Topic I :

Determinant and its Properties.

→ **Determinant :-**

"The determinent is a scalar value that can be compute from the elements of square matrix and en-codes certain properties of the linear transformation described by the matrix."

OR Simply,

The quantity obtained by addion of prododuct of element of square matrix according to specified rule.

It is denoted by det(A), det A or |A| . It occurs throughout mathematics

Generall, If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= ad - bc = \det A .$$

This determinant may be positive, negative or zero i-e it is not associated with absolute value at all

Example :

If $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$, then what is det A.

Solution:-

$$\det A = \begin{vmatrix} 2 & 5 \\ 0 & 1 \end{vmatrix}$$

$$= 2(1) - (0)(5)$$

$$= 2 - 0$$

$$\boxed{\det A. = 2}$$

→ **Properties of Determinant:-**

There are the following main properties of determinant that are useful as they permit us to generate same results with e simply computation of enteries.

Important Properties of Determinant:-

- **Reflection Property:**

The determinant remains unaltered when its rows are changed into colums and columns into rows i.e $|A| = |A^t|$.

Generally:- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$|A| = ad - bc$$

then $A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$|A^t| = ad - bc.$$

Example :-

If $A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3 - 2 = 1.$$

$$\Rightarrow \quad A^t = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$|A^t| = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 - 2 = 1$$

Hence, $|A| = |A^t|$.

- **All-zero Property :-**

If all elements of a row (or a column) are zero then determinant would be zero.

Generally, if $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$

$$|A| = \begin{vmatrix} 0 & 0 \\ a & b \end{vmatrix} = 0(b) - a(0) = 0.$$

Example :

If $A = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

then $|A| = \begin{vmatrix} 3 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 3 \end{vmatrix}.$

expand by row 1 :-

$= 3(0-0) - 1(0-0) + 2(0-0)$

$\det A = 0.$

Hence determinant would be zero.

- **Propotionality (Repitition) property:-**

If all elements of a row (or column) are propotional (identical) to elements of some other row (or column), then determinant would be zero.

Generally, if $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{bmatrix}$.

$$\det A = a_1(b_2a_3 - a_2b_3) - a_2(b_1a_3 - a_1b_3) + a_3(a_2b_1 - a_1b_2)$$

$$= a_1b_2a_3 - a_1a_2b_3 - a_2b_1a_3 + a_2a_1b_3 + a_3a_2b_1 - a_3a_1b_2 = 0.$$

Example:-

If $A = \begin{bmatrix} 3 & 2 & 3 \\ 1 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix}$

then $|A| = \begin{vmatrix} 3 & 2 & 3 \\ 1 & 2 & 0 \\ 3 & 2 & 3 \end{vmatrix}$.

$= 3(6-0) - 2(3-0) + 3(2-6).$

$= 18 - 6 - 12$

$= 18 - 18 = 0.$

- **Switching Property:-**

The interchange of any two rows (or columns), then determinant changes its sign. Generally, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $|A| = ad - bc$.

$\Rightarrow$ $A' = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ (interchange of R_1 & R_2)

$|A'| = -ad + bc = -(ad - bc)$

Hence $-|A| = +|A'|$.

Example, If $A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$.

then $|A| = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix}$

$= 2(2) - 3(1) = 4 - 3 = 1$.

$\Rightarrow$ $A' = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

$|A'| = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = 3 - 4 = -1$.

- **Scalar-multiple property:**

If all elements of a row (or column) of a determinant are multiplied by non-zero constant then determinant get multiplied with same Constant.

Generally, if $A = \begin{bmatrix} Ka_1 & b_1 \\ Ka_2 & b_2 \end{bmatrix}$.

then $|A| = \begin{vmatrix} Ka_1 & b_1 & c_1 \\ Ka_2 & b_2 & c_2 \\ Ka_3 & b_3 & c_3 \end{vmatrix}$

expand by C_1,

$$= Ka_1(b_2c_3 - b_3c_2) - Ka_2(b_2c_3 - c_2b_3) + Ka_3(c_2b_3 - b_2c_3).$$

$$= K[a_1(b_2c_3 - b_3c_2) - a_2(b_2c_3 - c_2b_3) + a_3(c_2b_3 - b_2c_3)].$$

$$= K|A|.$$

Example:- If $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$

then $|A| = \begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} = 8 - 3 = 5.$

now by multiplying C_1 by 2, A becomes

$$A' = \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix}$$

$$|A'| = 16 - 6 = 10 = 2 \times 5.$$

Hence, determinant is of 2 times.

