Math III

(40)

4/-

$\nabla_3^2 u = 0$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \longrightarrow (1)$$

Let $u = X(x)Y(y)Z(z) \longrightarrow (1a)$

$u_x = X'(x)Y(y)Z(z)$

$u_{xx} = X''YZ$

$u_y = XY'Z$

$u_{yy} = XY''Z$

$u_z = XYZ'$

$u_{zz} = XYZ''$

Using in (1) we have

$X''YZ + XY''Z + XYZ'' = 0$

$\div$ing by $XYZ$ we have

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - \frac{Z''}{Z} = -n^2 \text{ (separable constant)} \longrightarrow (2)$$

From (2) we have

$$\frac{X''}{X} = -n^2 \Rightarrow X'' + n^2 X = 0 \longrightarrow (3)$$

$$+\frac{Y''}{Y} = -\frac{Z''}{Z} + n^2 = -m^2$$

$$\Rightarrow \boxed{Y'' + m^2 Y = 0} \longrightarrow (4)$$

$-Z'' + Zn^2 = -m^2 Z$

$$\Rightarrow \boxed{Z'' - (m^2+n^2)Z = 0} \longrightarrow (5)$$

From (3)

$$D^2+n^2=0$$

$$D=\pm in$$

$$\therefore \boxed{X = A_1 \cos nx + B_1 \sin nx} \longrightarrow (6)$$

From Eq. (4) we can write

$$\boxed{Y = A_2 \cos my + B_2 \sin my} \longrightarrow (7)$$

From (5)

$$Z = A_3 e^{\sqrt{m^2+n^2}\, Z} + B_3 e^{-\sqrt{m^2+n^2}\, Z} \longrightarrow (8)$$

So the Eq. (1a) after using (6)–(8) yields

$$u(x,y,z) = \left.\begin{matrix}\cos nx \\ \sin nx\end{matrix}\right\} \left.\begin{matrix}\cos my \\ \sin my\end{matrix}\right\} e^{\pm\sqrt{m^2+n^2}\, Z}.$$

This is the general solution of the Laplace equation in three dimensions.

xample Consider the steady state temperature distribution in a thin rectangular slab. Two sides are insulated. One side is kept at zero temperature and the temperature of the remaining side is $f(x)$.

lution The diffusion equation (heat equation) n two dimensions is given by

$$\nabla_2^2 u = \frac{1}{k}\frac{\partial u}{\partial t}$$

Steady state means that temp $u$ is independent of time and thus

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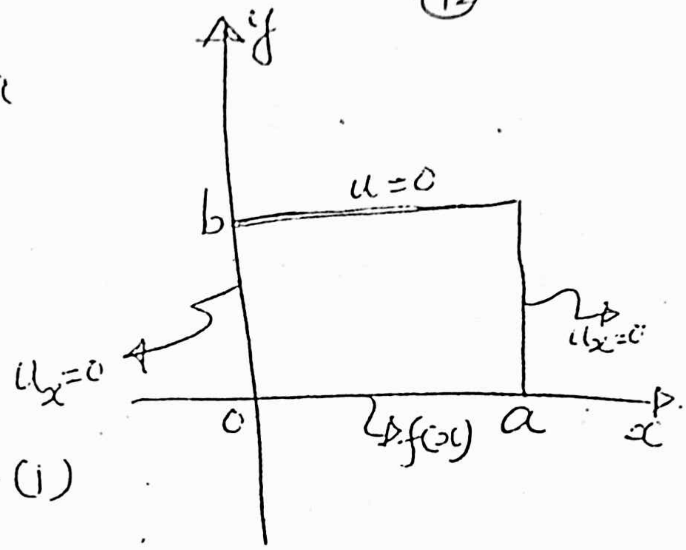
$\nabla_2^2 u = 0$

$u(x,0) = f(x), \quad 0 \le x \le a$

$u(x,b) = 0$

$u_x(0,y) = 0$

$u_x(a,y) = 0.$




$u_{xx} + u_{yy} = 0 \quad \text{——— (1)}$

Let $u(x,y) = X(x)Y(y)$

$\therefore$ (1) becomes

$X''Y + XY'' = 0$

Dividing by $XY$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\alpha^2 \quad \text{——— (2)}$$

$$\boxed{\begin{aligned} X'' + \alpha^2 X &= 0 \quad \text{——— (3)} \\ Y'' - \alpha^2 Y &= 0 \quad \text{——— (4)} \end{aligned}}$$

Solving (3) we have

$\boxed{X(x) = A\cos\alpha x + B\sin\alpha x} \quad \text{——— (4a)}$

Now $u_x(0,y) = 0$ $\qquad Y(y) \neq 0$

$\Rightarrow X'(0)Y(y) = 0$

$\Rightarrow \boxed{X'(0) = 0} \quad \text{——— (4b)}$

$u'_x(a,y) = 0 \Rightarrow X'(a)Y(y) = 0$ $\quad \text{——— (4c)}$ $\quad ; Y(y) \neq 0.$

$\Rightarrow \boxed{X'(a) = 0.}$

From (4a)

$$X'(x) = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$$

$$X'(0) = 0 \Rightarrow 0 + B\alpha = 0$$

$$\Rightarrow \boxed{B = 0}$$

$\therefore$ (4a) gives

$$X(x) = A \cos \alpha x \longrightarrow (4d)$$

$$X'(x) = -A\alpha \sin \alpha x$$

$$X'(a) = 0$$

$$\Rightarrow -A\alpha \sin \alpha a = 0$$

$A \neq 0$ for non-trivial solution. so $\alpha \neq 0$.

