Sol: By def

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \alpha^n e^{-\alpha a} \cos\alpha x \, d\alpha \quad \text{—} (1)$$

We know from calculus that

$$\int_0^\infty e^{-\alpha a} \cos\alpha x \, d\alpha = \frac{a}{a^2+x^2} \quad \text{—} (2)$$

Diff (2) n times w.r.t a, we have

$$(-1)^n \int_0^\infty \alpha^n e^{-\alpha a} \cos\alpha x \, d\alpha = \frac{d^n}{da^n}\left(\frac{a}{a^2+x^2}\right)$$

$$= \frac{1}{2}\frac{d^n}{da^n}\left(\frac{1}{a-ix} + \frac{1}{a+ix}\right)$$

$$= \frac{1}{2}\left[(-1)^n n!\,(a-ix)^{-n-1} + (-1)^n n!\,(a+ix)^{-n-1}\right]$$

$$= \frac{(-1)^n n!}{2}\left[(a-ix)^{-n-1} + (a+ix)^{-n-1}\right] \quad \text{—} (3)$$

Let $a+ix = r(\cos\theta + i\sin\theta)$

$a = r\cos\theta, \quad x = r\sin\theta$

$r^2 = a^2+x^2, \quad \tan\theta = \frac{x}{a}$

Thus,

$$(a+ix)^{-n-1} = r^{-n-1}\left[\cos(-n-1)\theta + i\sin(-n-1)\theta\right]$$

$$(a-ix)^{-n-1} = r^{-n-1}\left[\cos(-n-1)\theta - i\sin(-n-1)\theta\right]$$

Then

$$f(x) = \int_0^\infty \alpha^n e^{-\alpha a} \cos\alpha x \, d\alpha = \sqrt{\frac{2}{\pi}} \frac{n!\cos(n+1)\theta}{(a^2+x^2)^{(n+1)/2}}$$

from (3)

Ex Find the Fourier transform of

(i) $\partial^n u/\partial x^n$ of the function $u(x,t)$ assuming that u and its first $(n-1)$ derivatives with respect to x vanish as $x \to \pm\infty$.

(ii) $\partial u/\partial t$.

Also, find the sine and cosine transforms of $\dfrac{\partial^2 u}{\partial x^2}$ of the ftn $u(x,t)$.

Soln (i) $\mathcal{F}[u(x,t)] = \bar{u}(\alpha,t) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} e^{i\alpha x} u(x,t)\,dt.$

$$\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underset{\text{II}}{\frac{\partial u}{\partial x}}\, \underset{\text{I}}{e^{i\alpha x}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\left\{ \cancel{\left. u(x,t)\, e^{i\alpha x}\right|_{-\infty}^{\infty}} - i\alpha\int_{-\infty}^{\infty} u(x,t)\, e^{i\alpha x}\,dx \right\} \longrightarrow (1)$$

It is assumed that

$$\lim_{x \to \pm\infty} u(x,t) = 0$$

Hence from (1)

$$\boxed{\mathcal{F}\left[\frac{\partial u}{\partial x}\right] = -i\alpha\, \bar{u}(\alpha,t)} \longrightarrow (2)$$

Now

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2}\, e^{i\alpha x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underset{\text{I}}{e^{i\alpha x}}\, \underset{\text{II}}{d\left(\frac{\partial u}{\partial x}\right)}$$

(2B)

$$\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = \frac{1}{\sqrt{2\pi}}\left[\left(\frac{\partial u}{\partial x}e^{i\alpha x}\right)_{-\infty}^{\infty} - i\alpha\left(e^{i\alpha x}u\right)_{-\infty}^{\infty} + (i\alpha)^2\int_{-\infty}^{\infty} e^{i\alpha x}u\,dx\right]$$

Assuming that both u and $\partial u/\partial x$ tend to zero as $x \longrightarrow \pm\infty$, we have

$$\boxed{\mathcal{F}\left[\frac{\partial^2 u}{\partial x^2}\right] = (-1)^2 (i\alpha)^2\, \bar{u}(\alpha, t).} \longrightarrow (3)$$