- **Sum Property:-**

Generally,

$$\begin{vmatrix} a_1 + b_1 & c_1 & d_1 \\ a_2 + b_2 & c_2 & d_2 \\ a_3 + b_3 & c_3 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Example:-

$$\begin{vmatrix} 2+1 & 0 & 3 \\ 3+1 & 1 & 2 \\ 4+0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

L.H.S:-

$$\begin{vmatrix} 2+1 & 0 & 3 \\ 3+1 & 1 & 2 \\ 4+0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 3 \\ 4 & 1 & 2 \\ 4 & 2 & 1 \end{vmatrix}$$

$= 3(1-4) + 3(8-4) = -9+12$

$= 3.$

R.H.S:-

$$= \begin{vmatrix} 2 & 0 & 3 \\ 3 & 1 & 2 \\ 4 & 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 0 & 2 & 1 \end{vmatrix}$$

$= [2(1-3)+3(6-4)] + [1(1-4)+3(2)].$

$= [-6+6] + [-3+6) = 3.$

Hence L.H.S = R.H.S.

- **Property of Invariance:-**

Generally,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+\alpha b_1+\beta c_1 & b_1 & c_1 \\ a_2+\alpha b_2+\beta c_2 & b_2 & c_2 \\ a_3+\alpha b_3+\beta c_3 & b_3 & c_3 \end{vmatrix}$$

That is, a determinant remains unaltered under an operation of form:

$$C_i \longrightarrow C_i + \alpha C_j + \beta C_k.$$

Example:

$$\begin{vmatrix} 1 & 0 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 1(-1)+2(1) = 1.$$

Now if $\alpha = 2$, $\beta = 4$ then determinant becomes:-

$$\begin{vmatrix} 1+0+8 & 0 & 2 \\ 1+0+4 & 0 & 1 \\ 2+2+12 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 9 & 0 & 2 \\ 5 & 0 & 1 \\ 16 & 1 & 3 \end{vmatrix}$$

$= 9(-1)+2(5) = 1$

- **Multiplicative Property:-**

In multiplicative property of determinant were have $|AB| = |A||B|$

Generally,

if $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$, $B = \begin{bmatrix} a_3 & a_4 \\ b_3 & b_4 \end{bmatrix}$.

L.H.S:- $AB = \begin{bmatrix} a_1a_3 + a_2b_3 & a_1a_4 + a_2b_4 \\ b_1a_3 + b_2b_3 & b_1a_4 + b_2b_4 \end{bmatrix}$.

$$|AB| = \begin{vmatrix} a_1a_3 + a_2b_3 & a_1a_4 + a_2b_4 \\ b_1a_3 + b_2b_3 & b_1a_4 + b_2b_4 \end{vmatrix}$$

R.H.S:-

$$|A||B| = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_1 \end{vmatrix} \begin{vmatrix} a_3 & a_4 \\ b_3 & b_4 \end{vmatrix}.$$

$$= \begin{vmatrix} a_1a_3 + a_2b_3 & a_1a_4 + a_2b_4 \\ b_1a_3 + b_2b_3 & b_1a_4 + b_2b_4 \end{vmatrix}$$

Hence, from L.H.S & R.H.S

$$|AB| = |A||B|.$$

Example:-

If $A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$

L.H.S:-

$$AB = \begin{bmatrix} 2\times1 + 0\times1 & 2\times2 + 0\times3 \\ 1\times1 + 1\times1 & 1\times2 + 1\times3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+0 & 4+0 \\ 1+1 & 2+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 2 & 5 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 2 & 4 \\ 2 & 5 \end{vmatrix} = 10 - 8$$

$$= 2.$$

R.H.S.

$$|A||B| = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= [2(1) - 0][3 - 2].$$

$$= (2)(1)$$

$$= 2.$$

Hence L.H.S = R.H.S.

- **Triangle Property:-**

If all elements of determinant above or below main diagonal consist of zeros, then determinant is equal to product of diagonal elements.

Generally,

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{bmatrix}.$$

$$|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & c_3 \end{vmatrix}.$$

$$= a_1 b_2 c_3$$

Example

If $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

then $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

$= (1)(2)(1)$

$= 2.$

Topic II:

Improper Integral.

→ **Definition:-**

"An integral which is improper, is a definite integral that has either or both limits infinite or an integrand that approachs infinity for one or more points in range of integration"

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_{-\infty}^b f(x)\,dx.$$

They can however be defined as limits of integral;