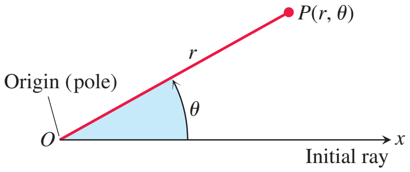

FIGURE 11.20 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

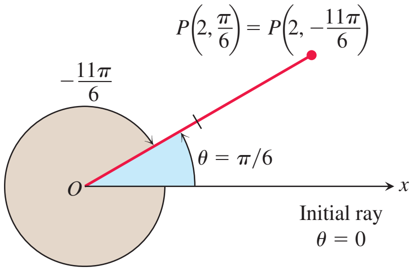

FIGURE 11.21 Polar coordinates are not unique.

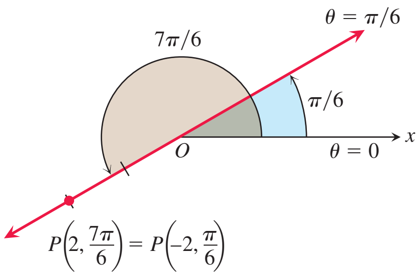

FIGURE 11.22 Polar coordinates can have negative r-values.

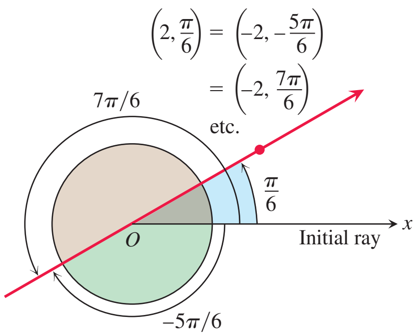

FIGURE 11.23 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

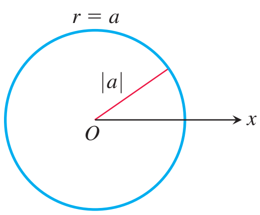

FIGURE 11.24 The polar equation for a circle is $r = a$.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 11.20). Usually the positive x-axis is chosen as the initial ray. Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as

$$P(r, \theta)$$

Directed distance from O to P — Directed angle from initial ray to OP

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2, \theta = -11\pi/6$ (Figure 11.21). In some situations we allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 11.22). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2, \theta = \pi/6$.

EXAMPLE 1 Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.23). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots.$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots.$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

Polar Equations and Graphs

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 11.24).

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray. (See Figure 11.22 for an example.)

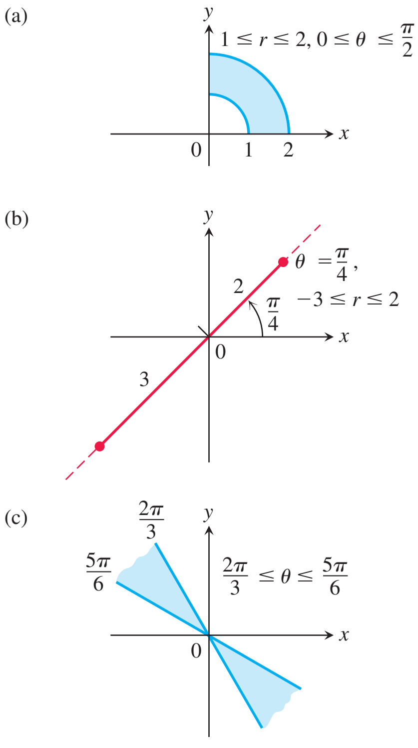

FIGURE 11.25 The graphs of typical inequalities in r and θ (Example 3).

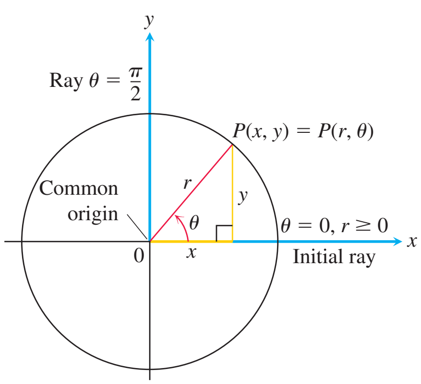

FIGURE 11.26 The usual way to relate polar and Cartesian coordinates.

