Solution of Wave Equation

Wave motions that occur in nature, viz (i) Sound waves, surface waves, transverse vibrations of an infinite string, and of mechanical systems are governed by the wave equation. As our first example, we shall consider the transverse displacements of an infinite string.

Ex Compute the displacement $u(x,t)$ of an infinite string using the method of Fourier transform given that the string is initially at rest and that the initial displacement is $f(x)$, $-\infty < x < \infty$.

Soln The governing problem is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty \quad \text{--- (1)}$$

$$\left.\begin{aligned} u(x,0) &= f(x), \\ \frac{\partial u}{\partial t}(x,0) &= 0 \end{aligned}\right\} -\infty < x < \infty \quad (2)$$

Taking Fourier transform of (1) and (2) we get

$$\frac{d^2 \bar{u}(\alpha,t)}{dt^2} = -c^2\alpha^2 \bar{u}(\alpha,t)$$

$$\Rightarrow \boxed{\frac{d^2\bar{u}}{dt^2} + c^2\alpha^2 \bar{u} = 0} \quad \text{--- (3)}$$

$$\bar{u}(\alpha,0) = \bar{f}(\alpha), \quad \frac{d\bar{u}(\alpha,0)}{dt} = 0. \quad \text{--- (4)}$$

$$\therefore \bar{u}(x,t) = A\cos(c\alpha t) + B\sin(c\alpha t) \longrightarrow (5)$$

$$\bar{u}(x,0) = \bar{f}(x) = A = \bar{f}(\alpha) \therefore$$

$$\boxed{\bar{u}(x,t) = \bar{f}(\alpha)\cos(c\alpha t) + B\sin(c\alpha t)} \longrightarrow (6)$$

$$\frac{d\bar{u}(x,t)}{dt} = -c\alpha\bar{f}(\alpha)\sin(c\alpha t) + Bc\alpha\cos(c\alpha t)$$

$$\frac{d\bar{u}(x,0)}{dt} = (0) + Bc\alpha = 0$$

$$\Longrightarrow \boxed{B = 0}.$$

Thus from (6)

$$\boxed{\bar{u}(x,t) = \bar{f}(\alpha)\cos(c\alpha t)} \longrightarrow (7)$$

Taking inverse Fourier transform, we obtain

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)\cos(c\alpha t)\,e^{-i\alpha x}\,d\alpha. \longrightarrow (8)$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{i\alpha u}\,du\right]\left(\frac{e^{ic\alpha t}+e^{-ic\alpha t}}{2}\right)e^{-i\alpha x}\,d\alpha \quad *$$

$$= \frac{1}{2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\,e^{-ius}\,du\right\}\right]\left(e^{-icst}+e^{icst}\right)e^{isx}\,ds$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-isu}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x-ct)}\,ds\right\}du\right.$$
$$\left. + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-isu}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{is(x+ct)}\,ds\right\}du\right]$$

Soln of (3) is

$\bar{u}(\alpha,t) = A\cos(c\alpha t) + B\sin(c\alpha t) \longrightarrow (5)$

$\bar{u}(\alpha,0) = \bar{f}(\alpha) = A = \bar{f}(\alpha)$ ∴

$$\boxed{\bar{u}(\alpha,t) = \bar{f}(\alpha)\cos(c\alpha t) + B\sin(c\alpha t)} \longrightarrow (6)$$

$$\frac{d\bar{u}(\alpha,t)}{dt} = -c\alpha\bar{f}(\alpha)\sin(c\alpha t) + Bc\alpha\cos(c\alpha t)$$

$$\frac{d\bar{u}(\alpha,0)}{dt} = (0) + Bc\alpha = 0$$

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$$\left.+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)e^{-isu}\left\{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{is(x+ct)}\,ds\right\}du\right]$$

$$u(x,t) = \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-isu}\, \bar{f}(x-ct)\, du\right.$$

$$\left. + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, e^{-isu}\, \bar{f}(x+ct)\, du\right]$$

Using Fourier integral formula, we have

$$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right].$$

which is well-known D' Alemberts solution of the wave equation.