$\alpha = 0$ gives trivial solution

Thus $\sin \alpha a = 0 = \sin n\pi \quad (n = 0, 1, 2, \cdots)$

$$\alpha_n a = n\pi$$

$$\boxed{\alpha_n = \frac{n\pi}{a}} \longrightarrow (4e)$$

With (4e), Eq. (4d) takes the following form

$$\boxed{X(x) = A_n \cos \frac{n\pi}{a} x} \longrightarrow (4f)$$

Now solving (4) we get.

$$Y = Ae^{\alpha y} + Be^{-\alpha y}$$

$$= \left( \frac{A}{2} e^{\alpha y} + \frac{A}{2} e^{\alpha y} + \frac{B}{2} e^{+\alpha y} - \frac{B}{2} e^{\alpha y} \right)$$

$$+ \left( \frac{B}{2} e^{-\alpha y} + \frac{B}{2} e^{-\alpha y} + \frac{A}{2} e^{-\alpha y} - \frac{A}{2} e^{-\alpha y} \right)$$

$$Y = \left(\frac{A+B}{2}\right)e^{\alpha y} + \left(\frac{A-B}{2}\right)e^{\alpha y} + \left(\frac{A+B}{2}\right)e^{-\alpha y} - \left(\frac{A-B}{2}\right)e^{-\alpha y} \quad (4)$$

$$Y = \frac{C}{2}e^{\alpha y} + \frac{D}{2}e^{\alpha y} + \frac{C e^{-\alpha y}}{2} - \frac{D}{2}e^{-\alpha y}$$

$$= C\left(\frac{e^{\alpha y} + e^{-\alpha y}}{2}\right) + D\left(\frac{e^{\alpha y} - e^{-\alpha y}}{2}\right)$$

$$\boxed{Y = C\cosh\alpha y + D\sinh\alpha y}$$

Let $C = E\sinh\alpha\epsilon$ ; $D = E\cosh\alpha\epsilon$

$$\Rightarrow Y = E\left[\sinh\alpha\epsilon\cosh\alpha y + \cosh\alpha\epsilon\sinh\alpha y\right]$$

$$Y = E\sinh(\alpha y + \alpha\epsilon)$$

$$\boxed{Y = E\sinh\alpha(y+\epsilon)} \longrightarrow (5)$$

$u(x, b) = 0 \Rightarrow X(x)\,Y(b) = 0 \qquad (X(x) \neq 0)$

$$\therefore \boxed{Y(b) = 0} \longrightarrow (6)$$

Making use of (6) in (5) we obtain

$Y(b) = 0 = E\sinh\alpha(b+\epsilon)$

$E \neq 0$ this is true only if $b + \epsilon = 0$

$b = -\epsilon$

$\Rightarrow \epsilon = -b$

$\therefore$ (5) gives

$$\boxed{Y_n = E_n\sinh\alpha_n(y-b)} \longrightarrow (7)$$

$$u(x,y) = \sum_{n=0}^{\infty} u_n(x,y)$$

$$= \sum_{n=0}^{\infty} X_n(x) Y_n(y)$$

$$\boxed{u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi}{a}(y-b)} \rightarrow (8)$$

$$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi(-b)}{a} \quad a_n = A_n E_n.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{a} \sinh\frac{n\pi}{a}(-b).$$

$$\Rightarrow a_0 = \frac{2}{a}\int_0^a f(x)\,dx$$

$$a_n = \frac{-2}{a \sinh\frac{n\pi}{a}b} \int_0^a f(x) \cos\frac{n\pi x}{a}\,dx.$$

Example

By separating the variables show that the equation

$$\nabla^2 u = 0$$

has solutions of the form $A \exp(\pm nx \pm iny)$ where $A$ and $n$ are constants. Deduce that the functions of the form

$$u(x,y) = \sum_{r} A_r e^{-\frac{r\pi x}{a}} \sin\frac{r\pi y}{a} \qquad x > 0, \quad 0 \le y \le a$$

where $A$'s are constant, are plane harmonic functions satisfying the conditions

$u(x,0) = 0$, $u(x,a) = 0$, $u(x,y) \to 0$ as $x \to \infty$.