In general

$$\boxed{\mathcal{F}\left[\frac{\partial^{(n)} u(x,t)}{\partial x^{(n)}}\right] = (-1)^n (i\alpha)^n\, \bar{u}(\alpha, t)} \longrightarrow (4)$$

$$\mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x}\frac{\partial u(x,t)}{\partial t}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\frac{d}{dt}\int_{-\infty}^{\infty} e^{i\alpha x}u(x,t)\,dt$$

$$= \frac{d\bar{u}}{dt}.$$

In the case of Fourier Sine and Cosine transforms, we have

$$\mathcal{F}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \sin\alpha x\,\frac{\partial^2 u}{\partial x^2}\,dx. \quad (5)$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{\partial u}{\partial x}\sin\alpha x\right]_0^{\infty} - \sqrt{\frac{2}{\pi}}\,\alpha\int_0^{\infty}\cos\alpha x\,\frac{\partial u}{\partial x}\,dx$$

We assume that $\frac{\partial u}{\partial x} \longrightarrow 0$ as $x \longrightarrow \infty$. Then the

RHS term of the above equation becomes

$$-\sqrt{\frac{2}{\pi}}\,\alpha\int_0^{\infty}\cos\alpha x\,\frac{\partial u}{\partial x}\,dx = -\sqrt{\frac{2}{\pi}}\,\alpha\left\{\left[u(x,t)\cos\alpha x\right]_0^{\infty} + \alpha\int_0^{\infty}u(x,t)\sin\alpha x\,dx\right\}$$

Also, assuming that $u(x,t)\to 0$ as $x\to\infty$, this equation becomes

$$\sqrt{\frac{2}{\pi}}\,\alpha\,u(x,t)\Big|_{x=0} - \alpha^2 F_s\left[u(x,t)\right]$$

Hence

$$F_s\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right] = \sqrt{\frac{2}{\pi}}\,\alpha\,u(x,t)\Big|_{x=0} - \alpha^2 F_s\left(u(x,t)\right)$$

Similarly, it can be shown that if

$u(x,t)\to 0$ and $\frac{\partial u}{\partial x}\to 0$ as $x\to\infty$

~~then~~

$$F_c\left[\frac{\partial^2 u}{\partial x^2}\right] = -\sqrt{\frac{2}{\pi}}\,\frac{\partial u(x,t)}{\partial x}\Big|_{x=0} - \alpha^2 F_c\left[u(x,t)\right]$$

Note Obviously, the choice of the sine or cosine transform is decided by the form of the boundary condition at the lower limit of the variable selected for exclusion. Thus we observe that for the exclusion of $\partial^2 u/\partial x^2$ from a given PDE, we require

$u|_{x=0}$ in the case of sine transform

$\frac{\partial u}{\partial x}\Big|_{x=0}$ in the case of cosine transform.

Convolution Thm (Faltung Thm)

If $\bar{f}(\alpha)$ and $\bar{g}(\alpha)$ are the Fourier transforms of functions $f(x)$ and $g(x)$, then the product $\bar{f}(\alpha)\bar{g}(\alpha)$ is the Fourier transform of the convolution product $f*g = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)g(x-u)\,du$

$f*g = g*f$ (convolution is commutative)

The product $f*g$ is called the convolution or Faltung of the functions f and g over the interval $(-\infty, \infty)$..

Special cases (Sine and cosine convolution integrals)

If $\bar{f}_c(\alpha)$ and $\bar{g}_c(\alpha)$ are the Fourier cosine transforms of $f(x)$ and $g(x)$, then

$$\int_0^{\infty} \bar{f}_c(\alpha)\,\bar{g}_c(\alpha)\cos\alpha x\,d\alpha = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} g(\eta)\,d\eta\int_0^{\infty}\bar{f}_c(\alpha)\Big[\cos|x-\eta|\alpha + \cos(x+\eta)\alpha\Big]\,d\alpha$$

If $\bar{f}_s(\alpha)$ and $\bar{g}_s(\alpha)$ are the Fourier sine transforms of $f(x)$ and $g(x)$, then

$$\int_0^{\infty}\bar{f}_s(\alpha)\bar{g}_s(\alpha)\sin\alpha x\,d\alpha = \frac{1}{2}\int_0^{\infty} f(\eta)\Big[g(|x-\eta|)-g(x+\eta)\Big]\,d\eta.$$

ii) $$\int_0^{\infty}\bar{f}_c(\alpha)\,\bar{g}_s(\alpha)\sin\alpha x\,d\alpha = \frac{1}{2}\int_0^{\infty} g(\eta)\Big[f(|x-\eta|) - f(x+\eta)\Big]\,d\eta.$$

iv) $$\int_0^{\infty}\bar{f}_s(\alpha)\,\bar{g}_c(\alpha)\sin\alpha x\,d\alpha = \frac{1}{2}\int_0^{\infty} f(\eta)\Big[g(|x-\eta|) - g(\eta+x)\Big]\,d\eta.$$