Ex $u_{tt} = c^2 u_{xx}$, $0 < x < \infty$, $t > 0$

$u(x,0) = f(x)$

$u_t(x,0) = g(x)$

Ans $$u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du.$$

Solution of Laplace Equation

One of the most important PDES that occurs in many applications is the Laplace equation. Steady state heat conduction, the electric potential in the steady flow of currents in solid conductors, the velocity potential of inviscid, irrotational fluids, the gravitational potential at any exterior point due to ellipsoidal Earth and so on, are all governed by Laplace equation. We shall now consider the following example.

Ex Solve the BVP (boundary value problem) in the half plane $y > 0$, described by

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \; y > 0 \quad \text{(1)}$$

$$u(x, 0) = f(x), \quad -\infty < x < \infty \quad \text{(2)}$$

u is bounded as $y \to \infty$; u and $\partial u/\partial x$ both vanish as $|x| \to \infty$.

Soln Taking Fourier exponential transform, we have

$$\mathcal{F}[u_{xx}] + \mathcal{F}[u_{yy}] = 0$$

$$-\alpha^2 \bar{u}(\alpha, y) + \frac{d^2 \bar{u}(\alpha, y)}{dy^2} = 0$$

$D^2 = \alpha^2$

$D = \pm\alpha$

Its general solution is

$$\boxed{\bar{u}(\alpha, y) = A(\alpha) e^{\alpha y} + B(\alpha) e^{-\alpha y}} \quad \text{(2)}$$

Since u must be bounded as $y \to \infty$, $\bar{u}(\alpha, y)$ and its Fourier transform also should be bounded as $y \to \infty$, implying $A(\alpha) = 0$ for $\alpha > 0$; but if $\alpha < 0$, $B(\alpha) = 0$; thus for any α.

$$\bar{u}(\alpha, y) = \text{Const}\left[e^{-|\alpha| y}\right] \longrightarrow (3)$$

Fourier transform of b.c.(2) gives

$$\bar{u}(\alpha, 0) = \mathcal{F}[f(x)] = \bar{F}(\alpha). \longrightarrow (4)$$

From (3) & (4) Const $= \bar{f}(\alpha)$

and thus from (3)

$$\bar{u}(\alpha, y) = \bar{f}(\alpha)\, e^{-|\alpha| y} \longrightarrow (5)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{+i\alpha\xi}\, d\xi\; e^{-|\alpha| y}$$

Taking inverse Fourier transform, we have

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-|\alpha| y}\, e^{i\alpha\xi}\, d\xi\right] e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\xi)\, d\xi \int_{-\infty}^{\infty} \exp\left[\left\{\alpha\left[i(\xi - x)\right] - |\alpha| y\right\}\right] d\alpha$$

But

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left\{\alpha\left[i(\xi - x)\right] - |\alpha| y\right\} d\alpha \longrightarrow (6)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{0} \exp\left\{\alpha\left[y + i(\xi - x)\right]\right\} d\alpha + \frac{1}{2\pi} \int_{0}^{\infty} \exp\left\{-\alpha\left[y - i(\xi - x)\right]\right\}$$

$$\frac{1}{2\pi}\left[\frac{\exp\{\alpha[y+i(\xi-x)]\}}{y+i(\xi-x)}\right]_{-\infty}^{0}$$

$$-\frac{1}{2\pi}\left[\frac{\exp\{-\alpha[y-i(\xi-x)]\}}{y-i(\xi-x)}\right]_{0}^{\infty}$$

$$=\frac{1}{2\pi}\left[\frac{1}{y+i(\xi-x)}+\frac{1}{y-i(\xi-x)}\right]$$

$$=\frac{1}{\pi}\frac{y}{(\xi-x)^2+y^2}. \longrightarrow (7)$$

Making use of (7) in (6) we get

$$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(\xi-x)^2+y^2}. \longrightarrow (8)$$

This solution is a well-known Poisson integral formula and is valid for $y>0$, when $f(x)$ is bounded and piecewise continuous for all real x.

Ex [Solve] $u_{xx}+u_{yy}(x,y)=0, \quad -\infty<x<\infty, \ y>0$ ——— (1)

$u_y(x,0)=f(x), \quad -\infty<x<\infty$

u is bounded as $y\longrightarrow\infty$

u and $\partial u/\partial x$ both vanish as $|x|\longrightarrow\infty$.

