(*) If $f(t)$ is a piecewise continuous function and satisfies the condition of exponential order C_0 such that $\lim_{t \to 0} f(t)/t$ exists, then for $s > C_0$,

$$L^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(x)\,dx.$$

Pf: Let $G(t) = \int_0^t f(x)\,dx$ —— (1)

$G(0) = 0$ (from (1)) —— (2)

$G'(t) = f(t)$

Taking Laplace

$$L[G'(t)] = L[f(t)]$$

$$sL[G(t)] - \cancel{G(0)} = \bar{f}(s)$$

by (2)

$$\Rightarrow L[G(t)] = \frac{\bar{f}(s)}{s}$$

$$G(t) = L^{-1}\left[\frac{\bar{f}(s)}{s}\right]$$

using (1) we can write

$$\boxed{\int_0^t f(x)\,dx = L^{-1}\left[\frac{\bar{f}(s)}{s}\right]}$$

This can be generalized as

$$L^{-1}\left[\frac{\bar{f}(s)}{s^n}\right] = \int_0^t \int_0^t \int_0^t \cdots \int_0^t f(t)\,dt^n.$$

(*) Change of Scale Property

If $L^{-1}[\bar{f}(s)] = f(t)$ then $L^{-1}[\bar{f}(\alpha s)] = \frac{1}{\alpha} f\left(\frac{t}{\alpha}\right)$.

$\boxed{Ex}$ Find the inverse Laplace transform of

(i) $\dfrac{1}{(s+a)^n}$ (ii) $\dfrac{s+2}{s^2-4s+13}$ (iii) $\dfrac{2s-3}{s^2-s-3/4}$

(iv) $\dfrac{s^3}{(s^2+a^2)^2}$

Soln: (i) Using the shifting property, we have

$$L^{-1}\left[\frac{1}{(s+a)^n}\right] = e^{-at} L^{-1}\left[\frac{1}{s^n}\right] = \frac{e^{-at}\, t^{n-1}}{(n-1)!}$$

(ii) $\dfrac{s+2}{s^2-4s+13} = \dfrac{(s-2)+4}{(s-2)^2+3^2}$

$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+3^2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+3^2}\right]$$

$$= e^{2t} L^{-1}\left[\frac{s}{s^2+3^2}\right] + 4e^{2t} L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

(by shifting property)

$$= e^{2t}\cos 3t + \frac{4}{3} e^{2t} \sin 3t$$

$\boxed{(iii)}$ $\dfrac{2s-3}{s^2-s-3/4} = \dfrac{2s-3}{\left(s-\frac{1}{2}\right)^2-1} = \dfrac{2\left(s-\frac{1}{2}\right)-2}{\left(s-\frac{1}{2}\right)^2-1}$

$$L^{-1}\left[\frac{2s-3}{s^2-s-3/4}\right] = 2L^{-1}\left[\frac{s-\frac{1}{2}}{\left(s-\frac{1}{2}\right)^2-1}\right] - 2L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2-1}\right]$$

$$= 2e^{t/2} L^{-1}\left[\frac{s}{s^2-1^2}\right] - 2e^{t/2} L^{-1}\left[\frac{1}{s^2-1^2}\right]$$

(by shifting property).

$$= 2e^{t/2}\cosh t - 2e^{t/2}\sinh t$$

(iv) $\dfrac{s^3}{(s^2+a^2)^2} = \dfrac{s(s^2+a^2-a^2)}{(s^2+a^2)^2}$

$\mathcal{L}[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n} \bar{f}(s)$

$= \dfrac{s}{(s^2+a^2)} - \dfrac{a^2 s}{(s^2+a^2)^2}$

$\mathcal{L}[\sin at] = \dfrac{a}{s^2+a^2} = \bar{f}(s)$

$L^{-1}\left[\dfrac{s^3}{(s^2+a^2)^2}\right] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] - a^2 L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

$\dfrac{d}{ds}\bar{f}(s) = \dfrac{-2as}{(s^2+a^2)}$

$= \cos at - \dfrac{a}{2} t \sin at.$

$= \mathcal{L}[t \sin at]$

Ex Find the inverse Laplace transform of

(i) $\ln \dfrac{s^2+1}{s(s+1)}$ (ii) $\cot^{-1}(s/k)$.

