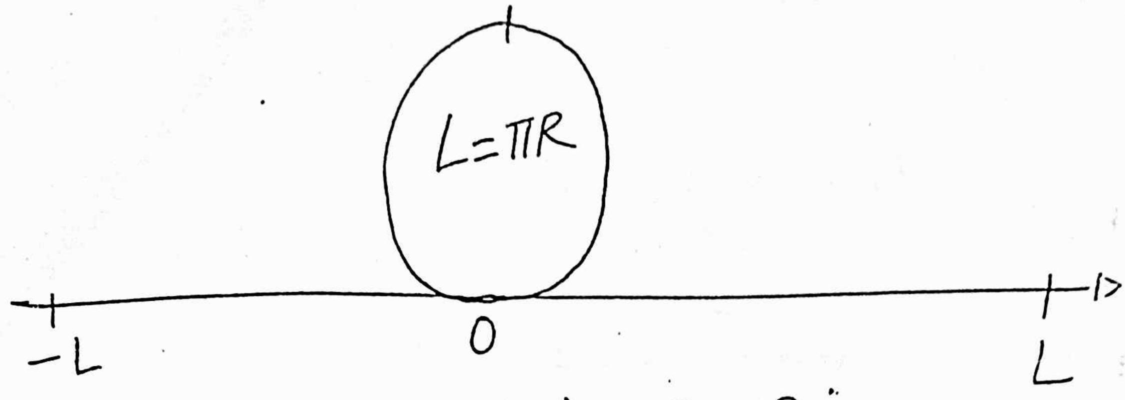
Heat conduction in a thin uniform circular ring

Physically, the ring of circumference $2L$ shown in the Fig. can be regarded as a rod of length $2L$ that, for continuity reasons, has the same temperature and heat flux at the two end points $x=-L$ and $x=L$.



Thus, the corresponding IBVP is

$$u_t(x,t) = k\,u_{xx}(x,t), \quad -L<x<L,\ t>0 \quad \text{(1)}$$

$$\left.\begin{aligned} u(-L,t) &= u(L,t), \\ u_x(-L,t) &= u_x(L,t), \end{aligned}\right\} t>0 \quad \text{(2)}$$

$$u(x,0) = f(x), \quad -L<x<L. \quad \text{(3)}$$

The PDE and BCs are homogeneous, so, as before, we seek a solution of the form

$$u(x,t) = X(x)T(t) \quad \text{(4)}$$

Using (4) in (1) we get:

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

$$\Rightarrow \boxed{\begin{aligned} X''+\lambda X &= 0, \ -L<x<L, \\ T'+\lambda k T &= 0, \ t>0, \end{aligned}} \quad \text{(5)}$$

From (2) and (4) we get:

$$u(-L,t) = u(L,t)$$

$$X(-L)\,T(t) = X(L)\,T(t)$$

$$\Rightarrow \boxed{X(-L) = X(L)}$$

$$u_x(-L,t) = u_x(L,t)$$

$$X'(-L)\,T(t) = X'(L)\,T(t)$$

$$\Rightarrow \boxed{X'(-L) = X'(L)}$$

$\therefore$
$$\boxed{\begin{aligned} X(-L) &= X(L) \\ X'(-L) &= X'(L) \end{aligned}} \longrightarrow (6)$$

From (5) for $\underline{\lambda > 0}$

$$(D^2 + \lambda)X = 0$$

$$X(x) = A_1 \cos\sqrt{\lambda}\,x + A_2 \sin\sqrt{\lambda}\,x \longrightarrow (7)$$

$$X(-L) = A_1 \cos\sqrt{\lambda}\,L - A_2 \sin\sqrt{\lambda}\,L$$

$$X(L) = A_1 \cos\sqrt{\lambda}\,L + A_2 \sin\sqrt{\lambda}\,L$$

using (6)

$$\cancel{A_1 \cos\sqrt{\lambda}\,L} - A_2 \sin\sqrt{\lambda}\,L = \cancel{A_1 \cos\sqrt{\lambda}\,L} + A_2 \sin\sqrt{\lambda}\,L$$

$$\Rightarrow 2A_2 \sin\sqrt{\lambda}\,L = 0 \longrightarrow$$

$$\boxed{A_2 = 0}$$

$$X(x) = A_1 \cos\sqrt{\lambda}\,x \longrightarrow (8)$$

$$X'(x) = -\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,x$$

$$X'(-L) = -\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}(-L)$$

$$= \sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,L \longrightarrow (9)$$

$$X'(L) = -\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,L \longrightarrow (10)$$

$X'(-L) = X'(L) \Rightarrow$ from (9) and (10) as

$$\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,L = -\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,L$$

$$2\sqrt{\lambda}\,A_1 \sin\sqrt{\lambda}\,L = 0.$$

(23)

$A_1 \neq 0$ for non-trivial solution so

$$2\sqrt{\lambda} \sin\sqrt{\lambda} L = 0$$

$2\sqrt{\lambda} \neq 0$ so $\sin\sqrt{\lambda} L = 0$

$$\Rightarrow \sin\sqrt{\lambda} L = \sin n\pi$$

$$\sqrt{\lambda_n} L = n\pi.$$

$$\boxed{\sqrt{\lambda_n} = \frac{n\pi}{L}} \longrightarrow (11)$$

$n = 0, 1, 2, \cdots$

$$\boxed{X_{1n}(x) = A_n \cos \frac{n\pi}{L} x} \longrightarrow (12)$$

from (8).

