# Average rate of change:

A rate of change describes how an output quantity changes relitive to the change in the input quantity.

The average rate of change b/w two input values is the total change of the function values (output values) divided by the change in the input values.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

or

$$\frac{f(x_1+h) - f(x_1)}{h}, \quad h \neq 0$$

## Question:

$f(x) = x^3 + 1$ , $[-1, 1]$

Find average rate of change.

Let $x_1 = -1$ , $x_2 = 1$

$f(-1) = (-1)^3 + 1 = -1 + 1 = 0$

$f(1) = (1)^3 + 1 = 1 + 1 = 2$

$$\because \quad \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$= \frac{2 - 0}{1 - (-1)} = \frac{2}{2} = 1$$

## Question:

$g(x) = x^2$ , $[-1, 1], [-2, 0]$

For $[-1, 1]$

$g(1) = 1^2 = 1$

$g(-1) = (-1)^2 = 1$

$$\frac{\Delta g}{\Delta x} = \frac{g(x_2) - g(x_1)}{x_2 - x_1}$$

$$= \frac{1 - 1}{1 - (-1)} = 0$$

For $[-2, 0]$

$x_1 = -2$ , $x_2 = 0$

$g(-2) = (-2)^2 = 4$

$g(0) = (0)^2 = 0$

$$\frac{\Delta g}{\Delta x} = \frac{0 - 4}{0 - (-2)} = \frac{-4}{2} = -2$$

## Question:

$$h(t) = \cot t \quad , \quad \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$x_1 = \pi/4, \; x_2 = 3\pi/4$

$$h\left(\frac{\pi}{4}\right) = \cot(\pi/4) = 1 \quad , \quad h\left(\frac{3\pi}{4}\right) = \cot\left(\frac{3\pi}{4}\right) = -1$$

$$\frac{\Delta h}{\Delta x} = \frac{h(x_2) - h(x_1)}{x_2 - x_1} = \frac{-1-1}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-2}{\frac{2\pi}{4}} = \frac{-2}{\pi/2} = -\frac{4}{\pi}$$

## Question:

$$R(\theta) = \sqrt{4\theta + 1} \qquad [0, 2]$$

$R(0) = \sqrt{(4)(0)+1} = 1$

$R(2) = \sqrt{4(2)+1} = 3$

$$\frac{\Delta R}{\Delta x} = \frac{R(x_2) - R(x_1)}{x_2 - x_1} = \frac{3-1}{2-0} = \frac{2}{2} = 1$$

## Slope of Curve

Average rate of change of function.

## Question: $y = x^2 - 3 \qquad (2, 1)$

Let $y = f(x)$

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1 + h) - f(x_1)}{h} \quad , \quad h \neq 0, \; h > 0$$

$x_1 = 2 \; , \; f(x_1 + h) = (2+h)^2 - 3 = 4 + 2h + h^2 - 3$
$= 1 + 2h + h^2$

$$f(x_1) = (2)^2 - 3 = 1$$

$$\frac{\Delta f}{\Delta x} = \frac{h^2 + 2h + 1 - 1}{h} = \frac{h^2 + 2h}{h} = \frac{\cancel{h}(h+2)}{\cancel{h}} = h + 2$$

$\Rightarrow \frac{\Delta f}{\Delta x} = h + 2 \Rightarrow h \to 0$
$= 2$

## Tangent Line: $\frac{y - y_0}{x - x_0} = m$

slope of curve $\Rightarrow m = 2$

$$\frac{y - y_0}{x - x_0} = \frac{y-1}{x-2} = 2$$

$\Rightarrow y - 1 = 2(x - 2) \Rightarrow y - 1 = 2x - 4$

$= y - 2x = -3$ (Eq. of tangent)

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# Concept of Limit:

Why we need a "limit"?

