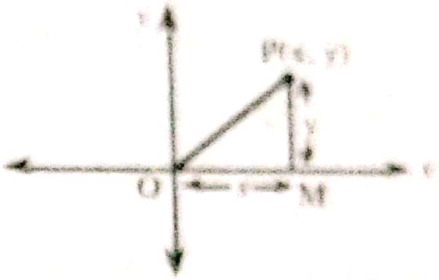
# 1

# ELEMENTARY MATHEMATICS

## 1. CO-ORDINATE SYSTEM AND GRAPHS

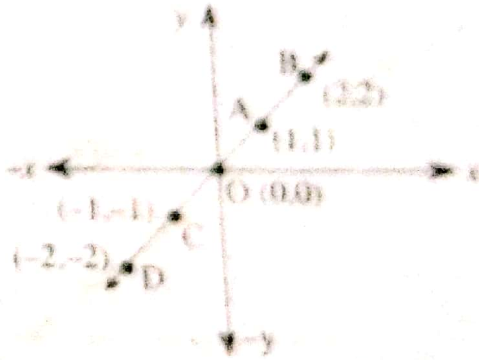
A plane surface is called a *plane*. We draw two perpendicular lines so that the plane is divided into FOUR parts called QUADRANTS. The intersection of the two perpendicular lines is called the ORIGIN. It is denoted by O. The horizontal line is called the x - axis and the vertical line is called the y-axis. The distances to the right of O on the x-axis are taken positive while those to the left are taken as negative. (These distances are called abscissae or x-coordinates). Similarly the distances to the upward direction of O are taken positive and those to the downward direction are taken as negative. (These distances are called *ordinates* or y-coordinates).

Each point on the plane can be described by two distances. For example, P(x, y) has horizontal distance $x$ and vertical distance $y$ as shown in the diagram. Likewise given two distances $x$ and y we can plot the point.




**GRAPH**. We plot different points and join them by a smooth line/curve, called the graph.

For example, if $f(x) = x$ or $y = x$, we give some values to $x$ and get corresponding values of y and getting pairs of values $(x, y)$ as ..., (–2,–2), (–1,–1), (0, 0), (1,1), 2,2), ....



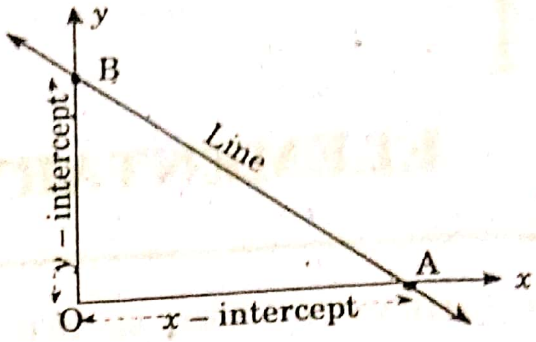

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$ | ... | –2 | –1 | 0 | 1 | 2 | ... |
| $y$ | ... | –2 | –1 | 0 | 1 | 2 | ... |

We plot these points as A, B, C, D , ... and join then and get the graph of $f(x) = x$ .

## 1.1 THE STRAIGHT LINE

A straight line is the locus ( path ) of a point P(x, y) which moves such that its co-ordinates $x$ and $y$ satisfy certain conditions. *For example*, if a point P(x, y) moves such that its ordinate $(y)$ is always double its abscissa $(x)$, then equation of the locus of the point will be : $y = 2x$ which is, therefore, the equation of the line traced by $P$.

**INTERCEPTS.** The distance from the origin to a point which a line cuts $x$-axis is called the **$x$-intercept**. Similarly **$y$-intercept** is defined as the distance from the origin to the point where the line cuts the $y$-axis as shown in the figure.



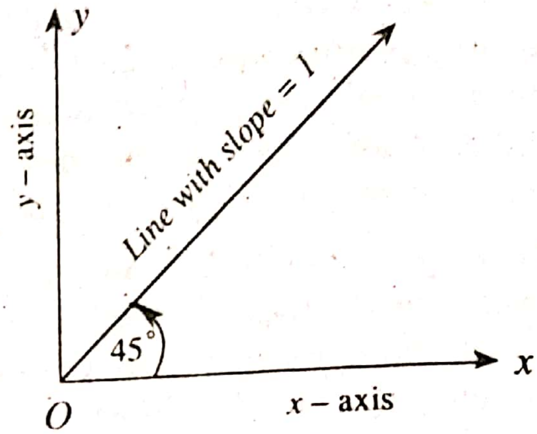

$OA = x-\text{intercept}$

$OB = y-\text{intercept}$

## INCLINATION AND SLOPE OF A LINE

The angle which a line makes with +ve direction of $x$ – axis is called the **inclination** of the line. The tangent of the inclination is called the **slope** of the line.