Let us define a ftn
$$\phi(x,y) = u_y(x,y) \longrightarrow (1a)$$
Then (1) becomes

$$\phi_{xx} + \phi_{yy} = \frac{\partial}{\partial y}\left[u_{xx} + u_{yy}\right] = 0$$

$$\Rightarrow \quad \phi_{xx} + \phi_{yy} = 0 \longrightarrow (2)$$

$$\phi(x,0) = f(x) \longrightarrow (3)$$

Note that (2) with (3) is solved in previous example and thus from (P-36, Eq. 8) we have
$$\phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(\xi-x)^2+y^2}, \; y>0 \longrightarrow (4)$$
or by (1a)
$$\frac{\partial u}{\partial y}(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{(\xi-x)^2+y^2}$$
Intg
$$u(x,y) = \int \frac{\partial u}{\partial y}(x,y)\,dy$$
$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\left[\int \frac{y\,dy}{(x-\xi)^2+y^2}\right]d\xi$$
$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\,\log\left[(x-\xi)^2+y^2\right]d\xi + \text{const}$$

(38)

(37) Using Fourier Cosine transform, find the temperature $u(x,t)$ in a semi-infinite rod determined by the PDE

$$u_t = k\,u_{xx}, \quad 0<x<\infty,\ t>0 \quad \text{--- (1)}$$

subject to IC: $u(x,0) = 0, \quad 0 \le x < \infty$ ——— (1a)

BC: $u_x(0,t) = -u_0$ (a constant) when $x=0$ and $t>0$.

u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \infty$.

Soln: Taking Fourier Cosine transform of (1) we have

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial t}\cos\alpha x\,dx = k\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha x\,dx$$

$$= k\sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\cos\alpha x\right]_0^\infty + k\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\partial u}{\partial x}\sin\alpha x\,dx$$

Since $\frac{\partial u}{\partial x} \to 0$ as $x \to \infty$

$\therefore$ we get

$$\frac{d\bar{u}_c(\alpha,t)}{dt} = -k\sqrt{\frac{2}{\pi}}\left(\frac{\partial u}{\partial x}\right)_{x=0} + k\alpha\sqrt{\frac{2}{\pi}}\left(u\sin\alpha x\right)_0^\infty - k\alpha^2\sqrt{\frac{2}{\pi}}\int_0^\infty u\cos\alpha x\,dx$$

If $u \to 0$ as $x \to \infty$, we have

$$\frac{d\bar{u}_c}{dt} = \sqrt{\frac{2}{\pi}}\,k u_0 - k\alpha^2\bar{u}_c$$

$$\Rightarrow \boxed{\frac{d\bar{u}_c}{dt} + k\alpha^2\bar{u}_c = \sqrt{\frac{2}{\pi}}\,k u_0} \longrightarrow (2)$$

For the general soln of (2) we can write

$$\frac{d}{dt}\left[e^{k\alpha^2 t}\,\bar{u}_c\right] = \sqrt{\frac{2}{\pi}}\,k u_0\, e^{k\alpha^2 t}.$$

Integ

$$e^{k\alpha^2 t}\,\bar{u}_c = \sqrt{\frac{2}{\pi}}\,k u_0 \int e^{k\alpha^2 t}\,dt + A \quad \text{—— (3)}$$

Taking the Fourier cosine transform of (1a) we get

$$\bar{u}_c(\alpha, 0) = 0 \quad \text{—— (4)}$$

From (3) + (4) we have

$$0 = \sqrt{\frac{2}{\pi}}\,\frac{\cancel{k} u_0}{\cancel{k}\alpha^2} + A$$

$$\Rightarrow \quad A = -\frac{u_0}{\alpha^2}\sqrt{\frac{2}{\pi}}$$

$\therefore$ From (3)

$$e^{k\alpha^2 t}\,\bar{u}_c = \sqrt{\frac{2}{\pi}}\,\cancel{k} u_0\,\frac{e^{k\alpha^2 t}}{\cancel{k}\alpha^2} - \frac{u_0}{\alpha^2}\sqrt{\frac{2}{\pi}}$$

$$= \frac{u_0}{\alpha^2}\sqrt{\frac{2}{\pi}}\left[e^{k\alpha^2 t} - 1\right]$$

$$\boxed{\bar{u}_c = \frac{u_0}{\alpha^2}\sqrt{\frac{2}{\pi}}\left[1 - e^{-k\alpha^2 t}\right]}$$