2.1 Solving First-Degree Equations

OBJECTIVES

1. Solve first-degree equations
2. Use equations to solve word problems

In Section 1.1, we stated that an equality (equation) is a statement in which two symbols, or groups of symbols, are names for the same number. It should be further stated that an equation may be true or false. For example, the equation $3 + (-8) = -5$ is true, but the equation $-7 + 4 = 2$ is false.

Algebraic equations contain one or more variables. The following are examples of algebraic equations.

$$3x + 5 = 8 \qquad 4y - 6 = -7y + 9 \qquad x^2 - 5x - 8 = 0$$

$$3x + 5y = 4 \qquad x^3 + 6x^2 - 7x - 2 = 0$$

An algebraic equation such as $3x + 5 = 8$ is neither true nor false as it stands, and we often refer to it as an "open sentence." Each time that a number is substituted for x, the algebraic equation $3x + 5 = 8$ becomes a numerical statement that is true or false. For example, if $x = 0$, then $3x + 5 = 8$ becomes $3(0) + 5 = 8$, which is a false statement. If $x = 1$, then $3x + 5 = 8$ becomes $3(1) + 5 = 8$, which is a true statement. **Solving an equation** refers to the process of finding the number (or numbers) that make(s) an algebraic equation a true numerical statement. We call such numbers the **solutions** or **roots** of the equation, and we say that they *satisfy* the equation. We call the set of all solutions of an equation its **solution set**. Thus $\{1\}$ is the solution set of $3x + 5 = 8$.

In this chapter, we will consider techniques for solving **first-degree equations in one variable**. This means that the equations contain only one variable and that this variable has an exponent of 1. The following are examples of first-degree equations in one variable.

$$3x + 5 = 8 \qquad \frac{2}{3}y + 7 = 9$$

$$7a - 6 = 3a + 4 \qquad \frac{x-2}{4} = \frac{x-3}{5}$$

Equivalent equations are equations that have the same solution set. For example,

1. $3x + 5 = 8$
2. $3x = 3$
3. $x = 1$

are all equivalent equations because $\{1\}$ is the solution set of each.

The general procedure for solving an equation is to continue replacing the given equation with equivalent but simpler equations until we obtain an equation of the form *variable* = *constant* or *constant* = *variable*. Thus in the example above, $3x + 5 = 8$ was simplified to $3x = 3$, which was further simplified to $x = 1$, from which the solution set $\{1\}$ is obvious.

To solve equations we need to use the various properties of equality. In addition to the reflexive, symmetric, transitive, and substitution properties we listed in Section 1.1, the following properties of equality are important for problem solving.

Addition Property of Equality

For all real numbers a, b, and c,

$$a = b \quad \text{if and only if } a + c = b + c$$

Multiplication Property of Equality

For all real numbers a, b, and c, where $c \neq 0$,

$$a = b \quad \text{if and only if } ac = bc$$

The addition property of equality states that when the same number is added to both sides of an equation, an equivalent equation is produced. The multiplication property of equality states that we obtain an equivalent equation whenever we multiply both sides of an equation by the same *nonzero* real number. The following examples demonstrate the use of these properties to solve equations.

Classroom Example
Solve $3x - 5 = 16$.

EXAMPLE 1 Solve $2x - 1 = 13$.

Solution

$$\begin{aligned} 2x - 1 &= 13 \\ 2x - 1 + 1 &= 13 + 1 && \text{Add 1 to both sides} \\ 2x &= 14 \\ \frac{1}{2}(2x) &= \frac{1}{2}(14) && \text{Multiply both sides by } \frac{1}{2} \\ x &= 7 \end{aligned}$$

The solution set is $\{7\}$.

To check an apparent solution, we can substitute it into the original equation and see if we obtain a true numerical statement.

✔ Check

$$\begin{aligned} 2x - 1 &= 13 \\ 2(7) - 1 &\stackrel{?}{=} 13 \\ 14 - 1 &\stackrel{?}{=} 13 \\ 13 &= 13 \end{aligned}$$

Now we know that $\{7\}$ is the solution set of $2x - 1 = 13$. We will not show our checks for every example in this text, but do remember that checking is a way to detect arithmetic errors.

Classroom Example
Solve $-5 = -4a + 8$.

EXAMPLE 2 Solve $-7 = -5a + 9$.

Solution

$$\begin{aligned} -7 &= -5a + 9 \\ -7 + (-9) &= 5a + 9 + (-9) && \text{Add } -9 \text{ to both sides} \\ -16 &= -5a \\ -\frac{1}{5}(-16) &= -\frac{1}{5}(-5a) && \text{Multiply both sides by } -\frac{1}{5} \\ \frac{16}{5} &= a \end{aligned}$$

The solution set is $\left\{\frac{16}{5}\right\}$.

Note that in Example 2 the final equation is $\frac{16}{5} = a$ instead of $a = \frac{16}{5}$. Technically, the symmetric property of equality (if $a = b$, then $b = a$) would permit us to change from $\frac{16}{5} = a$ to $a = \frac{16}{5}$, but such a change is not necessary to determine that the solution is $\frac{16}{5}$. Note that we could use the symmetric property at the very beginning to change $-7 = -5a + 9$ to $-5a + 9 = -7$; some people prefer having the variable on the left side of the equation.

Let's clarify another point. We stated the properties of equality in terms of only two operations, addition and multiplication. We could also include the operations of subtraction and division in the statements of the properties. That is, we could think in terms of subtracting the same number from both sides of an equation and also in terms of dividing both sides of an equation by the same nonzero number. For example, in the solution of Example 2, we could subtract 9 from both sides rather than adding -9 to both sides. Likewise, we could divide both sides by -5 instead of multiplying both sides by $-\frac{1}{5}$.

Classroom Example
Solve $8m - 7 = 5m + 8$.

EXAMPLE 3 Solve $7x - 3 = 5x + 9$.

Solution

$$7x - 3 = 5x + 9$$

$$7x - 3 + (-5x) = 5x + 9 + (-5x) \quad \text{Add } -5x \text{ to both sides}$$

$$2x - 3 = 9$$

$$2x - 3 + 3 = 9 + 3 \quad \text{Add 3 to both sides}$$

$$2x = 12$$

$$\frac{1}{2}(2x) = \frac{1}{2}(12) \quad \text{Multiply both sides by } \frac{1}{2}$$

$$x = 6$$

The solution set is $\{6\}$.

Classroom Example
Solve $2(x + 3) + 6(x - 4) = 5(x - 9)$.

EXAMPLE 4 Solve $4(y - 1) + 5(y + 2) = 3(y - 8)$.

Solution

$$4(y - 1) + 5(y + 2) = 3(y - 8)$$

$$4y - 4 + 5y + 10 = 3y - 24 \quad \text{Remove parentheses by applying the distributive property}$$

$$9y + 6 = 3y - 24 \quad \text{Simplify the left side by combining similar terms}$$

$$9y + 6 + (-3y) = 3y - 24 + (-3y) \quad \text{Add } -3y \text{ to both sides}$$

$$6y + 6 = -24$$

$$6y + 6 + (-6) = -24 + (-6) \quad \text{Add } -6 \text{ to both sides}$$

$$6y = -30$$

$$\frac{1}{6}(6y) = \frac{1}{6}(-30) \quad \text{Multiply both sides by } \frac{1}{6}$$

$$y = -5$$

The solution set is $\{-5\}$.