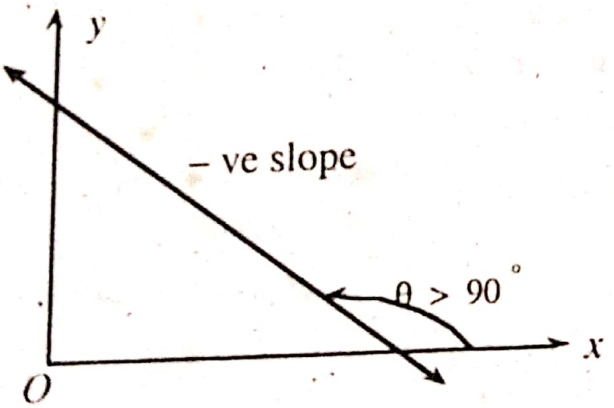
For example, if a line makes an angle of 45° with the $x$ – axis, then its

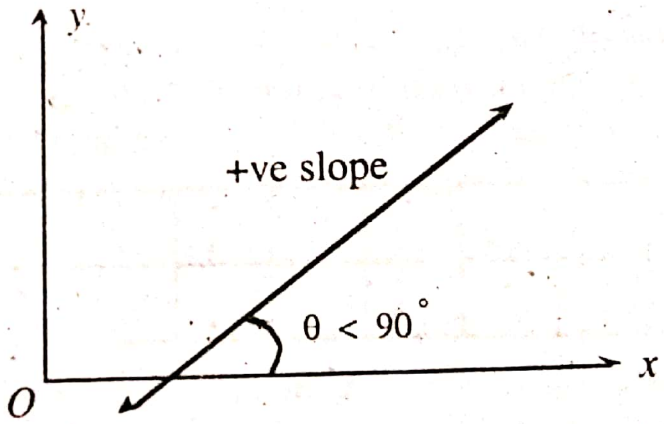



inclination = 45°

and its slope = tan 45° = 1.

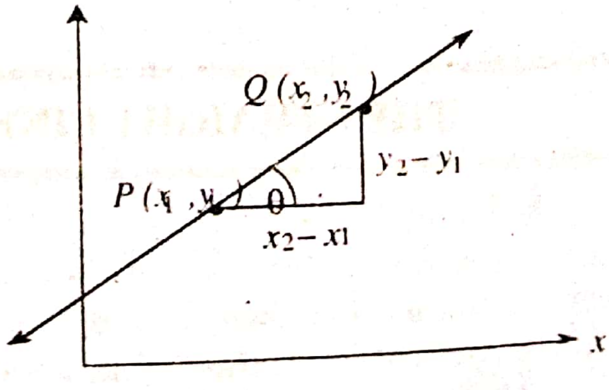
Again, if the inclination $\theta$ is such that $0 < \theta < 90$, then the $\tan\theta$ = + ve, hence the slope is +ve while if $90 < \theta < 180°$ then $\tan\theta$ = –ve, hence the slope is –ve.




## Slope of the Line joining two points

$P(x_1, y_1), Q(x_2, y_2)$.




From the figure we see that

Base $= x_2 - x_1$

and perpendicular $= y_2 - y_1$

$$\text{Slope} = \tan\theta = \frac{\text{perpendicular}}{\text{base}} = \frac{y_2 - y_1}{x_2 - x_1}$$

**Example . Find the slope of the line joining the points : (1, 2) , (4, 5).**

**Solution.** Here $(x_1, y_1) = (1, 2)$ , $(x_2, y_2) = (4, 5)$, that is

$x_1 = 1$ , $y_1 = 2$ , $x_2 = 4$ , $y_2 = 5$

Putting these values in the formula, we have

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{4-1} = \frac{3}{3} = 1$$

## 1.1.1 THE EQUATION OF A STRAIGHT LINE.

### 1. Slope - intercept Form : $y = mx + c$

$y = mx + c$ is called the slope-intercept form of the equation of a line where $m$ is the slope and $c$ is the intercept on y-axis.