Consider (7).

$$X'(x) = -\sqrt{\lambda} A_1 \sin\sqrt{\lambda} x + A_2 \sqrt{\lambda} \cos\sqrt{\lambda} x$$

$$X'(-L) = \sqrt{\lambda} A_1 \sin\sqrt{\lambda} L + A_2 \sqrt{\lambda} \cos\sqrt{\lambda} L$$

$$X'(L) = -\sqrt{\lambda} A_1 \sin\sqrt{\lambda} L + A_2 \sqrt{\lambda} \cos\sqrt{\lambda} L$$

$X'(-L) = X'(L)$ then

$$\sqrt{\lambda} A_1 \sin\sqrt{\lambda} L + \cancel{A_2 \sqrt{\lambda} \cos\sqrt{\lambda} L}$$

$$= -\sqrt{\lambda} A_1 \sin\sqrt{\lambda} L + \cancel{A_2 \sqrt{\lambda} \cos\sqrt{\lambda} L}$$

$$2\sqrt{\lambda} A_1 \sin\sqrt{\lambda} L = 0$$

$$\Rightarrow \boxed{A_1 = 0}$$

$\therefore$ From (7)

$$X(x) = A_2 \sin\sqrt{\lambda_n} x \longrightarrow (13)$$

$$X(-L) = -A_2 \sin\sqrt{\lambda_n} L$$

$$X(L) = A_2 \sin\sqrt{\lambda_n} L$$

$X(-L) = X(L)$ then $\quad -A_2 \sin\sqrt{\lambda_n} L = A_2 \sin\sqrt{\lambda_n} L$

$$\Rightarrow 2A_2 \sin\sqrt{\lambda_n} L = 0$$

$2A_2 \neq 0$ (for non-trivial soln)

$$\sin\sqrt{\lambda}L = 0 = \sin n\pi$$

$$\boxed{\sqrt{\lambda_n} = \frac{n\pi}{L}}$$

$\therefore$ $\boxed{X_{2n}(x) = B_n \sin\left(\frac{n\pi}{L}\right)x}$ from (13)

$n = 1, 2, \ldots \rightarrow (14)$

From (5)

$$T_n(t) = C_n e^{-\lambda_n kt}$$

$$= C_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \longrightarrow (15)$$

$\therefore$ From (4), (12), (14) and (15) we have

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right] e^{-\left(\frac{n\pi}{L}\right)^2 kt}} \rightarrow (16)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$

$2A_2 \neq 0$ (for non-trivial soln)

$$\sin\sqrt{\lambda}L = 0 = \sin n\pi$$

$$\boxed{\sqrt{\lambda_n} = \frac{n\pi}{L}}$$

$\therefore$

$$\boxed{X_{2n}(x) = B_n \sin\left(\frac{n\pi}{L}\right)x} \text{ from (13)}$$

$n = 1, 2, \cdots \rightarrow (14)$

From (5)

$$T_n(t) = C_n e^{-\lambda_n kt}$$

$$= C_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \rightarrow (15)$$

$\therefore$ From (4), (12), (14) and (15) we have

$$u(x,t) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi}{L}\right)^2 kt}$$

$$\boxed{u(x,t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right] e^{-\left(\frac{n\pi}{L}\right)^2 kt}} \rightarrow (16)$$

where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx,$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx, \quad n=1,2,\cdots,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx, \quad n=1,2,\cdots,$$

Example Consider the IBVP

$$\boxed{\begin{aligned} &u_t(x,t) = u_{xx}(x,t), \quad -1<x<1, \quad t>0, &\rightarrow (1)\\ &\left.\begin{aligned} u(-1,t) &= u(1,t),\\ u_x(-1,t) &= u_x(1,t), \end{aligned}\right\} t>0 &\rightarrow (2)\\ &u(x,0) = x+1, \quad -1<x<1, &\rightarrow (3)\end{aligned}}$$

Here $L=1$ and $f(x)=1$ so from the analysis of previous question the soln is

$$u(x,t) = a_0 + \sum_{n=1}^{\infty}\left(a_n\cos n\pi x + b_n\sin n\pi x\right)e^{-n^2\pi^2 kt}. \quad \rightarrow (4)$$

where

$$a_0 = \frac{1}{2}\int_{-1}^{1}(x+1)\,dx = \frac{1}{2}\left[\frac{1}{2}x^2+x\right]_{-1}^{1} = 1,$$

$$a_n = \int_{-1}^{1}(x+1)\cos(n\pi x)\,dx$$

$$= \left\{\cancel{\left[(x+1)\frac{1}{n\pi}\sin n\pi x\right]_{-1}^{1}}^{\,0} - \int_{-1}^{1}\frac{1}{n\pi}\sin n\pi x\,dx\right\}$$

$$= -\frac{1}{n\pi}\int_{-1}^{1}\sin n\pi x\,dx = \frac{-1}{n\pi}\left|-\frac{\cos n\pi x}{n\pi}\right|_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[\cos n\pi - \cos n\pi\right] = 0$$

$$b_n = \int_{-1}^{1} (x+1) \sin n\pi x \, dx$$

$$= \left\{ \left[(x+1)\left(-\frac{1}{n\pi}\right) \cos n\pi x \right]_{-1}^{1} + \int_{-1}^{1} \frac{1}{n\pi} \cos n\pi x \, dx \right\}$$

$$= -\frac{2}{n\pi} \cos n\pi + \frac{1}{n^2\pi^2} \cancelto{0}{\left[\sin n\pi x \right]_{-1}^{1}}$$

$$= (-1)^{n+1} \frac{2}{n\pi}.$$

Hence the soln (i) is

$$u(x,t) = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n\pi} \sin(n\pi x)\, e^{-n^2\pi^2 t}.$$