Let $f(x) = \frac{x^2-1}{x-1}$

$x=1 \quad F(1) = ?$

$x=0.1 \quad F(0.1) = \frac{(0.1)^2-1}{0.1-1} = \frac{0.01-1}{-0.9} = \frac{-0.99}{-0.9} = 1.09$

$x=0.2 \quad F(0.2) = \frac{(0.2)^2-1}{0.2-1} = \frac{0.04-1}{-0.8} = \frac{-0.96}{-0.8} = 1.2$

$x=0.4 \quad F(0.4) = \frac{(0.4)^2-1}{0.4-1} = \frac{0.16-1}{-0.6} = \frac{-0.84}{-0.6} = 1.4$

$x=0.6 \quad F(0.6) = \frac{0.36-1}{0.6-1} = \frac{-0.64}{-0.4} = 1.6$

$x=0.8 \quad F(0.8) = \frac{0.64-1}{0.8-1} = \frac{-0.36}{-0.2} = 1.8$

$x=0.9 \quad F(0.9) = \frac{0.81-1}{0.9-1} = \frac{-0.19}{-0.1} = 1.9$

$x=0.99 \quad F(0.99) = \frac{0.98-1}{0.99-1} = \frac{-0.02}{-0.01} = \frac{2}{1} = 2$

So,

$x \to 1$ and $f(x) \to 2$

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

## Definition:-

Let $f(x)$ be a function. The limit of $f(x)$ at point $x=a$ is denoted by

$$\lim_{x\to a} f(x)$$

and it mean's that when $x$ approches to $a$ then where $f(x)$ approches.

Example: At Point $x=a$, $f(x) = \frac{x^2-1}{x-1}$ is not defined

But; $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-1}{x-1} = 2$

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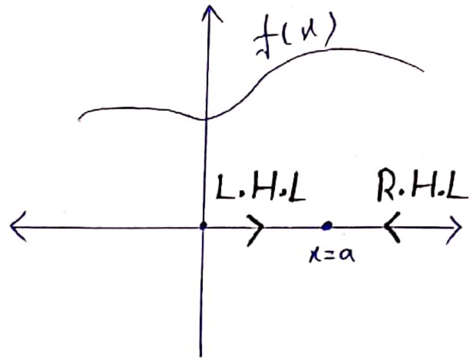
# Types of Limit's:-

i) Left hand Limit (L.H.L)
ii) Right hand Limit (R.H.L)

i) $\lim_{x \to a^-} f(x) \to$ L.H.L of $f(x)$ at $x=a$

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Put $x = a-h$
$a = a-h$
$h \to 0$




ii) $\lim_{x \to a^+} f(x) \to$ R.H.L of $f(x)$ at $x=a$

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Put $x = a+h$
$a = a+h$
$h \to 0$

# Rules and Method for finding limits

i) Factorization
ii) Rationalisation
iii) Subsitution

$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a-h) \to x = a-h$

$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h) \to x = a+h$

The following rules hold if $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ ($L$ and $M$ are real numbers)

1) Sum Rule:
$\lim_{x \to c} [f(x) + g(x)] = L + M$

2) Difference Rule:
$\lim_{x \to c} [f(x) - g(x)] = L - M$

3) Product rule:
$\lim_{x \to c} [f(x) \cdot g(x)] = L \cdot M$

4) Constant multiple rule
$\lim_{x \to c} k f(x) = kL$ (any no. $k$)

5) Quotient rule
$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

6): Power rule:
If $m$ and $n$ are integers then
$\lim_{x \to c} [f(x)]^{m/n} = L^{m/n}$
Provided $L^{m/n}$ is a real number.

# Question #1:

Evaluate: $\lim_{x\to 2} \frac{x^2-5x+6}{x^2-4}$

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{x-3}{x+2}$$

$$= \frac{2-3}{2+2} = -1/4$$

# Question #2:

Evaluate: $\lim_{x\to 3} (x^2-9)\left[\frac{1}{x+3} + \frac{1}{x-3}\right]$

$$\lim_{x\to 3} (x^2-9)\left[\frac{x\cancel{-3} + x\cancel{+3}}{(x+3)(x-3)}\right]$$

$$\lim_{x\to 3} \cancel{(x^2-9)}\left[\frac{2x}{\cancel{x^2-9}}\right]$$

$$\lim_{x\to 3} (2x) = 6$$

# Question #3:

Evaluate $\lim_{x\to 0} \frac{x}{1-\sqrt{1-x}} = \lim_{x\to 0} \frac{x}{1-\sqrt{1-x}} \times \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$