**Derivation.** Let P(x, y) be a point of the line. As told above, a line is the locus of a point P(x, y) which moves in a straight path under some constraints. For example

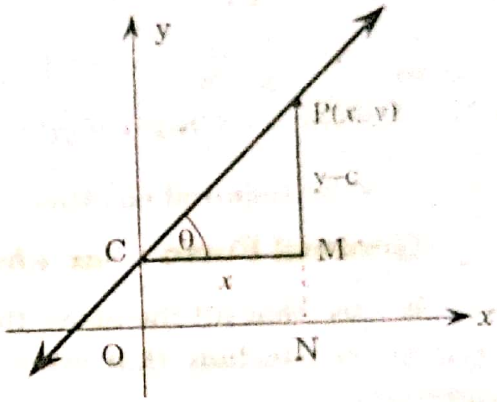
1. If a point P(x, y) moves so that its slope is always equal to $m$ then

$$\tan\theta = m = \frac{\text{perpendicular}}{\text{base}} = \frac{PM}{CM} = \frac{y-c}{x}$$

or $m = \frac{y-c}{x} \Rightarrow y - c = mx$

$\Rightarrow$ $\boxed{y = mx + c}$

is the required equation.




### 2. Intercepts Form : $\frac{x}{a} + \frac{y}{b} = 1$

$\frac{x}{a} + \frac{y}{b} = 1$ is called the Intercepts form of the equation of a line where $a$ is the intercept on $x$–axis and $b$ the intercept on $y$–axis.

**Derivation.** Let P(x, y) be a point of the line. As told above, a line is the locus of a point P(x, y) which moves in a straight path under some constraints. For example,

If a point P(x, y) moves so that it makes intercepts $a$ and $b$ on the axes respectively, then since A($a$,0), B(0,b) lie on the line, therefore putting them in $y = mx + c$, we get

$$0 = ma + c \ldots \text{(i)}, \quad b = m.0 + c \ldots \text{(ii)}$$

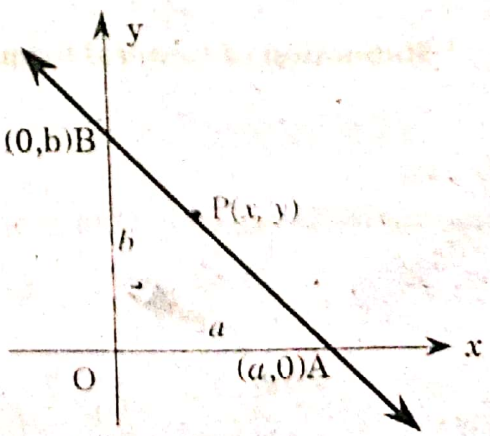
So from (ii) c = b, from (i) $m = -\frac{c}{a} = -\frac{b}{a}$, put in

$$y = mx + c$$

then $y = -\frac{b}{a}x + b \Rightarrow ay = -bx + ab$

Divide by $ab$ , then $\frac{ay}{ab} = -\frac{bx}{ab} = 1$

$\Rightarrow$ $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$ is the required equation.

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(ii) $4x + 3y - 12 = 0 \Rightarrow 4x + 3y = 12 \Rightarrow \frac{4x}{12} + \frac{3y}{12} = \frac{12}{12} \Rightarrow \frac{x}{3} + \frac{y}{4} = 1$

is the intercepts form with $a = 3$, $b = 4$

(iii) $4x + 3y - 12 = 0$ Here $a = 4$, $b = 3$ so that $\sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

So dividing by 5, required normal form is $\frac{4}{5}x + \frac{3}{5}y = \frac{12}{5}$, with $p = \frac{12}{5}$

## EXERCISE – 1.1

1. Find the slope of the line joining the points :

   (i) (3, 4), (7, 8) (ii) (7, 8), (3, 4) (iii) $(a, b)$, $(c, d)$

   Ans (i). 1 (ii). 1 (iii). $\frac{d-b}{c-a}$

2. Find the equation of line with

   (i) slope = 1, $y$-intercept = 3 (ii) $x$-intercept = 4, $y$-intercept = 5,

   (iii) length of perpendicular from origin = 5, inclination of perpendicular = 60°

   Ans. (i). $y = x + 3$ (ii). $5x + 4y = 20$ (iii). $x + \sqrt{3}y = 10$

3. Reduce the equation $3x + 4y - 12 = 0$ to

   (i) Slope-intercept form

   (ii) Intercepts form

   (iii) Perpendicular form.

