

Classification of 2nd order partial differential equation (PDE) ①

The general form of 2nd order PDE is

$$A u_{xx} + 2B u_{xy} + C u_{yy} + D u_x + E u_y + F u + G = 0$$

(where A, B, C are the linear functions of the independent variable x and y or constants)

Elliptic eq: $B^2 - AC < 0$ ✓

Parabolic: $B^2 - AC = 0$ ✓

Hyperbolic: $B^2 - AC > 0$ ✓

Examples

1. $u_{xx} + u_{yy} = 2$, $A=1$, $C=1$, $B=0$
 $B^2 - AC = -1 < 0$ (Elliptic)

2. $u_{xx} = x^2 u_t$ (Diffusion Eq.)

$A=1$, $B=0$, $C=0$

$B^2 - AC = 0$ (Parabolic Eq.)

3. $u_{xx} = \frac{1}{c^2} u_{tt}$

$u_{xx} - \frac{1}{c^2} u_{tt} = 0$ ✓

$A=1$, $B=0$, $C=-\frac{1}{c^2}$

$B^2 - AC = \frac{1}{c^2} > 0$ (Hyperbolic Eq.)

Exercises

Classify the following eqs.

1. $u_{xx} - 2u_{xy} + 2u_{yy} + 5u_x + 6u_y + 7 = 0$

2. $u_{xx} - 3u_{xy} + \frac{1}{2}u_{yy} + 16u_y = 0$

Reduction of 2nd order PDE into Canonical Form (2)

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Consider

$$A u_{xx} + 2B u_{xy} + C u_{yy} + F(x, y, u, u_x, u_y) = 0$$

where A, B, C are functions of x and y and u have continuous first and second order partial derivative. Also, A, B, C do not vanish at the same time.

Let us transform the independent variables by substituting?

$$\xi = \xi(x, y)$$

$$\eta = \eta(x, y)$$

ξ and η are such that

$$\frac{D(\xi, \eta)}{D(x, y)} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

In the new coordinate (ξ, η) (1) can be written as

$$\bar{A} \frac{\partial^2 u}{\partial \xi^2} + 2\bar{B} \frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C} \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, u_\xi, u_\eta) = 0$$

where

$$\bar{A} = A \left(\frac{\partial \xi}{\partial x} \right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2$$

$$\bar{B} = A \left(\frac{\partial \eta}{\partial x} \right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2$$

$$\bar{C} = A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}$$

Aside

(3)

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial x} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial x} \right)$$

$$= \left[\frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \xi} \right) \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \xi} \right) \frac{\partial \eta}{\partial x} \right] \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2}$$

$$+ \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2}$$

$$+ \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial x} \right)^2 + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}$$

⇐

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2 \frac{\partial^2 u}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial x} \right)^2 + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right] = \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right]$$

$$= \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \xi} \right) \left(\frac{\partial \xi}{\partial x} \right) \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \xi} \frac{\partial}{\partial x} \left(\frac{\partial \xi}{\partial x} \right) + \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial \eta} \right) \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial x} \right)$$

(2)

(4)

$$\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

$$= \frac{1}{\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}}$$

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now

$$\frac{1}{f_{eq}} = \frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}$$

$$= \frac{1}{\frac{1}{f_1} + \frac{1}{f_2} + \frac{1}{f_3}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \quad (5)$$

or

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial y} \right)^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \quad (4)$$

Using (2), (3) and (4) in (1) we have

$$\begin{aligned} & \left[\frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial x} \right)^2 \right. \\ & \left. + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} \right] + 2B \left[\frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} \right) \frac{\partial^2 u}{\partial \xi \partial \eta} \right. \\ & \left. + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} \right] \\ & + C \left[\frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial y} \right)^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right. \\ & \left. + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2 + \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right] + \\ & + F \left(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta} \right) = 0 \\ & = \left[A \left(\frac{\partial \xi}{\partial x} \right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2 \right] \frac{\partial^2 u}{\partial \xi^2} + \dots \quad (P.T.O) \end{aligned}$$

$$+ 2 \left[A \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} + B \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} \right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right] \frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$+ \left[A \left(\frac{\partial \eta}{\partial x} \right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2 \right] \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0$$

$$\bar{A} \frac{\partial^2 u}{\partial \xi^2} + 2\bar{B} \frac{\partial^2 u}{\partial \xi \partial \eta} + \bar{C} \frac{\partial^2 u}{\partial \eta^2} + F(\xi, \eta, u, \frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta}) = 0 \quad (5)$$

where $\bar{A} = A \left(\frac{\partial \xi}{\partial x} \right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2$,

$$\bar{B} = A \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial x} + B \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \eta}{\partial x} \frac{\partial \xi}{\partial y} \right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y}$$

$$\bar{C} = A \left(\frac{\partial \eta}{\partial x} \right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2$$

Note that

$$\bar{B}^2 - \bar{A}\bar{C} = (B^2 - AC) \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right)^2 \quad (7)$$

where A, B, C are not zero at the same time. Thus the transformation of independent variables does not change the type of eq.

Since $\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$

So $\left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right)^2$ is a positive expression.

Show that $\bar{B}^2 - \bar{A}\bar{C} = (B^2 - AC) \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right)^2$

A Side From (6)

$$\bar{B}^2 - \bar{A}\bar{C} = \left[A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right]^2$$

$$- \left[A \left(\frac{\partial \xi}{\partial x} \right)^2 + 2B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2 \right] \times \left[A \left(\frac{\partial \eta}{\partial x} \right)^2 + 2B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2 \right]$$

$$\bar{B}^2 - \bar{A}\bar{C} = A^2 \left(\frac{\partial \xi}{\partial x} \right)^2 \left(\frac{\partial \eta}{\partial x} \right)^2 + B^2 \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right)^2 + C^2 \left(\frac{\partial \xi}{\partial y} \right)^2 \left(\frac{\partial \eta}{\partial y} \right)^2$$

$$+ 2AB \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right)$$