$$= \lim_{x\to 0} \frac{x(1+\sqrt{1-x})}{1-(1-x)} \Rightarrow \lim_{x\to 0} \frac{x(1+\sqrt{1-x})}{\cancel{1}-\cancel{1}+x}$$

$$= \lim_{x\to 0} \frac{\cancel{x}(1+\sqrt{1-x})}{\cancel{x}} \Rightarrow \lim_{x\to 0} (1+\sqrt{1-x})$$

$$= 1+\sqrt{1-0} = 1+1 = 2$$

# Some formulas for limits

i) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

ii) $\lim_{x\to 0} \frac{\sin x}{x} = 1$

iii) $\lim_{x\to 0} \frac{x}{\sin x} = 1$

iv) $\lim_{x\to 0} \cos x = 1$

v) $\lim_{x\to 0} \frac{\tan x}{x} = 1$

vi) $\lim_{x\to 0} \frac{x}{\tan x} = 1$

## Q#4:

$$\lim_{x\to 2}\frac{x^4-2^4}{x-2} \Rightarrow (n=4,\ a=2) \Rightarrow 4.2^{4-1} = 4.2^3 = 32$$

## Q#5:

$$\lim_{x\to 3}\frac{x^6-729}{x-3} \Rightarrow \lim_{x\to 3}\frac{x^6-3^6}{x-3}$$

$$\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$$

$\therefore$ Put $n=6,\ a=3$

$$\lim_{x\to 3}\frac{x^6-3^6}{x-3} = 6.3^{6-1} = 6.3^5 = 1458$$

## Q#6:

$$\lim_{x\to 3}\frac{x^n-3^n}{x-3} = 27, \text{ Find } n?$$

$$\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$$

Put $a=3$

$$\lim_{x\to 3}\frac{x^n-3^n}{x-3} = n3^{n-1}$$

$$\Rightarrow 27 = n3^{n-1}$$

$$3.3^2 = n3^{n-1}$$

$$3.3^{3-1} = n3^{n-1}$$

$$\Rightarrow \boxed{n=3}$$

# Some Trigonometric Results:

1) $\sin 2x = 2\sin x\cos x$

2) $1+\cos 2x = 2\cos^2 x \Rightarrow 1+\cos x = 2\cos^2 x/2 \Rightarrow 1+\cos 4x = 2\cos^2 2x$

$1-\cos 2x = 2\sin^2 x \Rightarrow 1-\cos x = 2\sin^2 x/2 \Rightarrow 1-\cos 4x = 2\sin^2 2x$

3) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

4) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right) \cdot \cos\left(\frac{C-D}{2}\right)$

$\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right) \cdot \sin\left(\frac{D-C}{2}\right)$

5) $\sec^2\theta = 1 + \tan^2\theta \Rightarrow \sec^2\theta - \tan^2\theta = 1$

6) $\operatorname{cosec}^2\theta = 1 + \cot^2\theta \Rightarrow \operatorname{cosec}^2\theta - \cot^2\theta = 1$

7) $\sin(A \pm B) = \quad A\cos B \pm \sin B\cos A$

8) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

## Q#7:

$$\lim_{x\to 0} \frac{\sin x^2}{x} \qquad \text{Put } x^2 = y \Rightarrow x = \sqrt{y}$$

$$= \lim_{y\to 0} \frac{\sin y}{\sqrt{y}} \qquad \text{and When; } x \to 0 \Rightarrow y \to 0$$

$$= \lim_{y\to 0} \frac{\sin y}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}} \Rightarrow \lim_{y\to 0}\left(\frac{\sin y}{y}\right)\cdot\sqrt{y}$$

$$= \lim_{y\to 0}\left(\frac{\sin y}{y}\right) \cdot \lim_{y\to 0}(\sqrt{y})$$

$$= 1 \times 0 = 0$$

## Q#8:

Evaluate: $\lim_{x\to \pi/2} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2}$