EXAMPLE 2 A circle or line can have more than one polar equation.

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 11.23. ■

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2 \quad$ and $\quad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2 \quad$ and $\quad \theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6} \quad$ (no restriction on r)

Solution The graphs are shown in Figure 11.25. ■

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and let the initial polar ray be the positive x-axis. The ray $\theta = \pi/2, r > 0$, becomes the positive y-axis (Figure 11.26). The two coordinate systems are then related by the following equations.

> **Equations Relating Polar and Cartesian Coordinates**
>
> $$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each $(x, y) \ne (0, 0)$, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

EXAMPLE 4 Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not. ■

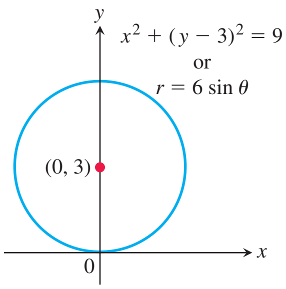

FIGURE 11.27 The circle in Example 5.

EXAMPLE 5 Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 11.27).

Solution We apply the equations relating polar and Cartesian coordinates:

$$\begin{aligned} x^2 + (y-3)^2 &= 9 \\ x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\ x^2 + y^2 - 6y &= 0 && \text{Cancelation} \\ r^2 - 6r \sin \theta &= 0 && x^2 + y^2 = r^2,\ y = r \sin \theta \\ r = 0 \quad \text{or} \quad r - 6 \sin \theta &= 0 \\ r &= 6 \sin \theta && \text{Includes both possibilities} \end{aligned}$$

EXAMPLE 6 Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r \cos \theta = -4$

(b) $r^2 = 4r \cos \theta$

(c) $r = \dfrac{4}{2 \cos \theta - \sin \theta}$

Solution We use the substitutions $r \cos \theta = x$, $r \sin \theta = y$, and $r^2 = x^2 + y^2$.

(a) $r \cos \theta = -4$

The Cartesian equation:
$$\begin{aligned} r \cos \theta &= -4 \\ x &= -4 && \text{Substitute.} \end{aligned}$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r \cos \theta$

The Cartesian equation:
$$\begin{aligned} r^2 &= 4r \cos \theta \\ x^2 + y^2 &= 4x && \text{Substitute.} \\ x^2 - 4x + y^2 &= 0 \\ x^2 - 4x + 4 + y^2 &= 4 && \text{Complete the square.} \\ (x-2)^2 + y^2 &= 4 && \text{Factor.} \end{aligned}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2 \cos \theta - \sin \theta}$

The Cartesian equation:
$$\begin{aligned} r(2 \cos \theta - \sin \theta) &= 4 \\ 2r \cos \theta - r \sin \theta &= 4 && \text{Multiply by } r. \\ 2x - y &= 4 && \text{Substitute.} \\ y &= 2x - 4 && \text{Solve for } y. \end{aligned}$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$

EXERCISES 11.3

Polar Coordinates

1. Which polar coordinate pairs label the same point?
   a. $(3, 0)$ **b.** $(-3, 0)$ **c.** $(2, 2\pi/3)$
   d. $(2, 7\pi/3)$ **e.** $(-3, \pi)$ **f.** $(2, \pi/3)$
   g. $(-3, 2\pi)$ **h.** $(-2, -\pi/3)$

2. Which polar coordinate pairs label the same point?
   a. $(-2, \pi/3)$ **b.** $(2, -\pi/3)$ **c.** (r, θ)
   d. $(r, \theta + \pi)$ **e.** $(-r, \theta)$ **f.** $(2, -2\pi/3)$
   g. $(-r, \theta + \pi)$ **h.** $(-2, 2\pi/3)$

3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(2, \pi/2)$ **b.** $(2, 0)$
c. $(-2, \pi/2)$ **d.** $(-2, 0)$

4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.

a. $(3, \pi/4)$ **b.** $(-3, \pi/4)$
c. $(3, -\pi/4)$ **d.** $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.