   Ans. (i). $y = -\frac{3}{4}x - \frac{5}{4}$ (ii). $\frac{x}{4} + \frac{y}{3} = 1$ (iii). $\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$

## 1.2 FUNCTIONS AND LIMITS

### 1. VARIABLES AND CONSTANTS.

In our daily life we observe that quantities like population of a country, annual rainfall, temperature, volume, pressure and so on have different values at different times or locations. Such quantities are called **variables** and all such quantities which do not change their values neither from time to time nor from place to place are termed as **constants**, e.g., the ratio of the circumference of a circle to its diameter is a constant and that is an irrational number $\pi$. Variable quantities are usually represented by English alphabets $x, y, z, u, v$ etc. whereas constants are represented by English alphabet $a, b, c,$ *etc.*

Now consider the following three variable quantities :

$y$ = Number of moles of a gas

$x$ = Number of atmospheres

$z$ = Absolute temperature (K)

We may relate variables $y$ and $x$ by saying that *volume of a gas depends on pressure*. We may also relate variables $y$ and $z$ by saying that *volume of a gas depends on temperature*. Here $y$ is the dependent variable while $x$ and $z$ are independent variables.

## 2. FUNCTIONS

If the variable $y$ depends upon the variable $x$ we symbolically write $y = f(x)$ and if to each value of $x$, there corresponds a unique value of $y$, we say that **y is a function of x.** In the above example, we may say that $y$ is a function of $x$ and $z$. As the area of a circle depends upon its radius, therefore, we may say that *Area of a circle is a function of its radius*. Also since the volume of a sphere depends upon its radius, so we may say that *volume of a sphere is a function of its radius.*

## 3. LIMIT

### Value of a Variable.

When a variable $x$ takes value 2, say, we say that the value of $x = 2$, i.e., $\boxed{x - 2 = 0}$.

### Limit of a Variable.

When a variable $x$ assumes values in such a way that it approaches a certain number 2, say, we say that $x$ approaches 2 or $x$ tends to 2 (written as $x \longrightarrow 2$), we mean that the limiting value of $x$ is 2 or the limit of $x$ is 2. In this case $x$ is very very close to 2 but not equal to 2. So, if $x$ approaches 2, then $x$ is *approximately* equal to 2 and $x \neq 2$, i.e., $\boxed{x - 2 \neq 0}$.

### Value of a function.

If $f(x)$ is a function of $x$, i.e., $f(x) = x^2$. Then value of $f(x)$ when $x = 2$, denoted by $f(2)$ is given by simply putting $x = 2$ in $y = x^2$. Thus, Value of $f(x) = f(2) = (2)^2 = 4$

**Limit of a function.** If $x \longrightarrow 2$ then $f(x) = x^2 \longrightarrow (2)^2 = 4$. In other words, if limit of $x$ is 2 then limit of $f(x)$ is 4.

**EXAMPLE.** Evaluate the following limits : (i) $\lim\limits_{x \to 1} \left(\dfrac{x^2 - 4}{x - 2}\right)$ (ii) $\lim\limits_{x \to \infty} \dfrac{x^2 + 7x + 6}{x^2 + 3}$

**Solution.** (i) We simply put $x = 1$ and get

$$\lim_{x \to 1} \left(\frac{x^2 - 4}{x - 2}\right) = \frac{1 - 4}{1 - 2} = \frac{-3}{-1} = 3$$

(ii) If we put $x = \infty$, we get $\frac{\infty}{\infty}$ which is indeterminate. Since as $x \longrightarrow \infty$, $\frac{1}{x} \longrightarrow 0$, so dividing by $x^2$, we write

$$\lim_{x \to \infty} \frac{x^2 + 7x + 6}{x^2 + 3} = \lim_{x \to \infty} \frac{1 + \frac{7}{x} + \frac{6}{x^2}}{1 + \frac{3}{x^2}} = \frac{1 + 0 + 0}{1 + 0} = 1$$

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

For example $\frac{d}{dx}(x^{10}) = 10\,x^{10-1} = 10x^9$

**Rules of Derivatives**

1. Derivative of constant is zero. i.e., $\frac{d}{dx}(c) = 0$, $\frac{d}{dx}(9) = 0$

(2) If $u(x)$ and $v(x)$ are two functions of $x$, then

$$\frac{d}{dx}[u+v] = \frac{d}{dx}(u) + \frac{d}{dx}(v)$$

$$\frac{d}{dx}[u-v] = \frac{d}{dx}(u) - \frac{d}{dx}(v)$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

**EXAMPLES.**

1. $\frac{d}{dx}[x^2+x^3] = \frac{d}{dx}(x^2) + \frac{d}{dx}(x^3) = 2x + 3x^2$

2. $\frac{d}{dx}[x^2-x^3] = \frac{d}{dx}(x^2) - \frac{d}{dx}(x^3) = 2x - 3x^2$

3. $\frac{d}{dx}[x^2 \times x^3] = x^2 \times \frac{d}{dx}(x^3) + x^3 \times \frac{d}{dx}(x^2) = x^2.(3x^2) + x^3.(2x) = 3x^4 + 2x^4 = 5x^4$