Put $\frac{\pi}{2} - x = y \Rightarrow x = \pi/2 - y$

when $x \to \pi/2 \Rightarrow y = 0$

$$\lim_{x\to\pi/2} \frac{1-\sin x}{\left(\frac{\pi}{2}-x\right)^2} \Rightarrow \lim_{y\to 0} \frac{1-\sin\left(\frac{\pi}{2}-y\right)}{y^2}$$

$$\lim_{y\to 0} = \frac{1-\cos y}{y^2} \Rightarrow \lim_{y\to 0} \frac{2\sin^2 y/2}{y^2}$$

$$\lim_{y\to 0} = 2\cdot\left(\frac{\sin y/2}{y}\right)^2 \Rightarrow \lim_{y\to 0} 2\left(\frac{\sin y/2}{\frac{y}{2}\cdot 2}\right)^2$$

$$= \lim_{y\to 0} 2 \cdot \left( \frac{\sin y/2}{y/2} \right) \cdot \frac{1}{4}$$

$$= 2 \cdot 1 \cdot \frac{1}{4}$$

$$= 1/2$$

# Concept of limit at infinity:

## Q#9

$$\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+3} - \sqrt{x}\right)$$

$$\lim_{x\to\infty} \sqrt{x}\left(\sqrt{x+3} - \sqrt{x}\right) \times \frac{\left(\sqrt{x+3} + \sqrt{x}\right)}{\left(\sqrt{x+3} + \sqrt{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}\,(\cancel{x}+3-\cancel{x})}{\sqrt{x+3} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{3\sqrt{x}}{\sqrt{x+3} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{3\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\left(\sqrt{1+\frac{3}{x}} + 1\right)}$$

$$= \lim_{x\to\infty} \frac{3}{\sqrt{1+3/x} + 1} = \frac{3}{\sqrt{1+0} + 1}$$

$$= \frac{3}{1+1} = 3/2$$

$$\lim_{x\to\infty} f(x) = l$$

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

$$\lim_{x\to\infty} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to\infty} (1 + 1/x)^x = e$$

$$\lim_{x\to 0} (1+x)^{1/x} = e$$

$$\lim_{x\to 0} \frac{1}{x} \log(1+x) = 1$$

$$\lim_{x\to 0} x \log\left(1+\frac{1}{x}\right) = 1$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$$

$$\lim_{x\to 0} e^x = 1$$

$$\lim_{x\to\infty} e^{-x} = 0$$

$$\lim_{x\to\infty} e^x = \infty$$

## Q #10

Evaluate:

$$\lim_{x\to 0} \frac{x - |x|}{x}$$

We know that

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

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$$\lim_{x\to 0^-} f(x) = 2 \qquad = \begin{cases} \frac{x-x}{x} & \text{if } x \geq 0 \\ \frac{x+x}{x} & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = 0 \qquad = \begin{cases} 0 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Limit of function doesn't exist.

**Q #11:**

$$f(x) = \begin{cases} x & , \; 0 \leq x < 1 \\ 3-x & , \; 1 \leq x < 2 \end{cases}$$

Show that $\lim_{x\to 1^+} f(x) = 2$. Does the limit of $f(x)$ at $x=1$ exist?

L.H.L at $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} (1-h) = 1 \rightarrow (1)$$

R.H.L at $x=1$

$$\lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} [3-(1+h)]$$

$$= \lim_{h\to 0} (3-1-h) = 2 \rightarrow (2)$$

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

$\therefore$ $\lim_{x\to 1} f(x)$ does not exist.

**Q #12:** Evaluate: $\lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$

Let $f(x) = \lim_{x\to 0} \frac{e^{1/x}}{e^{1/x}+1}$

Note:

$e^{\infty} = \infty$

$e^{0} = 1$

$e^{-\infty} = 0$

L.H.L at $x=0$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$\lim_{h\to 0} f(-h) = \lim_{h\to 0} \frac{e^{-1/h}}{1+e^{-1/h}}$$

$$= \frac{e^{-\infty}}{1+e^{-\infty}} = \frac{0}{1+0} = 0$$

R.H.L at $x=0$

$$\lim_{x \to 0^+} f(x) = \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{e^{1/h}}{1+e^{1/h}} = \lim_{h \to 0} \frac{\cancel{e^{1/h}}}{\cancel{e^{1/h}}(e^{-1/h}+1)} = \lim_{h \to 0} \frac{1}{e^{-1/h}+1}$$

$$= \frac{1}{0+1} = 1$$

$$\underline{\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)}$$