Soln We have $L[t^n f(t)] = (-1)^n \bar{f}^{(n)}(s)$

for $n=1$ $L[t f(t)] = -\bar{f}'(s)$

$+\bar{f}'(s) = -L[t f(t)]$ —— (1)

$L[f(t)] = \ln \dfrac{s^2+1}{s(s+1)} = \bar{f}(s)$

$\dfrac{d}{ds} \bar{f}(s) = \dfrac{d}{ds}\left[\ln(s^2+1) - \ln s - \ln(s+1)\right]$

$= -L[t f(t)]$

$= \dfrac{2s}{s^2+1} - \dfrac{1}{s} - \dfrac{1}{s+1}$

$L[t f(t)] = \dfrac{1}{s} + \dfrac{1}{s+1} - \dfrac{2s}{s^2+1}$

$t f(t) = L^{-1}\left[\dfrac{1}{s} + \dfrac{1}{s+1} - \dfrac{2s}{s^2+1}\right]$

$$t f(t) = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+1}\right] - 2L^{-1}\left[\frac{s}{s^2+1}\right]$$

$$= 1 + e^{-t} - 2\cos t$$

$$f(t) = \frac{1}{t}\left[1 + e^{-t} - 2\cos t\right]$$

$$\therefore \boxed{L^{-1}\left[\ln \frac{s^2+1}{s(s+1)}\right] = \frac{1}{t}\left[1 + e^{-t} - 2\cos t\right]}$$

ii,
$$L[f(t)] = \bar{f}(s) = \cot^{-1}(s/k)$$

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\left(\cot^{-1} s/k\right) = -\frac{k}{k^2+s^2}$$

$$-L[t f(t)] = \frac{k}{s^2+k^2} = \bar{f}(s) \qquad \frac{d}{ds}\bar{f}(s)$$

$$-t f(t) = L^{-1}\left[\frac{k}{s^2+k^2}\right] = \sin kt$$

$$f(t) = -\frac{\sin kt}{t}$$

$$\boxed{L^{-1}\left[\cot^{-1} s/k\right] = -\frac{\sin kt}{t}}$$

Convolution thm (Faltung thm)

We often come across functions which are not transforms of some known function, but then, they can possibly be

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expressed as a product of two functions, each of which is the transform of a known function. Thus we may be able to write the given function as $\bar{f}(s)\,\bar{g}(s)$, where $\bar{f}(s)$ and $\bar{g}(s)$ are known to be transforms of the functions $f(t)$ and $g(t)$, respectively.

$\boxed{\text{Thm}}$ If $\bar{f}(s)$ and $\bar{g}(s)$ are the Laplace transforms of $f(t)$ and $g(t)$ respectively, then $\bar{f}(s)\,\bar{g}(s)$ is the Laplace transform of

$$\int_0^t f(t-u)\,g(u)\,du.$$

i.e.

$$L^{-1}\left[\bar{f}(s)\,\bar{g}(s)\right] = f(t) * g(t)$$

$$= \int_0^t f(t-u)\,g(u)\,du$$

$$= \int_0^t f(u)\,g(t-u)\,du.$$

This integral is called the convolution of f and g and denoted by $*$.

$\boxed{*}$ Also $f*g = g*f$.

Ex Evaluate

(i) $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right] = \dfrac{1}{a}\left[\dfrac{s}{s^2+a^2} \cdot \dfrac{a}{s^2+a^2}\right]$

$\bar{f}(s) = \dfrac{s}{s^2+a^2}$, $\bar{g}(s) = \dfrac{a}{s^2+a^2}$.

$L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = f(t)$.

$L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{a}{s^2+a^2}\right] = \sin at = g(t)$

By convolution thm

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{a} f(t) * g(t)$$

$$= \frac{1}{a}\int_0^t f(t-u)\, g(u)\, du$$

$$= \frac{1}{a}\int_0^t \cos a(t-u) \sin au\, du$$

$$= \frac{1}{a}\int_0^t [\cos at \cos au + \sin at \sin au] \sin au\, du$$

$$= \frac{1}{2a}\cos at \int_0^t \sin 2au\, du$$

$$+ \frac{\sin at}{2a}\int_0^t (1-\cos 2au)\, du$$

$$= \frac{\cos at}{2a}\left(-\frac{\cos 2au}{2a}\right)_0^t + \frac{\sin at}{2a}\left(u - \frac{\sin 2au}{2a}\right)_0^t$$

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$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{-1}{4a^2}\left[\cos at \cos 2at + \sin at \sin 2at\right] + \frac{\cos at}{4a^2} + \frac{t \sin at}{2a}$$

$$= \frac{t \sin at}{2a}.$$

(ii) $$L^{-1}\left[\frac{s}{(s+a)(s^2+1)}\right] = L^{-1}\left[\frac{1}{s+a} \cdot \frac{s}{s^2+1}\right]$$