4. $\frac{d}{dx}\left[\frac{x^5}{x^2}\right] = \frac{x^2.\frac{d}{dx}(x^5) - x^5.\frac{d}{dx}(x^2)}{(x^2)^2} = \frac{x^2.\,5x^4 - x^5.\,2x}{x^4} = \frac{5x^6 - 2x^6}{x^4} = \frac{3x^6}{x^4} = 3x^2$

**EXAMPLE**. Differentiate the following

1. $\sqrt{x} + \frac{1}{\sqrt{x}}$ 2. $\frac{x-1}{x+1}$ 3. $\sqrt{\frac{a+x}{a-x}}$ 4. $a + bx + cx^3$ 5. $\frac{2}{x}$

**Solution. (1).** Let $y = \sqrt{x} + \frac{1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$

then $\frac{dy}{dx} = \frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(x^{-1/2}) = \frac{1}{2}x^{-1/2} + \left(-\frac{1}{2}x^{-3/2}\right) = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

**2.** $y = \frac{x-1}{x+1}$ then $\frac{dy}{dx} = \frac{(x+1)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x+1)}{(x+1)^2}$

$$= \frac{(x+1).(1) - (x-1).(1)}{(x+1)^2} = \frac{x+1-x+1}{(x+1)^2} = \frac{2}{(x+1)^2}$$

3. $y = \sqrt{\dfrac{a+x}{a-x}} = \left(\dfrac{a+x}{a-x}\right)^{1/2}$, then $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\dfrac{a+x}{a-x}\right)^{-1/2} \dfrac{d}{dx}\left(\dfrac{a+x}{a-x}\right)$

$$= \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2} \frac{(a-x)\frac{d}{dx}(a+x) - (a+x)\frac{d}{dx}(a-x)}{(a-x)^2}$$

$$= \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2} \frac{(a-x)(1) - (a+x).(-1)}{(a-x)^2}$$

$$= \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2} \frac{a-x+a+x}{(a-x)^2} = \frac{1}{2}\left(\frac{a+x}{a-x}\right)^{-1/2} \cdot \frac{2a}{(a-x)^2}$$

4. $y = a + bx + cx^2$ then $\dfrac{dy}{dx} = \dfrac{d}{dx}(a) + \dfrac{d}{dx}(bx) + \dfrac{d}{dx}(cx^2) = 0 + b + 2cx$

5. $y = \dfrac{2}{x} = 2\,x^{-1}$, then $\dfrac{dy}{dx} = \dfrac{d}{dx}(2\,x^{-1}) = 2\dfrac{d}{dx}(x^{-1}) = 2\,.(-1)x^{-2} = -2x^{-2} = -\dfrac{2}{x^2}$

## EXERCISE – 1.4

Find the derivative of the following functions :

1. $x^8$
2. $x^3 + 2x + 2$
3. $x^6 - x^8$
4. $\dfrac{3}{2}x^2$
5. $\dfrac{x-1}{x-2}$
6. $\dfrac{x^2+1}{x^2-1}$
7. $(3-5x^2)^{-\frac{7}{2}}$
8. $\sqrt[5]{x}$

Ans. (1) $8x^7$ 2. $3x^2+2$ 3. $6x^5 - 8x^7$ 4. $3x$

5. $\dfrac{-1}{(x-2)^2}$ 6. $\dfrac{-4x}{(x^2-1)^2}$ 7. $35(3-5x^2)^{-\frac{9}{2}}$ 8. $\dfrac{1}{5}x^{-4/5}$

## 1.3.2 DIFFERENTIATION OF TRIGONOMETRIC FUNCTIONS

Following are the formulas for trigonometric functions :

1. $\dfrac{d}{dx}(\sin x) = \cos x$
2. $\dfrac{d}{dx}(\cos x) = -\sin x$
3. $\dfrac{d}{dx}(\tan x) = \sec^2 x$
4. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$
5. $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
6. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

*Note.* When we have some function of $x$ instead of $x$, then we multiply by derivative of $f(x)$.

