta} \tag{2.65}$$

Take $B_\theta, B, v_\parallel, v_d$ to be constants, then we can integrate this orbit equation:

$$[\ln r] = [-\ln|\frac{B_\theta v_\parallel}{B} + v_d\cos\theta|]\ . \tag{2.66}$$

Take $r = r_0$ when $\cos\theta = 0$ ($\theta = \frac{\pi}{2}$) then

$$r = r_0/\left[1 + \frac{Bv_d}{b_\theta v_\parallel}\cos\theta\right] \tag{2.67}$$

If $\frac{Bv_d}{B_\theta v_\parallel} << 1$ this is approximately

$$r = r_0 - \Delta\cos\theta \tag{2.68}$$

where $\Delta = \frac{Bv_d}{B_\theta v_\parallel}r_0$

This is approximately a circular orbit shifted by a distance Δ:

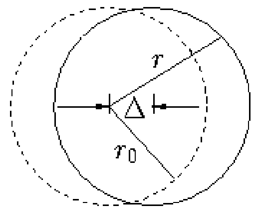

Figure 2.12: Shifted, approximately circular orbit

Substitute for v_d

$$\begin{aligned}\Delta &\simeq r_0 \frac{B}{B_\theta}\frac{1}{q}\frac{(mv_\parallel^2 + \frac{1}{2}mv_\perp^2)}{v_\parallel}\frac{1}{B_\phi R} &(2.69)\\ &\simeq \frac{1}{qB_\theta}\frac{mv_\parallel^2 + \frac{1}{2}mv_\perp^2}{v_\parallel}\frac{r_p}{R} &(2.70)\end{aligned}$$

$$\text{If } v_\perp = 0 \qquad \Delta = \frac{mv_\parallel}{qB_\theta}\frac{r_0}{R} = r_{L\theta}\frac{r_0}{R}, \qquad (2.71)$$

where $r_{L\theta}$ is the Larmor Radius in a field $B_\theta \times r/R$.

Provided Δ is small, particles *will* be confined. Obviously the important thing is the poloidal rotation of the field lines: Rotational Transform.

Rotational Transform

$$\text{rotational transform} \equiv \frac{\text{poloidal angle}}{\text{1 toroidal rotation}} \qquad (2.72)$$

$$(\text{transform}/2\pi =) \quad \iota \equiv \frac{\text{poloidal angle}}{\text{toroidal angle}} \quad . \qquad (2.73)$$

(Originally, ι was used to denote the transform. Since about 1990 it has been used to denote the transform divided by 2π which is the inverse of the safety factor.)

'Safety Factor'

$$\text{`}q_s' = \frac{1}{\iota} = \frac{\text{toroidal angle}}{\text{poloidal angle}} \quad . \qquad (2.74)$$

Actually the value of these ratios may vary as one moves around the magnetic field. Definition strictly requires one should take the limit of a large no. of rotations.

q_s is a topological number: number of rotations the long way per rotation the short way.

Cylindrical approx.:

$$q_s = \frac{rB_\phi}{RB_\theta} \qquad (2.75)$$

In terms of safety factor the orbit shift can be written

$$|\Delta| = r_{L\theta}\ \frac{r}{R} = r_{L\phi}\frac{B_\phi r}{B_\theta R} = r_L q_s \tag{2.76}$$

(assuming $B_\phi >> B_\theta$).

2.6 The Mirror Effect of Parallel Field Gradients: $\mathbf{E} = 0$, $\nabla B \parallel \mathbf{B}$

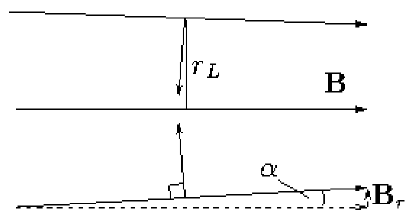

Figure 2.13: Basis of parallel mirror force

In the above situation there is a net force along **B**.
Force is

$$\begin{aligned} < F_\parallel > &= -|q\mathbf{v}\wedge\mathbf{B}|\ \sin\alpha = \ -|q|v_\perp B\sin\alpha \\ \sin\alpha &= \frac{-B_r}{B} \end{aligned} \tag{2.77, 2.78}$$

Calculate B_r as function of B_z from $\nabla.\mathbf{B} = 0$.

$$\nabla.\mathbf{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0\ . \tag{2.79}$$

Hence

$$rB_r = -\int r\frac{\partial B_z}{\partial z}dr \tag{2.80}$$

Suppose r_L is small enough that $\frac{\partial B_z}{\partial z} \simeq$ const.

$$[rB_r]_0^{r_L} \simeq \int_0^{r_L} rdr\ \frac{\partial B_z}{\partial z} = -\frac{1}{2}\ r_L^2\frac{\partial B_z}{\partial z} \tag{2.81}$$

So

$$B_r(r_L) = -\frac{1}{2}r_L\frac{\partial B_z}{\partial z} \tag{2.82}$$

$$\sin\alpha = -\frac{B_r}{B} = +\frac{r_L}{2}\frac{1}{2}\frac{\partial B_z}{\partial z} \tag{2.83}$$