$$\bar{f}(s) = \frac{1}{s+a}, \quad \bar{g}(s) = \frac{s}{s^2+1}$$

$$L^{-1}[\bar{f}(s)] = f(t) = e^{-at}$$

$$L^{-1}[\bar{g}(s)] = g(t) = \cos t$$

using convolution thm

$$L^{-1}\left[\frac{s}{(s+a)(s^2+1)}\right] = f(t) * g(t)$$

$$= \int_0^t e^{-a(t-u)} \cos u \, du$$

$$= e^{-at}\int_0^t e^{au} \cos u \, du.$$

$$= e^{-at}\left[\frac{e^{au}}{a^2+1}(\sin u + a\cos u)\right]_0^t$$

$$\Rightarrow \frac{1}{a^2+1}\left[a\cos t + \sin t - a e^{-at}\right].$$

$$\frac{e^{-at}}{a^2+1}\left[e^{at}(\sin t + a\cos t) - a\right]$$

$$I = \int e^{au} \cos u \, du$$

$$I = \frac{e^{au}}{a}\cos u - \int \frac{e^{au}}{a}(-\sin u)\,du$$

$$= \;\; '' \;\; + \frac{1}{a}\int e^{au} \sin u \, du$$

$$= \;\; '' \;\; '' + \frac{1}{a}\left[\frac{e^{au}}{a}\sin u - \int \frac{e^{au}}{a}\cos u\right]$$

$$= \;\; '' \;\; '' \;\; \frac{1}{a^2} e^{au}\sin u - \frac{1}{a^2} I$$

$$I + \frac{1}{a^2} I = \frac{e^{au}}{a}\cos u + \frac{1}{a^2}e^{au}\sin u$$

$$\frac{a^2+1}{a^2} I = [$$

$$= \frac{a^2}{a^2+1}[$$

$$= \frac{e^{au}}{a^2+1}[a\cos u + \sin u]$$

(iii) $\bar{L}^{1}\left[\frac{1}{s^2(s+1)^2}\right] = \bar{L}^{1}\left[\frac{1}{s^2} \cdot \frac{1}{(s+1)^2}\right]$.

$\bar{f}(s) = \frac{1}{s^2} \Rightarrow f(t) = t$

$\bar{g}(s) = \frac{1}{(s+1)^2} \Rightarrow g(t) = t\,\bar{e}^{t}$

Using convolution thm

$$\bar{L}^{1}\left[\frac{1}{s^2(s+1)^2}\right] = f(t) * g(t) = \int_0^t (t-u)\, u \bar{e}^{u}\, du$$

$$= t\int_0^t \bar{e}^{u} u\,du - \int_0^t u^2 \bar{e}^{u}\, du$$

$$= -t\left[\left[u e^{-u}\right]_0^t - \int_0^t \bar{e}^{u} du\right]$$

$$+ \left[u^2 \bar{e}^{u}\right]_0^t - 2\int_0^t u e^{-u} du$$

$$= (t+2)\, e^{-t} + t - 2.$$

(iv) Show that $\int_0^t J_0(u)\, J_0(t-u)\, du = \sin t$.

$$\bar{L}^{1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$\frac{1}{s^2+1} = \frac{1}{\sqrt{s^2+1}}\,\frac{1}{\sqrt{s^2+1}}$$

$$\bar{f}(s) = \frac{1}{\sqrt{s^2+1}}, \quad \bar{g}(s) = \frac{1}{\sqrt{s^2+1}}$$

