Properties of Fourier transform

In many practical situations, determination of Fourier transform of certain functions is very complex. Once we know the transform of some elementary functions, we can find the transform of many other ftns with the help of the properties associated with the Fourier transform. We now discuss some of the important properties of the Fourier transform.

Linearity Property

If $\bar{F}(\alpha)$ and $\bar{g}(\alpha)$ are the Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = c_1 \mathcal{F}[f(x)] + c_2 \mathcal{F}[g(x)]$$

$$= c_1 \bar{f}(\alpha) + c_2 \bar{g}(\alpha).$$

where c_1 and c_2 are constants.

Pf: $$\mathcal{F}\left[c_1 f(x) + c_2 g(x)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x}\left[c_1 f(x) + c_2 g(x)\right] dx$$

$$= \frac{c_1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} f(x)\, dx + \frac{c_2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\alpha x} g(x)\, dx$$

$$= c_1 \bar{f}(\alpha) + c_2 \bar{g}(\alpha)$$

Change of Scale

If $\bar{f}(\alpha)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(ax)$ is

$$\frac{1}{a}\bar{f}\left(\frac{\alpha}{a}\right).$$

By def.

$$F[f(x)] = \bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(ax)\,dx. \longrightarrow (1)$$

Letting $ax = t \Rightarrow dx = dt/a$. Therefore from (1)

$$F[f(ax)] = \frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\left(\frac{\alpha}{a}\right)t} f(t)\,dt$$

$$\boxed{F[f(ax)] = \frac{1}{a}\bar{f}(\alpha/a),\ a>0.} \longrightarrow (2)$$

Similarly,

$$\boxed{F[f(ax)] = \frac{-1}{a}\bar{f}(\alpha/a),\ a<0} \longrightarrow (3)$$

Combining Eqs. (2) and (3) we have

$$\boxed{F[f(ax)] = \frac{1}{|a|}\bar{f}(\alpha/a),} \ a\neq 0 \longrightarrow (4)$$

Also

$$F_s[f(ax)] = \frac{1}{a}\bar{f}_s(\alpha/a), \quad a>0 \quad \rightarrow (5)$$

$$F_c[f(ax)] = \frac{1}{a}\bar{f}_c(\alpha/a), \quad a>0 \quad \rightarrow (6)$$

(3) Shifting Property

If $\bar{f}(\alpha)$ is the Fourier transform of $f(x)$, then the Fourier transform of $f(x-a)$ is

$$F[f(x-a)] = e^{i\alpha a}\bar{f}(\alpha).$$

Pf. $\bar{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x-a)\,dx \quad — (1)$$

Setting $x-a=t \Rightarrow dx=dt$

$$F[f(x-a)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha(a+t)} f(t)\,dt$$

$$= e^{i\alpha a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha t} f(t)\,dt$$

$$= e^{i\alpha a}\bar{f}(\alpha).$$

$$\Rightarrow F^{-1}[e^{i\alpha a}\bar{f}(\alpha)] = f(x-a).$$

(4) Modulation Property

If $\bar{f}(\alpha)$ is the Fourier transform of $f(x)$, then

Fourier transform of $f(x)\cos ax$ is
$$\frac{1}{2}\left[\bar{f}(\alpha-a)+\bar{f}(\alpha+a)\right].$$

Pf:.. By def.
$$\bar{f}(\alpha)=\mathcal{F}[f(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\,dx$$

$$\mathcal{F}[f(x)\cos ax]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\left[\frac{e^{iax}+e^{-iax}}{2}\right]dx$$

$$=\frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\alpha+a)x} f(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\alpha-a)x} f(x)\,dx\right]$$

$$=\frac{1}{2}\left[\bar{f}(\alpha+a)+\bar{f}(\alpha-a)\right].$$

In the same fashion it can be established that

$$\mathcal{F}_s[f(x)\cos ax]=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \sin\alpha x\,dx.$$

$$=\sqrt{\frac{2}{\pi}}\,\frac{1}{2}\int_0^{\infty} f(x)\left[\sin(\alpha+a)x+\overset{-\sin(a-\alpha)}{\sin(\alpha-a)x}\right]dx$$

$\because 2\sin\alpha\cos\beta=\sin(\alpha+\beta)+\sin(\alpha-\beta)$

$$=\frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\left\{\int_0^{\infty} f(x)\sin(\alpha+a)x\,dx+\int_0^{\infty} f(x)\sin(\alpha-a)x\,dx\right\}\right].$$

$$=\boxed{\frac{1}{2}\left[\bar{f}_s(\alpha+a)\pm\bar{f}_s(\alpha-a)\right]}$$

and

$$\mathcal{F}_c\left[f(x)\cos ax\right] = \frac{1}{2}\left[\bar{f}_c(\alpha+a) + \bar{f}_c(\alpha-a)\right]$$

$$\mathcal{F}_s\left[f(x)\sin ax\right] = \frac{1}{2}\left[\bar{f}_c(\alpha-a) - \bar{f}_c(\alpha+a)\right]$$

$$\mathcal{F}_c\left[f(x)\sin ax\right] = \frac{1}{2}\left[\bar{f}_s(\alpha+a) - \bar{f}_s(\alpha-a)\right].$$