**EXAMPLE.**

1. $\dfrac{d}{dx}(\sin x^2) = \cos x^2 . \dfrac{d}{dx}(x^2) = \cos x^2 . (2x) = 2x \cos x^2$

2. $\dfrac{d}{dx}(\cos x^5) = -\sin x^5 . \dfrac{d}{dx}(x^5) = -\sin x^5 . (4x^2) = -4x^2 \sin x^5$

3. $\frac{d}{dx}(\tan x^4) = \sec^2 x^4 \cdot \frac{d}{dx}(x^4) = \sec^2 x^4 \cdot (4x^3) = 4x^3 \sec^2 x^4$

4. $\frac{d}{dx}(\cot x^2) = -\text{cosec}^2 x^2 \cdot \frac{d}{dx}(x^2) = -\text{cosec}^2 x^2 \cdot (2x) = -2x\, \text{cosec}^2 x^2$

5. $\frac{d}{dx}(\sec x^2) = \sec x^2 \tan x^2 \cdot \frac{d}{dx}(x^2) = \sec x^2 \tan x^2 \cdot (2x) = 2x \sec x^2 \tan x^2$

6. $\frac{d}{dx}(\text{cosec}\, x^2) = -\text{cosec}\, x^2 \cot x^2 \cdot \frac{d}{dx}(x^2) = -\text{cosec}\, x^2 \cot x^2 \cdot (2x) = -2x\, \text{cosec}\, x^2 \cot x^2$

**EXAMPLE. Find the derivative of $x \sin x$.**

**Solution.** $y = x \sin x$, then $\frac{dy}{dx} = x \cdot \frac{d}{dx}(\sin x) + \sin x \cdot \frac{d}{dx}(x)$

$$= x \cdot (\cos x) + \sin x \cdot (1) = x \cdot \cos x + \sin x$$

**EXAMPLE. Find the derivative of $\sqrt{\sin \sqrt{x}}$.**

**Solution.** $y = \sqrt{\sin \sqrt{x}} = \left(\sin \sqrt{x}\right)^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}\left(\sin \sqrt{x}\right)^{-1/2} \cdot \frac{d}{dx}\left(\sin \sqrt{x}\right) = \frac{1}{2}\left(\sin \sqrt{x}\right)^{-1/2}\left(\cos \sqrt{x}\right)\frac{d}{dx}\left(\sqrt{x}\right)$$

$$= \frac{1}{2}\left(\sin \sqrt{x}\right)^{-1/2}\left(\cos \sqrt{x}\right)\frac{1}{2}x^{-1/2} = \frac{1}{2}\left(\sin \sqrt{x}\right)^{-1/2} \cdot \left(\cos \sqrt{x}\right) \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\cos \sqrt{x}}{4\sqrt{x}\sqrt{\sin \sqrt{x}}}$$

**EXAMPLE. Find the derivative of $\sin^4 x$.**

**Solution.** Using $\frac{d}{dx}(f^n(x)) = n\, f^{n-1}(x) \cdot f'(x)$,

we have $y = \sin^4 x = (\sin x)^4$

$$\therefore \frac{dy}{dx} = 4(\sin x)^3 \cdot \frac{d}{dx}(\sin x) = 4 \sin^3 x \cdot \cos x$$

## EXERCISE – 1.5

Find the derivative of the following functions :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | $\sin x^4$ | 2. | $\sec x^5$ | 3. | $\tan x^9$ | 4. | $\cos x^{11}$ |
| 5. | $\sqrt{\sin x}$ | 6. | $\sin(\tan x)$ | 7. | $\cos^2 x^2$ | 8. | $\cos^2(ax+b)$ |

*Ans.* 1. $4x^3 \cos x^4$ 2. $5x^4 \sec x^5 \tan x^5$ 3. $9x^8 \sec^2 x^9$ 4. $-11x^{10} \sin x^{11}$

5. $\frac{\cos x}{2\sqrt{\sin x}}$ 6. $\sec^2 x \cdot \cos(\tan x)$ 7. $-4x \cos x^2 \sin x^2$

8. $-2a \cos(ax+b) \cdot \sin(ax+b)$

## 1.3.3 DIFFERENTIATION OF LOGARITHMIC and EXPONENTIAL FUNCTIONS

### LOGARITHMIC FUNCTIONS

**DEFINITION.**

The ***logarithm*** to the base $a$ ($a > 0, a \neq 1$) of the number N (N > 0) is the number $x$ such that $a^x = N$. Thus

$$x = \log_a N \quad \text{if and only if} \quad a^x = N$$

For example

$$\log_2 8 = 3 \quad \text{because} \quad 2^3 = 8$$

**Properties of Natural Logarithms.**

In calculus, all logarithms are to the base $e$, the irrational number whose value is $\lim\limits_{n \to \infty}\left(1 + \frac{1}{n}\right)^n$ was found to be 2.7182...., unless some other base is explicitly mentioned

The fundamental properties of logarithms to the base $e$ are as given below

1. $ln\, 1 = 0$
2. $ln\, e = 1$
3. $ln\,(xy) = ln\, x + ln\, y \qquad x, y > 0$
4. $ln\left(\frac{x}{y}\right) = ln\, x - ln\, y \qquad x, y > 0$
5. $ln\, x^m = m\, ln\, x$

**Natural Logarithms.**

In calculus, the most important logarithmic functions are those whose base equal $e$ i.e., of the form : $\log_e x$.