Solution The given equation is

$$\nabla^2 u = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{———} (1)$$

Let $u(x,y) = X(x)\,Y(y)$

$$\frac{\partial^2 u}{\partial x^2} = X''(x)\,Y(y)$$

$$\frac{\partial^2 u}{\partial y^2} = X(x)\,Y''(y)$$

using in (1) we have

$$X''Y + XY'' = 0$$

Dividing by $XY$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = n^2 \quad \text{———} (2)$$

From (2), we have

$$X'' - n^2 X = 0$$

$$(D^2 - n^2)X = 0$$

$$\boxed{X(x) = A_1 e^{nx} + B_1 e^{-nx}} \quad \text{———} (2a)$$

($X(\infty)$ should be bounded $\Rightarrow A_1 = 0$).

Again from (2)

$$Y'' + n^2 Y = 0$$

$$\Rightarrow Y(y) = A_2 e^{\pm iny}$$

$$\therefore \boxed{u(x,y) = A e^{\pm nx \pm iny}} \quad \text{———} (3)$$

$u(x,0)=0$ and $u(x,a)=0$

$X(x)Y(0)=0$ | $X(x)Y(a)=0$

$\Rightarrow \boxed{Y(0)=0}$ as $X(x)\neq 0$ | $\Rightarrow \boxed{Y(a)=0}$ ; $X(x)\neq 0$

Now
$$Y = A_2 e^{\pm iny}$$

$$Y = B_1 \cos ny + B_2 \sin ny$$

$$Y(0)=0 \Rightarrow B_1 = 0$$

$\therefore$
$$Y = B_2 \sin ny$$

$$Y(a)=0$$

$$\Rightarrow B_2 \sin na = 0$$ so

$B_2 \neq 0$ for non-trivial solution

$$\Rightarrow \sin na = 0 = \sin r\pi$$

$$n = \frac{r\pi}{a}, \quad r = 1, 2, \ldots$$

$\therefore$
$$\boxed{Y_r = B_r \sin \frac{r\pi}{a} y}$$

Now as from (2a) $A_1 = 0$ after using $X(\infty) \to 0$

$\therefore$
$$\boxed{X_r(x) = B_{1r} e^{-\frac{r\pi}{a} x}}$$

so
$$\boxed{u = \sum_r A_r e^{-\frac{r\pi x}{a}} \sin \frac{r\pi}{a} y \; ; \; x > 0, \; 0 \leq y \leq a}$$

$$A_r = B_r B_{1r}$$

This solution is called plane harmonic function.

Q Solve

$u_{xx} = \alpha^2 u_t$ ——— (1)

$u(x,t) = X(x)T(t)$

(1) becomes

$$X''T = \alpha^2 X T'$$

Dividing by $XT$

$$\frac{X''}{X} = \alpha^2 \frac{T'}{T} = -n^2 \text{ (say)} \longrightarrow (2)$$

$$\Rightarrow \boxed{X'' + n^2 X = 0}$$

$$\boxed{T' + \frac{n^2}{\alpha^2} T = 0}$$

$$\boxed{X(x) = A\cos nx + B\sin nx} \longrightarrow (3)$$

$$\left(D + \frac{n^2}{\alpha^2}\right) T = 0$$

$$D = -\frac{n^2}{\alpha^2} \Longrightarrow \boxed{T = c\, e^{-\frac{n^2}{\alpha^2}t}} \longrightarrow (4)$$

$\therefore$

$$u(x,t) = (A\cos nx + B\sin nx)\, c\, e^{-n^2/\alpha^2 t}$$

$$= A_1 e^{\pm inx - n^2/\alpha^2 t}.$$

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Q Solve $u_{xx} + u_{yy} = \frac{1}{c^2} u_{tt}$ ——— (1)

Let $u(x,y,t) = X(x)Y(y)T(t)$.

$u_{xx} = X''YT$

$u_{yy} = XY''T$

$u_{tt} = XY\ddot{T}$.

(1) gives

$$X''YT + XY''T = \frac{1}{c^2} XY\ddot{T}$$

$$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

$$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} + \frac{1}{c^2}\frac{\ddot{T}}{T} = -n^2 \longrightarrow (2)$$

From (2) $X'' + n^2 X = 0$

$D = \pm in$

$$\therefore \boxed{X = A e^{\pm inx}} \longrightarrow (2a)$$

From (2) $-\frac{Y''}{Y} + \frac{1}{c^2}\frac{\ddot{T}}{T} = -n^2$

$$\Rightarrow \frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{Y''}{Y} - n^2 = m^2 \longrightarrow (3)$$

$$\ddot{T} - m^2 c^2 T = 0$$

$$\boxed{T = B e^{\pm mct}} \longrightarrow (4)$$

From (3)

$$y'' - (m^2 + n^2) y = 0$$

$$Y = C e^{\pm\sqrt{m^2+n^2}\, y} \longrightarrow (5)$$

from Eqs. (2a), (4) and (5) we get

$$u = \left.\begin{matrix} \cos nx \\ \sin nx \end{matrix}\right\} e^{\pm mct} \Big\} e^{\pm\sqrt{m^2+n^2}\, y}.$$