Differentiation Property

If $f(x)$ and its first $(r-1)$ derivatives are continuous, and if its rth derivative is piecewise continuous, then

$$\mathcal{F}\left[f^{(r)}(x)\right] = \bar{f}^{(r)}(\alpha) = (-i\alpha)^r \mathcal{F}[f(x)]$$

$$= (-i\alpha)^r \bar{f}(\alpha),\ r = 0, 1, 2, \cdots$$

provided f and its derivatives are absolutely integrable. In addition, we assume that $f(x)$ and its first $(r-1)$ derivatives vanish as $x \to \pm\infty$.

Pf By def.

$$\mathcal{F}\left[f^{(r)}(x)\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^r f}{dx^r} e^{i\alpha x}\,dx$$

$$= \bar{f}^{(r)}(\alpha) \quad \text{——— (1)}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{d^{r-1}f}{dx^{r-1}} e^{i\alpha x}\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{d^{r-1}f}{dx^{r-1}}(i\alpha)\, e^{i\alpha x}\,dx. \quad \longrightarrow (2)$$

(Integ. by parts)

(10)

assume that d^{r-1}/dx^{r-1} tends to zero as $\pm\infty$, we may write the above result in form

$$\bar{f}^{(r)}(\alpha) = -(i\alpha)\bar{f}^{(r-1)}(\alpha) = (-i\alpha)^2 \bar{f}^{(r-2)}(\alpha)$$
$$= (-i\alpha)^r \bar{F}(\alpha).$$

Hence $\boxed{\bar{f}^{(r)}(\alpha) = (-i\alpha)^r \bar{F}(\alpha)}$ → (3)

From (1) and (3)

$$\boxed{F\left[f^{(r)}(x)\right] = (-i\alpha)^r \bar{F}(\alpha)}$$

Similarly

$$\bar{f}_c^{(2r)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1}\, \alpha^{2n} + (-1)^{r+1} \alpha^{2r} \bar{f}_c(\alpha),$$

$$\bar{f}_c^{(2r+1)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n}\, \alpha^{2n} + (-1)^{r+1} \alpha^{2r+1} \bar{f}_s(\alpha),$$

where

$$a_{r-1} = \lim_{x\to 0} \sqrt{\frac{2}{\pi}}\, \frac{d^{r-1}f}{dx^{r-1}} \text{ (say)}.$$

Note (i) When $x=0$ and $\frac{df}{dx} = \frac{d^3f}{dx^3} = 0$, then

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^2f}{dx^2} \cos\alpha x\, dx = -\alpha^2 \bar{f}_c(\alpha)$$

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{d^4f}{dx^4} \cos\alpha x\, dx = \alpha^4 \bar{f}_c(\alpha).$$

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$$\bar{f}^{(r)}(\alpha) = -(i\alpha)\bar{f}^{(r-1)}(\alpha) = (-i\alpha)^2 \bar{f}^{(r-2)}(\alpha)$$

$$= (-i\alpha)^r \bar{F}(\alpha).$$

Hence $\boxed{\bar{f}^{(r)}(\alpha) = (-i\alpha)^r \bar{F}(\alpha)} \longrightarrow (3)$

From (1) and (3)

$$\boxed{F\left[f^{(r)}(x)\right] = (-i\alpha)^r \bar{F}(\alpha)}$$

Similarly

$$\bar{f}_c^{(2r)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n-1}\,\alpha^{2n} + (-1)^{r+1}\alpha^{2r}\bar{f}_c(\alpha),$$

$$\bar{f}_c^{(2r+1)}(\alpha) = -\sum_{n=0}^{r-1} (-1)^n a_{2r-2n}\,\alpha^{2n} + (-1)^{r+1}\alpha^{2r+1}\bar{f}_s(\alpha),$$

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Note (i) When $x=0$ and $\frac{df}{dx} = \frac{d^3f}{dx^3} = 0$, then

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$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d^4f}{dx^4}\cos\alpha x\,dx = \alpha^4\bar{f}_c(\alpha).$$

(4)

(ii) When $x=0$ and $f=\frac{d^2f}{dx^2}=0$, then

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d^2f}{dx^2}\sin\alpha x\,dx = -\alpha^2\bar{f}_s(\alpha),$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d^4f}{dx^4}\sin\alpha x\,dx = \alpha^4\bar{f}_s(\alpha).$$