Logarithms to the base $e$ are called Natural logarithms.

$$\boxed{\ln x = \log_e x}$$

**DERIVATIVE OF $\ln x$.**

Let $\quad y = \ln x$

then $\quad y + \delta y = \ln(x + \delta x)$

$$\delta y = \ln(x + \delta x) - \ln x = \ln\left(\frac{x + \delta x}{x}\right) = \ln\left(1 + \frac{\delta x}{x}\right)$$

Divide both sides by $\delta x$, then

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x}\ln\left(1 + \frac{\delta x}{x}\right) = \frac{1}{x}\cdot\frac{x}{\delta x}\ln\left(1 + \frac{\delta x}{x}\right) = \frac{1}{x}\ln\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

When $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ and $\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \to e$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x} \ln e = \frac{1}{x} \qquad \{ \text{as } \ln e = 1 \}$$

i.e.
$$\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

**DIFFERENTIAL COEFFICIENT OF $\log_a x$.**

Let $\quad y = \log_a x$

then $\quad y + \delta y = \log_a (x + \delta x)$

$$\delta y = \log_a (x + \delta x) - \log_a x = \log_a \left(\frac{x + \delta x}{x}\right) = \log_a \left(1 + \frac{\delta x}{x}\right)$$

Divide both sides by $\delta x$, then

$$\frac{\delta y}{\delta x} = \frac{1}{\delta x} \log_a \left(1 + \frac{\delta x}{x}\right) = \frac{1}{x} \cdot \frac{x}{\delta x} \log_a \left(1 + \frac{\delta x}{x}\right) = \frac{1}{x} \log_a \left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}}$$

When $\delta x \to 0$, $\frac{\delta y}{\delta x} \to \frac{dy}{dx}$ and $\left(1 + \frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \to e$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x} \log_a e = \frac{1}{x \log_e a} = \frac{1}{x \ln a}$$

i.e.
$$\boxed{\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}}$$

**EXAMPLE 1.**

**Differentiate $\ln (\ln x)$.**

**Solution.**

Let $\quad y = \ln(\ln x)$

Put $\quad u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$, $\frac{du}{dx} = \frac{1}{u}$

Now $\quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{1}{x} = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$

$$\therefore \quad \frac{d}{dx}\{(\ln \ln x)\} = \frac{1}{x \ln x}$$

**EXAMPLE 2.**

**Differentiate : $\frac{1}{x} . \ln x$.**

Solution.

Let $y = \frac{1}{x} . \ln x$

then $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{x} . \ln x\right) = \frac{1}{x}\frac{d}{dx}(\ln x) + \ln x . \frac{d}{dx}\left(\frac{1}{x}\right)$

$= \frac{1}{x} . \frac{1}{x} + \ln x . \left(\frac{-1}{x^2}\right) = \frac{1}{x^2} - \frac{1}{x^2}\ln x = \frac{1}{x^2}(1 - \ln x)$

**EXAMPLE 3.**

**Differentiate : $[\ln (x+3)]^2$.**

**Solution.**

Let $y = [\ln (x+3)]^2$

then $\frac{dy}{dx} = \frac{d}{dx}\left[\{\ln (x+3)\}^2\right]$

$= 2\ln (x+3) . \frac{d}{dx}[\ln (x+3)] = 2\ln (x+3) . \frac{1}{x+3} . \frac{d}{dx}(x+3)$

$= 2\ln (x+3) . \frac{1}{x+3} . 1 = \frac{2\ln (x+3)}{x+3}$

**EXAMPLE 4.**

**Differentiate : $\ln (x^3+2)(x^2+3)$.**

**Solution.**

Let $y = \ln (x^3+2)(x^2+3)$

$= \ln (x^3+2) + \ln (x^2+3)$

$\frac{dy}{dx} = \frac{d}{dx}\left[\ln (x^3+2)\right] + \frac{d}{dx}\left[\ln (x^2+3)\right]$

