First order Partial differential Eq.

An equation written in the form

$$f(x, y, z, p, q) = 0 \quad \text{———} (1)$$

is called 1st order PDE. where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$; x and y are independent variables and z is dependent variable.

Origins of First Order PDEs.

Here we shall examine the interesting question of how eqs. of the type (1) arise.

Let us consider an equation

$$x^2 + y^2 + (z - c)^2 = a^2 \quad \text{———} (2)$$

with a and c are arbitrary constants. Eq. (2) represents the set of all spheres whose centers lie along the z-axis.
Differentiation of Eq. (2) w.r.t x and y, respectively yields

$$2x + 2(z - c)\frac{\partial z}{\partial x} = 0$$

$$2y + 2(z - c)\frac{\partial z}{\partial y} = 0$$

$2 \neq 0 \quad \therefore$

$$x + (z - c)\frac{\partial z}{\partial x} = 0$$

$$y + (z - c)\frac{\partial z}{\partial y} = 0$$

or

$$\boxed{\begin{aligned} x + (z - c)p &= 0 \\ y + (z - c)q &= 0 \end{aligned}} \longrightarrow (3)$$

Eliminate c

From (3)

$$\frac{x}{p} = z - c$$

$$\frac{y}{q} = z - c$$

$$\therefore \frac{x}{p} = \frac{y}{q} \Rightarrow x.q = yp$$

$$\Rightarrow \boxed{yp - xq = 0} \longrightarrow (4)$$

1st order PDE.

i.e The Set of all spheres with centers on the z-axis is characterized by the PDE (4)

2)

Let us consider another equation

$$x^2 + y^2 = (z-c)^2 \tan^2\alpha \longrightarrow (5)$$

with c and α as arbitrary constants. Note that Eq. (5) represents the set of all right circular cones whose axis coincide with the line OZ.

Differentiate Eq. (5) w.r.t x and y we have

$$2(z-c)\tan^2\alpha \frac{\partial z}{\partial x} = 2x$$

$$2(z-c)\tan^2\alpha \frac{\partial z}{\partial y} = 2y$$

or

$$\boxed{\begin{aligned} p(z-c)\tan^2\alpha &= x \\ q(z-c)\tan^2\alpha &= y \end{aligned}} \longrightarrow (6)$$

Eliminate c and α

From (6) $(z-c)\tan^2\alpha = \dfrac{x}{p}$

$$(z-c)\tan^2\alpha = \frac{y}{q}.$$

or

$$\frac{x}{p} = \frac{y}{q}.$$

$$\Rightarrow \boxed{yp - xq = 0}$$

which is same as Eq. (4).

Note It means that more than one geometrical entities can be described by the same Eq.

(*) Now what the spheres and cones have in common is that they are surface of revolution which have the line oz as axes of symmetry. All surfaces of revolution with this property are characterized by an equation of the form

$$z = f(x^2+y^2) \longrightarrow (7)$$

where the function f is arbitrary.

If we write

$$x^2 + y^2 = u \longrightarrow (8)$$

then from (7)

$$2x\, f'(u) = \frac{\partial z}{\partial x}$$

$$2y\, f'(u) = \frac{\partial z}{\partial y}$$

or

$$\boxed{\begin{aligned} p &= 2x\, f'(u) \\ q &= 2y\, f'(u) \end{aligned}} \longrightarrow (9)$$

where $f'(u) = df/du$.

From (9) (after eliminating $f'(u)$) we get

$\boxed{yp - xq = 0}$ (Same as (4))

Thus we see that the function Z defined by each of the equations (2), (5) and (7) is in "some sense" solution of Eq. (4).

We now generalize this argument slightly. The relations (2) and (5) are both of the type

$$F(x, y, z, a, b) = 0 \longrightarrow (10)$$

where a and b are arbitrary constants. Differentiate Eq. (10) w.r.t x and y we obtain

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial y} = 0$$

or

$$\boxed{\begin{aligned} &\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z} = 0 \\ &\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z} = 0 \end{aligned}} \longrightarrow (11)$$

The set of equations (10) and (11) constitute three equations involving three arbitrary constants a and b, and, in the general case, it will be possible to eliminate a and b from these equations to obtain a relation of the type

$$\boxed{f(x, y, z, p, q) = 0} \longrightarrow (8)$$

Showing that the system of surfaces give rise to a PDE. of the first order.

Note PDEs may be linear or nonlinear.

Linear Equations of the First order

An equation of the form

$$Pp + Qq = R \longrightarrow (*)$$

where P, Q and R are given functions of x, y and z (which do not involve p or q). $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

is known as 1st order Linear PDE (as p and q have 1 power)
This equation (1) is referred to as Lagrange's equation.

(*) We wish to find a relation between x, y, and z involving an arbitrary function.

Result: The general solution of the linear partial differential equation

$$Pp + Qq = R$$

is

$$F(u, v) = 0$$

where F is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}.$$

Example Find the general solution of

$$x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z.$$

Soln

$$P = x^2, \; Q = y^2; \; R = (x+y)z.$$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \longrightarrow (**)$$

Using values we have

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \longrightarrow (1)$$

From (1) $\frac{dx}{x^2} = \frac{dy}{y^2}$

Integ $\boxed{x^{-1} - y^{-1} = c_1} \longrightarrow (2)$

$$\frac{dx - dy}{x^2 - y^2} \longrightarrow (3)$$

From (1) & (3)

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$$

as $x+y \neq 0$ so

$$\frac{dx - dy}{(x-y)} = \frac{dz}{z}$$

Integ $\ln(x-y) = \ln z + \ln c_2$

$\Rightarrow \ln(x-y) - \ln z = \ln c_2$

$\Rightarrow \ln\left(\frac{x-y}{z}\right) = \ln c_2$

$\Rightarrow \boxed{\frac{x-y}{z} = c_2} \longrightarrow (4)$

Since $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ form a solution of (**). In this case we have after using (2) and (4)

$$x^{-1} - y^{-1} = u$$

$$\frac{x-y}{z} = v.$$

$\therefore$ the general solution is $F(u, v) = 0$

$$\Rightarrow \boxed{F\left(x^{-1} - y^{-1}, \frac{x-y}{z}\right) = 0}.$$

General result: If $u_i(x_1, x_2, \cdots, x_n, z) = c_i$ $(i = 1, 2, \ldots, n)$ are independent solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$$

then the relation $\Phi(u_1, u_2, \ldots, u_n) = 0$, in which the function ϕ is arbitrary, is a general solution of the linear partial differential equation

$$P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R.$$

Example: If u is a function of x, y, and z which satisfies the partial differential equation

$$(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$$

Show that u contains x, y and z only in combinations $x+y+z$ and $x^2+y^2+z^2$.

Soln: Auxiliary equations are

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0}$$

$\Rightarrow$ $du = 0$

$dx + dy + dz = 0$

$x\,dx + y\,dy + z\,dz = 0$

Integrating above equations we get

$u = c_1, \quad x + y + z = c_2, \quad x^2 + y^2 + z^2 = c_3.$

General solution is

$u = f(x + y + z, \ x^2 + y^2 + z^2)$