Parseval's Relation

$$\int_{-\infty}^{\infty} |\bar{F}(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(u)|^2 du.$$

Pf From Fourier convolution, we have

$$f*g = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} \bar{F}(\alpha)\bar{g}(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du. \to (1)$$

If we set $x=0$ in (1) we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(\alpha)\bar{g}(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(-u)\, du \to (2)$$

For the case $g(-u) = f^*(u)$ [where $f^*(u)$ is the complex conjugate of the function $f(u)$], we have

$$\bar{g}(\alpha) = \mathcal{F}[g(u)] = \mathcal{F}[f^*(-u)] = \bar{F}^*(\alpha) \quad (3)$$

Thus (2) yields

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(\alpha)\bar{F}^*(\alpha)\, d\alpha = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, f^*(u)\, du$$

complex conjugate property $|z\bar{z}| = |z|^2$

Therefore,

$$\int_{-\infty}^{\infty} |\bar{F}(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |f(u)|^2 du.$$

which is known as Parseval's relation.

Example Using the Fourier cosine transform of e^{-ax} and e^{-bx}, show that

$$\int \frac{dx}{(a^2+x^2)(b^2+x^2)} = \frac{\pi}{2ab(a+b)}, \quad a>0,\ b>0.$$

Let $f(x) = e^{-bx}$, $g(x) = e^{-ax}$, then

$$\bar{f}_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-bx} \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \frac{b}{b^2+\alpha^2}$$

Similarly

$$\bar{g}_c(\alpha) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+\alpha^2}$$

Now as $\displaystyle\int_0^\infty \bar{f}_c(\alpha) \bar{g}_c(\alpha) \, d\alpha = \int_0^\infty \bar{f}_c(\alpha) \, d\alpha \left[\sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos \alpha x \, dx \right]$

$$= \int_0^\infty g(x) \, dx \left[\sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\alpha) \cos \alpha x \, d\alpha \right]$$

$$= \int_0^\infty g(x) f(x) \, dx \longrightarrow (1)$$

Using $f(x)$ and $g(x)$ in (1) we have

$$\int_0^\infty \bar{f}_c(\alpha) \bar{g}_c(\alpha) \, d\alpha = \int_0^\infty e^{-(a+b)x} \, dx = \frac{1}{a+b}$$

$$\Rightarrow \int_0^\infty \frac{d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} \frac{(2ab)}{\pi} = \boxed{\frac{1}{a+b}}$$

Ans

$$\Rightarrow \int_0^\infty \frac{d\alpha}{(a^2+\alpha^2)(b^2+\alpha^2)} = \frac{\pi}{2ab(a+b)}, \quad a>0,\ b>0$$

Transform of Dirac Delta ftn.

$$\mathcal{F}[\delta(t-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha t}\,\delta(t-a)\,dt = \frac{e^{i\alpha a}}{\sqrt{2\pi}}$$

when $a=0$ then $\left[\text{as } \int_{-\infty}^{\infty}\delta(t-a)f(t)\,dt = f(a)\right]$

$$\boxed{\mathcal{F}[\delta(t)] = \frac{1}{\sqrt{2\pi}}}$$

$$\mathcal{F}^{-1}[1] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\alpha t}(1)\,d\alpha = \sqrt{2\pi}\,\delta(t).$$

Solution of Diffusion Equation

Let us consider the problem of flow of heat in an infinite medium $-\infty < x < \infty$, when the initial temperature distribution $f(x)$ is known and no heat sources are present. Mathematically, we have to solve the problem described in the following example.

Ex Solve

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\ t > 0 \qquad (1)$$

$u(x,t)$ and $u_x(x,t)$ both $\to 0$ as $|x| \to \infty$.

$u(x,0) = f(x), \quad -\infty < x < \infty.$

Soln: Taking Fourier transform, we have

$$\frac{d\bar{u}(\alpha,t)}{dt} = -k\alpha^2\,\bar{u}(\alpha,t).$$