$= \frac{1}{x^3+2} . \frac{d}{dx}(x^3+2) + \frac{1}{x^2+3} . \frac{d}{dx}(x^2+3)$

$= \frac{1}{x^3+2} . (3x^2) + \frac{1}{x^2+3} . (2x) = \frac{3x^2}{x^3+2} + \frac{2x}{x^2+3}$

## EXERCISE – 1.6

1. Differentiate the following :

   (i) $\frac{1}{\ln x} + \ln \frac{1}{x}$

   (ii) $\ln \sqrt{x} + \sqrt{\ln x}$

   (iii) $\ln \frac{x}{\sqrt{1+x^2}}$

   (iv) $x \ln(4 - x^2)$

2. Find the differential coefficient of the following :

   (i) $\ln \cos x$

   (ii) $\ln \tan x$

   (iii) $\sin (\ln \tan x)$

   (iv) $\ln (\sec x + \tan x)$

## 1.4 INTEGRATION

The inverse process of differentiation is called Integration. It is denoted by the symbol $\int$.

For example, $\frac{d}{dx}(x^2) = 2x \Rightarrow \int (2x)\, dx = x^2$

From the above formulas of derivatives, we have the following formulas for integration:

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$
2. $\int (ax+b)^n\, dx = \frac{(a+bx)^{n+1}}{a(n+1)}$
3. $\int \sin x\, dx = -\cos x$
4. $\int \cos x\, dx = \sin x$
5. $\int \sec^2 x\, dx = \tan x$
6. $\int \operatorname{cosec}^2 x\, dx = -\cot x$
7. $\int \sec x \tan x\, dx = \sec x$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$
9. $\int e^x\, dx = e^x$
10. $\int \frac{1}{x}\, dx = \ln x$
11. $\int \frac{1}{ax+b}\, dx = \frac{1}{a} \ln (ax+b)$

## ...HOD OF SUBSTITUTION.

We make substitution in such a way that one of the above-mentioned formulas becomes applicable.

**EXAMPLES.**

1. $\int (x+1)^2 dx = \frac{(x+1)^{2+1}}{(2+1)} = \frac{(x+1)^3}{3}$

2. $\int \frac{2x}{x^2+1} dx = \int \frac{du}{u}$ [ Put $x^2 = u$, then $2x\,dx = du$ ]

$= \ln u = \ln (x^2+1)$

3. $\int \frac{1}{(a-x)} dx = -\int \frac{du}{u}$ [ Put $a - x = u$, then $-dx = du$ or $dx = -du$ ]

$= -\ln u = -\ln (a-x)$

4. $\int \frac{1}{(a-x)^3} dx = \int (a-x)^{-3} dx = \frac{(a-x)^{-3+1}}{(-1)(-3+1)} = \frac{(a-x)^{-2}}{2} = \frac{1}{2(a-x)^2}$

5. $\int \sin 2x\, dx.$

Put $2x = y$, then $2 = \frac{dy}{dx} \Rightarrow 2\,dx = dy$

$\therefore \int \sin 2x\, dx = \int \sin y.\frac{dy}{2} = \frac{1}{2}\int \sin y\, dy = \frac{1}{2}.(-\cos y) = -\frac{1}{2}\cos y = -\frac{1}{2}\cos 2x$

6. $\int \sec^2 2x\, dx.$

Put $2x = y$, then $2 = \frac{dy}{dx} \Rightarrow 2\,dx = dy$

$\therefore \int \sec^2 2x\, dx = \int \sec^2 y.\frac{dy}{2} = \frac{1}{2}\int \sec^2 y\, dy = \frac{1}{2}.(\tan y) = -\frac{1}{2}\tan 2x$

7. **Vapour Pressure Depends on Temperature.** We have

$$\frac{dP}{dT} = \frac{\Delta H_V}{TV_V} \quad \ldots (2$$

If the vapour is treated as an ideal gas, then

$$V_V = \frac{RT}{P}$$

Substituting the value of $V_V$ in equation(2), we get

$$\frac{1}{P}\left(\frac{dP}{dT}\right) = \frac{\Delta H_V}{RT^2} \quad \text{or} \quad \frac{dP}{P} = \frac{\Delta H_V}{RT^2}\, dT \quad \ldots$$

If we assume that $\Delta H_V$ remains constant over a moderate temperature range. With approximation, integration of Eq.(3) gives, using

$$\int \frac{dx}{x} = \ln x \quad \text{and} \quad c\int \frac{dx}{x^2} = c\int x^{-2} dx = c\,\frac{x^{-2+1}}{-2+1} = -c\,\frac{1}{x},$$