Ex Find the Fourier cosine transform of e^{-at^2}.

$$F_c\left[e^{-at^2}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at^2}\cos\alpha t\,dt = I_{(1)}\ (\text{say}) \longrightarrow (1)$$

Differentiating (1) w.r.t α we get.

$$\frac{dI}{d\alpha} = -\sqrt{\frac{2}{\pi}}\int_0^\infty t e^{-at^2}\sin\alpha t\,dt$$

$$= \frac{1}{2a}\sqrt{\frac{2}{\pi}}\int_0^\infty \sin\alpha t\, d(e^{-at^2})$$

$$= \frac{1}{2a}\sqrt{\frac{2}{\pi}}\left[\cancel{e^{-at^2}\sin\alpha t}\Big|_0^\infty - \alpha\int_0^\infty e^{-at^2}\cos\alpha t\,dt\right]$$

$d\int = \int d$ only when the function is uniform convergent as in this case $\cos\alpha t$ uniform converge

$$\Rightarrow \frac{dI}{d\alpha} + \frac{\alpha}{2a}I = 0 \text{ using } (1).$$

$$\frac{dI}{I} = -\frac{\alpha}{2a}d\alpha.$$

Integ

$$\boxed{I = Ce^{-\alpha^2/4a}} \longrightarrow (2)$$

when $\alpha=0$ then $I(0)=C$. ∴

$$I(\alpha) = I(0)e^{-\alpha^2/4a} \longrightarrow (3)$$

From (1) $I(0) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-at^2}(1)\,dt = \sqrt{\frac{2}{\pi}}\cdot\frac{\sqrt{\pi}}{2\sqrt{a}} = \frac{1}{\sqrt{2a}}$ (4)

(5) (4) in (3) we get.

$$F_c\left[e^{-at^2}\right] = \frac{1}{\sqrt{2a}} e^{-x^2/4a}.$$

If the Fourier sine transform of $f(x)$ is $\alpha/1+\alpha^2$, find $f(x)$.

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha \sin \alpha x\, d\alpha}{1+\alpha^2} - \sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha^2 \sin \alpha x + \sin \alpha x}{\alpha(1+\alpha^2)} d\alpha - \sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x(\alpha^2+1)}{\alpha(\alpha^2+1)} d\alpha - \sqrt{\frac{\pi}{2}}$$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha} - \sqrt{\frac{\pi}{2}}$$

Soln By def.

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha}{1+\alpha^2} \sin \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(\alpha^2+1)-1}{\alpha(1+\alpha^2)} \sin \alpha x \, d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha} d\alpha - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)} d\alpha. \longrightarrow (1)$$

As $\displaystyle\int_0^\infty \frac{\sin \alpha x}{\alpha} = \pi/2.$

$\therefore$ from (1)

$$f(x) = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)} d\alpha$$

$$= \sqrt{\frac{\pi}{2}} - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha(1+\alpha^2)} d\alpha \longrightarrow (2)$$

$$\frac{df}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos \alpha x}{1+\alpha^2} d\alpha \longrightarrow (3)$$

$$\frac{d^2f}{dx^2} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2} d\alpha. \longrightarrow (4)$$

Using (2) and (4), it follows that

$$\frac{d^2f}{dx^2} - f = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin \alpha x}{\alpha}\, d\alpha - \sqrt{\frac{\pi}{2}}$$

$$\sim \sqrt{\frac{\pi}{2}} - \sqrt{\frac{\pi}{2}}$$

$$\Rightarrow \frac{d^2f}{dx^2} - f = 0$$

soln is

$$f = c_1 e^{x} + c_2 e^{-x} \longrightarrow (5)$$

$$\frac{df}{dx} = c_1 e^{x} - c_2 e^{-x} \longrightarrow (6)$$

when $x=0$ then $f(0) = \sqrt{\pi/2}$ from (2)

and from (3)

$$\frac{df(0)}{dx} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{d\alpha}{1+\alpha^2} = -\sqrt{\frac{\pi}{2}}. \longrightarrow (7)$$

From (5) & (6)

$$c_1 + c_2 = f(0) = \sqrt{\frac{\pi}{2}}$$

$$c_1 - c_2 = -\sqrt{\frac{\pi}{2}}$$

Solving $c_1 = 0$, $c_2 = \sqrt{\pi/2}$.

∴ from (5)

$$\boxed{f(x) = \sqrt{\frac{\pi}{2}}\, e^{-x}}$$

(*) If the Fourier cosine transform of $f(x)$ is $\alpha^n e^{-a\alpha}$, find $f(x)$.