Complex Inversion Formula (Mellin-Fourier Integral)

In solving some complicated problems using the Laplace transform method, the direct approach so far followed may not be helpful in finding the inverse Laplace transform. Methods based on complex variable theory may come in handy for finding the inverse transform. Also, it can be noted that the Laplace transform of $f(t)$ is expressed as an integral. Similarly, the inverse Laplace transform of $\bar{f}(s)$ can be expressed as an integral which is known as

(4.1)

inverse integral. This integral can be evaluated using contour integration method. The complex inversion formula is stated in the following thm.

$\boxed{\text{Thm}}$ Let $f(t)$ and $f'(t)$ be conts ftns on $t \geq 0$ and $f(t) = 0$ for $t < 0$. In addition, if $f(t)$ is $O(e^{\gamma_0 t})$ and

$$\bar{f}(s) = L[f(t)]$$

then

$$L^{-1}[\bar{f}(s)] = f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \bar{f}(s)\, ds,$$

$t > 0$ and γ is a positive constant.

$\boxed{\text{Ex}}$ Find the inverse Laplace transform of $\dfrac{1}{(s+1)(s-2)^2}$.

$$L^{-1}\left[\frac{1}{(s+1)(s-2)^2}\right] = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}\, ds}{(s+1)(s-2)^2}$$

$$= \text{sum of the residues of } \left(\frac{e^{st}}{(s+1)(s-2)^2}\right).$$

At the simple pole $s = -1$ and double pole at $s = 2$.

Therefore,

$$L^{-1}\left[\frac{1}{(s+1)(s-2)^2}\right] = \lim_{s \to -1} \frac{\cancel{(s+1)}\, e^{st}}{\cancel{(s+1)}\,(s-2)^2} + \lim_{s \to 2} \frac{d}{ds}\left(\frac{e^{st}}{s+1}\right).$$

$$f(z_0) = \frac{\varphi^{m-1}(z_0)}{(m-1)!}$$

$$= \frac{e^{-t}}{9} + \lim_{s \to 2} \left[\frac{e^{st}[(s+1)t - 1]}{(s+1)^2}\right]$$

$$= \frac{e^{-t}}{9} + \frac{t}{3} e^{2t} - \frac{1}{9} e^{2t}.$$

Find the inverse Laplace transform of

$$\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}, \quad 0<x<l$$

$\frac{p(s)}{q(s)}$ $q(s)=0$ $\frac{p(s)}{q'(s)}$

Soln Let $F(s) = \dfrac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}$

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} e^{st}\,\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}\,ds$$

$$= \frac{1}{2\pi i}\int_{C} e^{st}\,\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}\,ds$$

= Sum of the residues of the integrand at infinitely many simple poles given by the roots of the transcendental equation $\sinh(l\sqrt{s}) = 0$.

i.e. $e^{l\sqrt{s}} - e^{-l\sqrt{s}} = 0$

$\sinh l\sqrt{s} = \sinh n\pi i$

$l\sqrt{s} = n\pi i$

$\sqrt{s} = \frac{n\pi i}{l}$

$\Rightarrow e^{2l\sqrt{s}} - 1 = 0$

$\Rightarrow e^{2l\sqrt{s}} = 1 = e^{2n\pi i}$

$\Rightarrow \sqrt{s}\,l = n\pi i$

$s l^2 = -n^2\pi^2$

$$\boxed{s = -\frac{n^2\pi^2}{l^2}, \quad n = 0, 1, 2, \ldots}$$

Thus we have to compute the residues at the poles

$s = 0$, $s = s_n = \dfrac{-n^2\pi^2}{l^2}$, $n = 1, 2, \ldots$

Now the residue at $s=0$ is

$$\underset{s\to 0}{Lt}\; s\, e^{st} \frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})} = 0$$

The residues at $s=s_n$ are obtained as

$$\underset{s\to s_n}{Lt}\; (s-s_n)\, e^{st} \frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}$$

$\frac{P(z)}{\psi'(z)}$

$$= \left. \frac{2\sqrt{s}}{l\cosh(l\sqrt{s})} e^{st} \sinh(x\sqrt{s}) \right|_{s=s_n}$$

$$= \frac{2\sqrt{\frac{-n^2\pi^2}{l^2}}}{l\cosh\left[l\sqrt{\frac{-n^2\pi^2}{l^2}}\right]} \exp\left(-\frac{n^2\pi^2 t}{l^2}\right) \times \sinh\left[x\sqrt{-n^2\pi^2/l^2}\right]$$

$$= \frac{2in\pi}{l^2} \exp\left(-\frac{n^2\pi^2}{l^2}t\right) \frac{\sinh\left[i(n\pi/l)x\right]}{\cosh(in\pi)}$$

$$= \frac{2n\pi(-1)}{l^2} \exp\left(-\frac{n^2\pi^2}{l^2}t\right) \frac{\sin(n\pi x/l)}{\cos n\pi}$$

$$= \frac{(-1)^{n+1} 2n\pi}{l^2} \sin\left(\frac{n\pi}{l}x\right) \exp\left(\frac{-n^2\pi^2 t}{l^2}\right); \quad n=1,2,\ldots$$

Therefore

$$L^{-1}\left[\frac{\sinh(x\sqrt{s})}{\sinh(l\sqrt{s})}\right] = \frac{2\pi}{l^2} \sum_{n=1}^{\infty} (-1)^{n+1} n \sin\left(\frac{n\pi x}{l}\right) \exp\left(-\frac{n^2\pi^2}{l^2}t\right).$$

~lution of Ordinary differential Eqs. (4.5)

The Laplace transform technique is one of the powerful tools for solving physical systems involving ODEs, particularly initial value problems. It reduces the solution of ODE to the soln of algebraic equation. This method has a particular advantage in finding the soln of an initial value problem, without first finding the general soln and then using the given ICs for evaluating the arbitrary constants. We shall first apply the Laplace transform technique to find the solution of a typical initial value problem and demonstrate the steps involved.