$= J_0(t)$, $g(t) = J_0(t)$

convolution thm

$$L^{-1}\left[\frac{1}{s^2+1}\right] = \int_0^t J_0(t)\, J_0(t-u)\, du$$

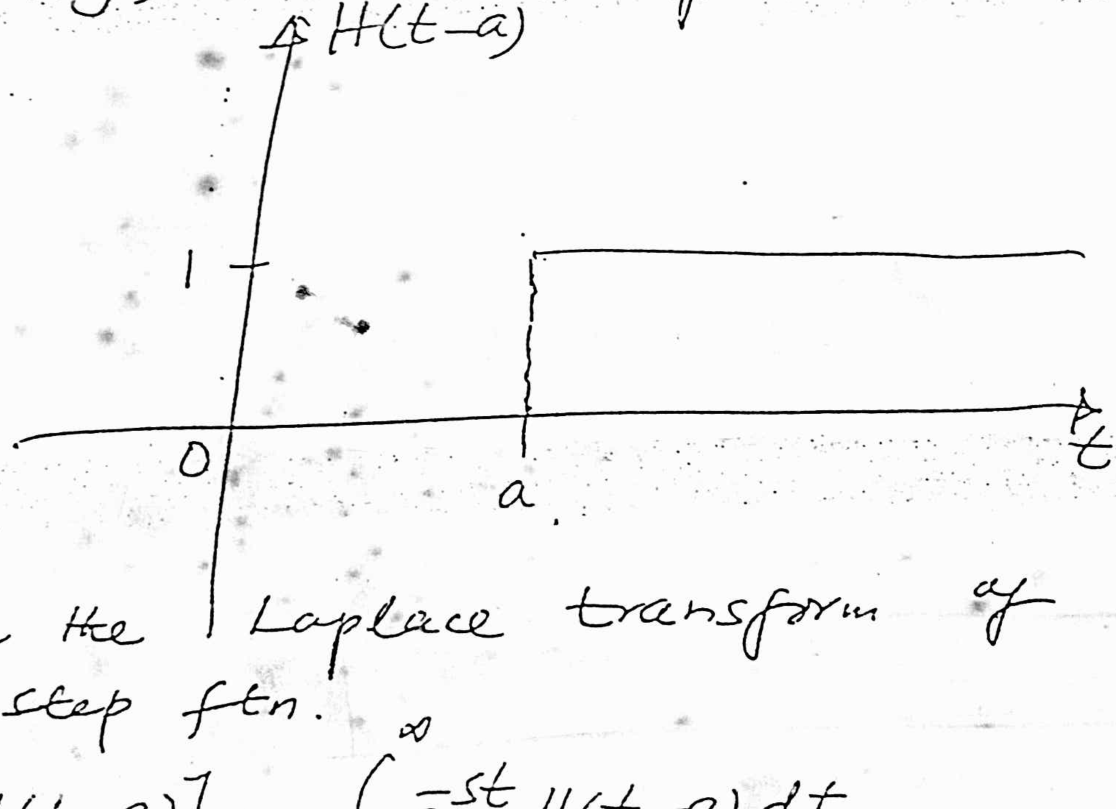
$$= \sin t.$$

Transform of Unit Step function

Def. The unit step ftn or Heaviside unit step ftn is defined as

$$H(t-a) = \begin{cases} 0 & \text{for } t<a \\ 1 & \text{for } t \geq a,\ a \geq 0 \end{cases}$$

Graphically, it can be depicted as



Ex Find the Laplace transform of unit step ftn.

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st}\, 0\, dt + \int_a^\infty e^{-st} (1)\, dt$$

$$L[H(t-a)] = \frac{e^{-as}}{s}, \quad s>0.$$

Thm (Second Shifting property)

If $L[f(t)] = \bar{f}(s)$ then

$$L[f(t-a)\,H(t-a)] = e^{-as}\bar{f}(s)$$

or $$L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)\,H(t-a).$$

Ex Find the inverse Laplace transform of

$$\frac{e^{-as}}{s^2+1}, \quad a>0$$

Soln As $L^{-1}[e^{-as}\bar{f}(s)] = f(t-a)H(t-a)$.

Here $\bar{f}(s) = \frac{1}{s^2+1}$

$$f(t) = L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$$

$$L^{-1}\left[e^{-as}\cdot\frac{1}{s^2+1}\right] = \sin(t-a)\,H(t-a)$$

$$= \begin{cases} 0, & t<a \\ \sin(t-a), & t\geq a \end{cases}$$

Thm (Heaviside Expansion Thm)

Let $\bar{f}(s)$ and $\bar{g}(s)$ be two polynomials in s where the degree of $\bar{f}(s)$ is lower than that of $\bar{g}(s)$, and if $\bar{g}(s)$ has n distinct roots α_i $(i=1,2,\dots n)$ then.

$$L^{-1}\left[\frac{\bar{f}(s)}{\bar{g}(s)}\right] = \sum_{i=1}^{n} \frac{\bar{f}(\alpha_i)}{\bar{g}'(\alpha_i)} e^{\alpha_i t}.$$

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Find the Laplace inverse of $\dfrac{s^2+1}{s^3+3s^2+2s}$.

Soln $\bar{f}(s) = s^2+1$, $\bar{g}(s) = s^3+3s^2+2s$
$= s(s+1)(s+2)$.

$\bar{g}(s)$ has three distinct roots $0, -1, -2$ and degree of $\bar{f}(s)$ is lower than that of $\bar{g}(s)$. Hence using the Heaviside expansion them, we have.

$$L^{-1}\left[\frac{s^2+1}{s^3+3s^2+2s}\right] = \frac{\bar{f}(0)}{\bar{g}'(0)} e^{0t} + \frac{\bar{f}(-1)}{\bar{g}'(-1)} e^{-t} + \frac{\bar{f}(-2)}{\bar{g}'(-2)} e^{-2t} \quad \text{—— (1)}$$

$\bar{g}'(s) = 3s^2+6s+2$; $\bar{f}(s) = 1+s^2$

using in (1) we get

$$L^{-1}\left[\frac{s^2+1}{s^3+3s^2+2s}\right] = \frac{1}{2} - 2e^{-t} + \frac{5}{2} e^{-2t}$$