6. Find the Cartesian coordinates of the following points (given in polar coordinates).

a. $(\sqrt{2}, \pi/4)$ **b.** $(1, 0)$
c. $(0, \pi/2)$ **d.** $(-\sqrt{2}, \pi/4)$
e. $(-3, 5\pi/6)$ **f.** $(5, \tan^{-1}(4/3))$
g. $(-1, 7\pi)$ **h.** $(2\sqrt{3}, 2\pi/3)$

Cartesian to Polar Coordinates

7. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(1, 1)$ **b.** $(-3, 0)$
c. $(\sqrt{3}, -1)$ **d.** $(-3, 4)$

8. Find the polar coordinates, $-\pi \le \theta < \pi$ and $r \ge 0$, of the following points given in Cartesian coordinates.

a. $(-2, -2)$ **b.** $(0, 3)$
c. $(-\sqrt{3}, 1)$ **d.** $(5, -12)$

9. Find the polar coordinates, $0 \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(3, 3)$ **b.** $(-1, 0)$
c. $(-1, \sqrt{3})$ **d.** $(4, -3)$

10. Find the polar coordinates, $-\pi \le \theta < 2\pi$ and $r \le 0$, of the following points given in Cartesian coordinates.

a. $(-2, 0)$ **b.** $(1, 0)$
c. $(0, -3)$ **d.** $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Graphing Sets of Polar Coordinate Points

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 11–26.

11. $r = 2$ **12.** $0 \le r \le 2$
13. $r \ge 1$ **14.** $1 \le r \le 2$
15. $0 \le \theta \le \pi/6, \quad r \ge 0$ **16.** $\theta = 2\pi/3, \quad r \le -2$
17. $\theta = \pi/3, \quad -1 \le r \le 3$ **18.** $\theta = 11\pi/4, \quad r \ge -1$
19. $\theta = \pi/2, \quad r \ge 0$ **20.** $\theta = \pi/2, \quad r \le 0$
21. $0 \le \theta \le \pi, \quad r = 1$ **22.** $0 \le \theta \le \pi, \quad r = -1$
23. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$
24. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$
25. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$
26. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 27–52 with equivalent Cartesian equations. Then describe or identify the graph.

27. $r\cos\theta = 2$ **28.** $r\sin\theta = -1$
29. $r\sin\theta = 0$ **30.** $r\cos\theta = 0$
31. $r = 4\csc\theta$ **32.** $r = -3\sec\theta$
33. $r\cos\theta + r\sin\theta = 1$ **34.** $r\sin\theta = r\cos\theta$
35. $r^2 = 1$ **36.** $r^2 = 4r\sin\theta$
37. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$ **38.** $r^2\sin 2\theta = 2$
39. $r = \cot\theta\csc\theta$ **40.** $r = 4\tan\theta\sec\theta$
41. $r = \csc\theta\, e^{r\cos\theta}$ **42.** $r\sin\theta = \ln r + \ln\cos\theta$
43. $r^2 + 2r^2\cos\theta\sin\theta = 1$ **44.** $\cos^2\theta = \sin^2\theta$
45. $r^2 = -4r\cos\theta$ **46.** $r^2 = -6r\sin\theta$
47. $r = 8\sin\theta$ **48.** $r = 3\cos\theta$
49. $r = 2\cos\theta + 2\sin\theta$ **50.** $r = 2\cos\theta - \sin\theta$
51. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$
52. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 53–66 with equivalent polar equations.

53. $x = 7$ **54.** $y = 1$ **55.** $x = y$
56. $x - y = 3$ **57.** $x^2 + y^2 = 4$ **58.** $x^2 - y^2 = 1$
59. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **60.** $xy = 2$
61. $y^2 = 4x$ **62.** $x^2 + xy + y^2 = 1$
63. $x^2 + (y - 2)^2 = 4$ **64.** $(x - 5)^2 + y^2 = 25$
65. $(x - 3)^2 + (y + 1)^2 = 4$ **66.** $(x + 2)^2 + (y - 5)^2 = 16$

67. Find all polar coordinates of the origin.

68. Vertical and horizontal lines

a. Show that every vertical line in the xy-plane has a polar equation of the form $r = a\sec\theta$.

b. Find the analogous polar equation for horizontal lines in the xy-plane.

11.4 Graphing Polar Coordinate Equations

It is often helpful to graph an equation expressed in polar coordinates in the Cartesian xy-plane. This section describes some techniques for graphing these equations using symmetries and tangents to the graph.